Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2026, Volume 32, Number 4, 678–685
DOI: 10.7546/nntdm.2026.32.4.678-685

Stable solutions to generalizations of Fibonacci relations of higher order

Timmy Ma [1] **and Richard Vernon** [2]

[1] Department of Mathematics, Xavier University of Louisiana
1 Drexel Dr, New Orleans, LA 70125, USA
e-mail: tma@xula.edu

[2] Department of Mathematics, Xavier University of Louisiana
1 Drexel Dr, New Orleans, LA 70125, USA
e-mail: rvernon@xula.edu

Received: 17 April 2026 **Revised:** 30 October 2026
Accepted: 13 November 2026 **Online First:** 14 November 2026

Abstract: In [9], Özban and Özdemir consider a generalized Fibonacci sequence, i.e., a sequence $\{q_n\}$ such that $q_n = q_{n-1} + q_{n-2}$ for all n. In particular, they give conditions on the initial values for which the resulting sequence is eventually strictly increasing or eventually strictly decreasing and provide an index in terms of the initial values after which the sequence is strictly monotone. In this paper, we extend their results to a generalized k-bonacci sequence.
Keywords: Fibonacci sequences, k-Fibonacci, k-bonacci, Monotone, Stability, Sequences.
2020 Mathematics Subject Classification: 11B39.

1 Introduction

We begin with the *classical k-bonacci sequence* $\{q_n\}$ indexed by the nonnegative integers, as defined in [6]:

$$\begin{cases} q_0 = q_1 = \cdots = q_{k-2} = 0, \\ q_{k-1} = 1, \\ q_n = q_{n-1} + q_{n-2} + \cdots + q_{n-k} \quad \text{for } n \geq k. \end{cases} \tag{1}$$

Copyright © 2026 by the Authors. This is an Open Access paper distributed under the terms and conditions of the Creative Commons Attribution 4.0 International License (CC BY 4.0). https://creativecommons.org/licenses/by/4.0/

If $k = 2$, we have the classical Fibonacci sequence and if $k = 3$, we have the classical Tribonacci sequence. Note that some authors start with $q_1 = 1$ for their sequence as opposed to the $k-1$th term or the kth term [1, 10], but aside from a reindexing, the sequences remain the same.

The Binet formula for the classical k-bonacci sequence (1) is

$$q_n = \sum_{i=1}^{k} A_i \alpha_i^n \tag{2}$$

for each n, where

- α_i is a distinct solution of $x^k - x^{k-1} - \cdots - 1 = 0$ for $1 \leq i \leq k$,
- without loss of generality, $1 < \alpha_1 < 2, A_1 > 0$, and $|\alpha_i| < 1$ for $i > 1$, and
- $\alpha_1 = \lim_{n\to\infty} \frac{q_n}{q_{n-1}}$ and $A_1 = \lim_{n\to\infty} \frac{q_n}{\alpha_1^n}$.

The formulation can be found in [1, 7, 8, 11, 13].

A *generalized k-bonacci sequence* $\{F_n\}$ is indexed by the nonnegative integers and satisfies the following relation for each $n \geq k$:

$$F_n = F_{n-1} + \cdots + F_{n-k}. \tag{3}$$

This formulation is discussed in [1–4, 6, 7, 11–13]. When $k = 2$, we call this a generalized Fibonacci sequence [5]. When $k = 3$, we call this a generalized Tribonacci sequence.

A sequence $\{F_n\}$ is *eventually positive (negative)* if there exists N such that for all $n \geq N$ we have $F_n > 0$ ($F_n < 0$). A sequence $\{F_n\}$ is *eventually strictly increasing (decreasing)* if there exists N such that for all $n \geq N$ we have $F_{n+1} > F_n$ ($F_{n+1} < F_n$). We now introduce a new definition: define a generalized k-bonacci sequence to be *stable* if the sequence does not diverge to infinity or negative infinity. In the next section, we will provide two additional equivalent definitions of stability for a generalized k-bonacci sequence.

