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Some properties of a Padovan sequence

Anthony G. Shannon [1], **Peter J.-S. Shiue** [2] **and Christopher Saito** [3]

[1] Warrane College, the University of New South Wales
Kensington, NSW 2033, Australia
e-mails: tshannon@warrane.unsw.edu.au, tshannon38@gmail.com

[2] Department of Mathematical Sciences, University of Nevada, Las Vegas
Las Vegas, NV, 89154-4020, USA
e-mail: shiue@unlv.nevada.edu

[3] Department of Mathematical Sciences, University of Nevada, Las Vegas
Las Vegas, NV, 89154-4020, USA
e-mail: saitoc1@unlv.nevada.edu

In memory of Peter G. Anderson, 1940–2025,
Graduate of Massachusetts IT, Emeritus of Rochester IT,
Member of Editorial Board of "Notes on Number Theory
and Discrete Mathematics".

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Abstract: This paper explores a connection between third order Padovan sequences and generalized continued fractions by analogy with second order recursive sequences and ordinary two-dimensional continued fractions. It does this with a partial recurrence relation which is related to the original third order Padovan relation.

Keywords: Fibonacci numbers, Tribonacci numbers, Partial quotients, Euclidean Algorithm, Padovan sequences.
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1 Introduction

We define three basic 3rd order linear recursive sequences $\{t_{m,n}\}$ $m = 0, 1, 2$ by initial values for $n = 0, 1, 2$ and the 3rd order linear homogeneous recurrence relation Shannon [15]

$$t_{m,n} = P_1 t_{m,n-1} + P_2 t_{m,n-2} + P_3 t_{m,n-3}, n \geq 3, \tag{1.1}$$

in which the P_j are arbitrary integers so that (1.1) is essentially analogous to Horadam's generalized sequence for his 2nd order $\{w_n\}$ Feinberg [10]. For later convenience, we shall take the initial values as $t_{m,0} = t_{m,1} = t_{m,2} = 1$ when $P_0 = 1, P_1 = 0, P_2 = P_3 = 1$, a Padovan sequence Borwein *et al.* [5, §2.4] – sequence A000931 in Sloane [20]. We also define related arrays of sequences $\{v_{m,n}\}$ for any integer m by a partial recurrence relation

$$v_{m,n} = v_{m-1,n-1} + P_{3-n} v_{m-1,2}, 0 < m < 3, n > 0, \tag{1.2}$$

with boundary conditions

$$v_{m,0} = P_3 t_n \text{ and } v_{0,n} = P_{3-n}. \tag{1.3}$$

The purpose of this paper is to explore some of the properties of this array as a non-standard recurrence Anderson *et al.* [3].

2 Partial recurrence relation

We first show that the characteristic equation associated with (1.1) is related to the partial recurrence relation by

$$x^3 = \sum_{n=0}^{2} P_{3-n} x^n \to x^{m+3} = \sum_