2 Trichotomy

Theorem 2.1. *Let $\{F_n\}$ be generalized k-bonacci sequence. Then exactly one of the following three is true:*

1. *$\{F_n\}$ is eventually strictly increasing, eventually positive, and diverges to infinity;*
2. *$\{F_n\}$ is eventually strictly decreasing, eventually negative, and diverges to negative infinity;*
3. *$\{F_n\}$ is neither eventually positive nor eventually negative, is neither eventually strictly increasing nor eventually strictly decreasing, and does not diverge to infinity or negative infinity.*

Proof. Suppose $\{F_n\}$ is eventually strictly increasing. Then there exists N such that for all $n \geq N$ we have $F_{n+1} > F_n$. Then $F_{N+k-1} + \cdots + F_N = F_{N+k} > F_{N+k-1}$, so $F_M > 0$ for some M between N and $N + k - 2$. Then for all $n > M \geq N$ we have $F_n > F_{n-1} > \cdots > F_M > 0$, meaning $\{F_n\}$ is eventually positive.

Next, suppose $\{F_n\}$ is eventually positive. Then there exists N such that for all $n \geq N$ we have $F_n > 0$. Let $M = N + k$ and suppose $n > M$. Then $F_n = F_{n-1} + \cdots + F_{n-k} > F_{n-1}$, meaning $\{F_n\}$ is eventually strictly increasing. Additionally, let $j \geq 1$ and suppose $F_{M+n} \geq nF_M$ for all $0 \leq n \leq j$. Then $F_{M+j-i} > 0$ for all i between 0 and $k-1$, and $F_{M+j-1} \geq F_M$. So $F_{M+j+1} = F_{M+j} + \cdots + F_{M+j-(k-1)} \geq F_{M+j} + F_{M+j-1} \geq jF_M + F_M = (j+1)F_M$. Then by induction, $F_{M+n} > nF_M$ for all $n \geq 1$, so $\{F_n\}$ diverges to infinity.

If $\{F_n\}$ diverges to infinity, then clearly $\{F_n\}$ is eventually positive. Then the three statements in the first case of the theorem are equivalent.

For the second case, given a k-bonacci sequence $\{F_n\}$, define a new k-bonacci sequence $\{G_n\}$ such that $G_n = -F_n$ for all n. Then $\{F_n\}$ being eventually strictly decreasing is equivalent to $\{G_n\}$ being eventually strictly increasing, eventually positive, and diverging to infinity, which is equivalent to $\{F_n\}$ being eventually negative and diverging to negative infinity.

Last, note that each of the statements in the third case is the negation of the corresponding pair of statements in the first two cases. □

By the previous theorem, a generalized k-bonacci sequence is stable exactly when the sequence is not eventually strictly monotone, which is also exactly when the sequence is neither eventually positive nor eventually negative. At the end of the next section, we will give one more equivalent definition of stability.

3 Conditions for stability

We first construct a Binet formula for a generalized k-bonacci sequence. We give three equal expressions, each of which will be used later.

Lemma 3.1. *If $\{F_n\}$ is a generalized k-bonacci sequence, then*

$$\begin{aligned} F_n &= \sum_{i=1}^{k} \left(F_0(\alpha_i^{k-1} - \alpha_i^{k-2} - \cdots - 1) + F_1(\alpha_i^{k-2} - \alpha_i^{k-3} - \cdots - 1) + \cdots + F_{k-1}(1)\right) A_i \alpha_i^n \\ &= \sum_{i=1}^{k} A_i \alpha_i^{n-k} \left(F_0(\alpha_i^{k-1}) + F_1(\alpha_i^{k-1} + \alpha_i^{k-2}) + \cdots + F_{k-1}(\alpha_i^{k-1} + \cdots + 1)\right). \end{aligned}$$

Additionally, for $n \geq k$,

$$F_n = q_{n-1}(F_0 + F_1 + \cdots + F_{k-1}) + q_{n-2}(F_1 + \cdots + F_{k-1}) + \cdots + q_{n-k}(F_{k-1}).$$

Proof. Let g be the generating function for the classical k-bonacci sequence; that is, $g(x) = \sum_{n \geq 0} q_n x^n$. Then

$$g(x) = x^{k-1}/(1 - x - \cdots - x^k) = \sum_{n \geq 0} (A_1 \alpha_1^n + \cdots + A_k \alpha_k^n) x^n.$$

From Wolfram [13], we know that α_i are distinct solutions of $x^k - x^{k-1} - \cdots - 1 = 0$ for $1 \leq i \leq k$, $1 < \alpha_1 < 2$, $A_1 > 0$, and $|\alpha_i| < 1$ for $i > 1$. Next, let G be the generating function for a generalized k-bonacci sequence. Then

$$
\begin{aligned}
G(x) &= \sum_{n\geq 0} F_n x^n, \\
G(x)(1 - x - \cdots - x^k) &= F_0 x^0 + (F_1 - F_0)x^1 + \cdots + (F_{k-1} - \cdots - F_0)x^{k-1}, \\
G(x) &= (F_0 x^{1-k} + (F_1 - F_0)x^{2-k} + \cdots + (F_{k-1} - \cdots - F_0)x^0)g(x) \\
&= \sum_{n\geq 0} F_0 q_n x^{n-k+1} + \sum_{n\geq 0}(F_1 - F_0)q_n x^{n-k+2} + \cdots + \sum_{n\geq 0}(F_{k-1} - \cdots - F_0)q_n x^n \\
&= \sum_{n\geq 1-k} F_0 q_{n+k-1} x^n + \sum_{n\geq 2-k} (F_1 - F_0)q_{n+k-2} x^n + \cdots + \sum_{n\geq 0}(F_{k-1} - \cdots - F_0)q_n x^n.
\end{aligned}
$$

Recall that $q_m \neq 0$ only if $m \geq k-1$. Then for any $i \geq 1$ we know $q_{n+k-i} \neq 0$ only if $n + k - i \geq k - 1$, or $n \geq i - 1 \geq 0$. Then

$$
\begin{aligned}
G(x) &= \sum_{n\geq 0} F_0 q_{n+k-1} x^n + \sum_{n\geq 0}(F_1 - F_0)q_{n+k-2} x^n + \cdots + \sum_{n\geq 0}(F_{k-1} - \cdots - F_0)q_n x^n \\
&= \sum_{n\geq 0} \left(F_0 q_{n+k-1} + (F_1 - F_0)q_{n+k-2} + \cdots + (F_{k-1} - \cdots - F_0)q_n\right) x^n = \sum_{n\geq 0} F_n x^n, \\
F_n &= F_0 q_{n+k-1} + (F_1 - F_0)q_{n+k-2} + \cdots + (F_{k-1} - \cdots - F_0)q_n \\
&= F_0(A_1\alpha_1^{n+k-1} + \cdots + A_k\alpha_k^{n+k-1}) + (F_1 - F_0)(A_1\alpha_1^{n+k-2} + \cdots + A_k\alpha_k^{n+k-2}) + \cdots \\
&\quad + (F_{k-1} - \cdots - F_0)(A_1\alpha_1^n + \cdots + A_k\alpha_k^n) \\
&= \sum_{i=1}^{k} \left(F_0(\alpha_i^{k-1} - \alpha_i^{k-2} - \cdots - 1) + F_1(\alpha_i^{k-2} - \alpha_i^{k-3} - \cdots - 1) + \cdots + F_{k-1}(1)\right) A_i\alpha_i^n.
\end{aligned}
$$

Next, we make use of the following identity for each α_i. Let j be between 0 and k.

$$
\begin{aligned}
\alpha_i^k - \alpha_i^{k-1} - \cdots - 1 &= 0, \\
\alpha_i^k - \alpha_i^{k-1} - \cdots - \alpha_i^{k-j} &= \alpha_i^{k-j-1} + \alpha_i^{k-j-2} + \cdots + 1, \\
\alpha_i^j - \alpha_i^{j-1} - \cdots - 1 &= \alpha_i^{-k}(\alpha_i^{k-1} + \alpha_i^{k-2} + \cdots + \alpha_i^j).
\end{aligned}
$$

The above substitution gives us the lemma's second formulation. Lastly, we regroup our expression by the common powers of each α_i:

$$
\begin{aligned}
F_n &= \sum_{i=1}^{k} A_i\alpha_i^{n-k}\left(F_0(\alpha_i^{k-1}) + F_1(\alpha_i^{k-1} + \alpha_i^{k-2}) + \cdots + F_{k-1}(\alpha_i^{k-1} + \cdots + 1)\right) \\
&= \sum_{i=1}^{k} A_i\alpha_i^{n-k}\left(\alpha_i^{k-1}(F_0 + F_1 + \cdots + F_{k-1}) + \alpha_i^{k-2}(F_1 + \cdots + F_{k-1}) + \cdots + \alpha_i^0(F_{k-1})\right) \\
&= \sum_{i=1}^{k} A_i\left(\alpha_i^{n-1}(F_0 + F_1 + \cdots + F_{k-1}) + \alpha_i^{n-2}(F_1 + \cdots + F_{k-1}) + \cdots + \alpha_i^{n-k}(F_{k-1})\right) \\
&= q_{n-1}(F_0 + F_1 + \cdots + F_{k-1}) + q_{n-2}(F_1 + \cdots + F_{k-1}) + \cdots + q_{n-k}(F_{k-1}).
\end{aligned}
$$

□

Theorem 3.1. *Let $\{F_n\}$ be a generalized k-bonacci sequence. Let*

$$C = F_0(\alpha_1^{k-1}) + F_1(\alpha_1^{k-1} + \alpha_1^{k-2}) + \cdots + F_{k-1}(\alpha_1^{k-1} + \cdots + 1).$$

If $C > 0$, then $\{F_n\}$ diverges to infinity. If $C < 0$, then $\{F_n\}$ diverges to negative infinity. If $C = 0$, then $\{F_n\}$ is stable.

Proof. This follows from the second expression in the statement of Lemma 3.1, the fact that $A_1 > 0$ and $\alpha_1 > 1$, and the fact that every term other than the first in the Binet formula vanishes since $|\alpha_i| < 1$ whenever $i \neq 1$. □

In the case where $k = 2$, the stability or divergence type reduces to whether the expression $F_0(\varphi) + F_1(\varphi + 1)$ is positive, negative, or zero, where $\varphi = (1 + \sqrt{5})/2$, which is analogous to the main result of Özban and Özdemir in [9]. We now list a result which provides one more equivalent definition of stability and follows immediately from Theorem 3.1.

Corollary 3.1. *$\{F_n\}$ is a stable generalized k-bonacci sequence if and only if* $\lim_{n\to\infty} F_n = 0$.

As an example, if $k = 3$, the stability condition becomes

$$F_0(\alpha_1^2) + F_1(\alpha_1^2 + \alpha_1) + F_2(\alpha_1^2 + \alpha_1 + 1) = 0,$$

where $\alpha_1 \approx 1.839$. If we additionally include initial values $(F_0, F_1) = (4.5, -3.2)$, then we would need $F_2 = -\frac{F_0(\alpha_1^2)+F_1(\alpha_1^2+\alpha_1)}{\alpha_1^2+\alpha_1+1} \approx 0.239117$ for $\{F_n\}$ to be stable.

An immediate consequence of a stable generalized k-bonacci sequence is to discuss its sum.

Corollary 3.2. *If $\{F_n\}$ is a stable generalized k-bonacci sequence, then*

$$\sum_{n=0}^{\infty} F_n = \frac{\sum_{i=0}^{k-1} F_i(k-2-i)}{k-1}.$$

Proof. This follows immediately from the generating function $G(x) = \sum_{n\geq 0} F_n x^n$.

$$\begin{aligned}\sum_{n\geq 0} F_n = G(1) &= \frac{F_0 + (F_1 - F_0) + \cdots + (F_{k-1} - F_{k-2} - \cdots - F_0)}{1 - 1 - \cdots - 1} \\ &= \frac{F_0(1-(k-1)) + F_1(1-(k-2)) + \cdots + F_{k-1}(1)}{1-k} \\ &= \frac{\sum_{i=0}^{k-1} F_i(k-2-i)}{k-1}.\end{aligned}$$

□

For example, for $k = 3$, we can see that $\sum_{n\geq 0} F_n = \frac{F_0-F_2}{2}$. Combining this with the fact that $F_2 = -(F_0(\alpha_1^2 - \alpha_1 - 1) + F_1(\alpha_1 - 1))$ for stable 3-bonacci sequences gives us the alternative formula $\sum_{n\geq 0} F_n = F_0(\frac{\alpha_1(\alpha_1-1)}{2}) + F_1(\frac{\alpha_1-1}{2})$.

4 Sufficient bounds

In Theorem 2.1 we showed that any eventually positive (negative) k-bonacci sequence is eventually strictly increasing (decreasing) and diverges to infinity (negative infinity), and in Theorem 3.1 we gave conditions on initial values that will yield this result. We now provide for any unstable generalized k-bonacci sequence a value of N in terms of the initial values such that for all $n \geq N$ we have F_n is positive (negative).

Theorem 4.1. *If $\{F_n\}$ is an unstable k-bonacci sequence and*

$$N = \max\left\{k, \log_{\alpha_1}\left|\frac{\sum_{i=0}^{k-1}|F_i + F_{i+1} + \cdots + F_{k-1}|}{2A_1\sum_{i=0}^{k-1}\alpha_1^{-(i+1)}(F_i + F_{i+1} + \cdots + F_{k-1})}\right|\right\},$$

then either F_n is positive for all $n \geq N$ or F_n is negative for all $n \geq N$.

Proof. Suppose $\{F_n\}$ is eventually positive.

From [1], we know that the distance between q_n and $A_1\alpha_1^n$ is less than $1/2$ for any n. Then for $n \geq k$,

$$\begin{aligned}
F_n &= q_{n-1}(F_0 + F_1 + \cdots + F_{k-1}) + q_{n-2}(F_1 + \cdots + F_{k-1}) + \cdots + q_{n-k}(F_{k-1}) \\
&> A_1\alpha_1^{n-1}(F_0 + F_1 + \cdots + F_{k-1}) + A_1\alpha_1^{n-2}(F_1 + \cdots + F_{k-1}) + \cdots + A_1\alpha_1^{n-k}(F_{k-1}) \\
&\quad - \frac{1}{2}|F_0 + F_1 + \cdots + F_{k-1}| - \frac{1}{2}|F_1 + \cdots + F_{k-1}| - \cdots - \frac{1}{2}|F_{k-1}| \\
&= A_1\sum_{i=0}^{k-1}\alpha_1^{n-(i+1)}(F_i + F_{i+1} + \cdots + F_{k-1}) - \frac{1}{2}\sum_{i=0}^{k-1}|F_i + F_{i+1} + \cdots + F_{k-1}|.
\end{aligned}$$

Since $\{F_n\}$ is eventually positive, from Theorem 2.1 and Theorem 3.1 we know that

$$\sum_{i=0}^{k-1}\alpha_1^{n-(i+1)}(F_i + F_{i+1} + \cdots + F_{k-1}) = C\alpha_1^{n-k} > 0.$$

So if $n \geq k$, then a sufficient condition for F_n to be positive is:

$$A_1\sum_{i=0}^{k-1}\alpha_1^{n-(i+1)}(F_i + F_{i+1} + \cdots + F_{k-1}) - \frac{1}{2}\sum_{i=0}^{k-1}|F_i + F_{i+1} + \cdots + F_{k-1}| \geq 0,$$

$$\begin{aligned}
\alpha_1^n\sum_{i=0}^{k-1}\alpha_1^{-(i+1)}(F_i + F_{i+1} + \cdots + F_{k-1}) &\geq \frac{1}{2A_1}\sum_{i=0}^{k-1}|F_i + F_{i+1} + \cdots + F_{k-1}|, \\
n &\geq \log_{\alpha_1}\left(\frac{\sum_{i=0}^{k-1}|F_i + F_{i+1} + \cdots + F_{k-1}|}{2A_1\sum_{i=0}^{k-1}\alpha_1^{-(i+1)}(F_i + F_{i+1} + \cdots + F_{k-1})}\right) \\
&= \log_{\alpha_1}\left|\frac{\sum_{i=0}^{k-1}|F_i + F_{i+1} + \cdots + F_{k-1}|}{2A_1\sum_{i=0}^{k-1}\alpha_1^{-(i+1)}(F_i + F_{i+1} + \cdots + F_{k-1})}\right|
\end{aligned}$$

In the case where $\{F_n\}$ is eventually negative, we can consider the corresponding eventually positive k-bonacci sequence $\{-F_n\}$. Note that replacing each F_n with $-F_n$ will not change our final expression. □

As an example, if $k = 4$, then by [13], we have $\alpha_1 \approx 1.927561975$ and $A_1 \approx 0.079078$. If we let $(F_0, F_1, F_2, F_3) = (-3, -3.1, 2, 2.2)$ we have the following table:

n	F_n	q_n	q_n/q_{n-1}	q_n/α_1^n
0	-3	0	–	–
1	-3.1	0	–	–
2	2	0	–	–
3	2.2	1	–	–
4	-1.9	1	1	$0.07243\ldots$
5	-0.8	2	2	$0.07516\ldots$
6	1.5	4	2	$0.07798\ldots$
7	1	8	2	$0.08091\ldots$
8	-0.2	15	1.875	$0.07870\ldots$
9	1.5	29	$1.9333\ldots$	$0.07894\ldots$
10	3.8	56	$1.93103\ldots$	$0.07890\ldots$
11	6.1	108	$1.92857\ldots$	$0.07912\ldots$
12	11.2	208	$1.92592\ldots$	$0.07906\ldots$
13	22.6	401	$1.92768\ldots$	$0.07907\ldots$
14	43.7	773	$1.92755\ldots$	$0.07907\ldots$
15	83.6	1490	$1.92757\ldots$	$0.07907\ldots$
16	161.1	2872	$1.92756\ldots$	$0.07907\ldots$
17	311	5536	$1.92756\ldots$	$0.07907\ldots$
18	599.4	10671	$1.92756\ldots$	$0.07907\ldots$
19	1155.1	20569	$1.92756\ldots$	$0.07907\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	∞	∞	α_1	A_1

According to Theorem 4.1, we have $N \approx 10.612$; thus, for all $n \geq 11$ we have $F_n > 0$, which we can see from the table above.

5 Open question

We conclude with an open question. In [9], the authors observed in the case where $k = 2$ that for a nonzero stable sequence, all even terms must be positive and all odd terms must be negative, or vice versa. Is there a way to predict the sign of the nth term of a nonzero stable k-bonacci sequence for arbitrary k in terms of the initial values?

Acknowledgments

We thank the reviewers for providing their time and efforts to provide crucial feedback to the structure and presentation of our work.

References

[1] Dresden, G., & Du, Z. (2014). A simplified Binet formula for k-generalized Fibonacci numbers. *Journal of Integer Sequences*, 17(4), Article ID 14.4.7.

[2] Ferguson, D. E. (1966). An expression for generalized Fibonacci numbers. *The Fibonacci Quarterly*, 4(3), 270–272.

[3] Flores, I. (1967). Direct calculation of k-generalized Fibonacci numbers. *The Fibonacci Quarterly*, 5(3), 259–266.

[4] Gabai, H. (1970). Generalized Fibonacci k-sequences. *The Fibonacci Quarterly*, 8(1), 31–38.

[5] Horadam, A. F. (1961). A generalized Fibonacci sequence. *The American Mathematical Monthly*, 68(5), 455–459.

[6] Lee, G. Y., Lee, S. G., Kim, J. S., & Shin, H. K. (2001). The Binet formula and representations of k-generalized Fibonacci numbers. *The Fibonacci Quarterly*, 39(2), 158–164.

[7] Miles, E. P. (1960). Generalized Fibonacci numbers and associated matrices. *The American Mathematical Monthly*, 67(8), 745–752.

[8] Miller, M. D. (1971). On generalized Fibonacci numbers. *The American Mathematical Monthly*, 78(10), 1108–1109.

[9] Özban, A. Y., & Özdemir, H. (2025). Eventually monotonic solutions of the generalized Fibonacci equations. *Open Mathematics*, 23(1), Article ID 20250198.

[10] Spickerman, W. R. (1982). Binet's formula for the Tribonacci sequence. *The Fibonacci Quarterly*, 20(2), 118–120.

[11] Spickerman, W. R., & Joyner, R. N. (1984). Binet's formula for the recursive sequence of order k. *The Fibonacci Quarterly*, 22(4), 327–331.

[12] Verma, K. L. (2025). The mathematics of generalized Fibonacci sequences: Binet's formula and identities. *Mathematica Moravica*, 29(1), 113–124.

[13] Wolfram, D. A. (1998). Solving generalized Fibonacci recurrences. *The Fibonacci Quarterly*, 36(2), 129–145.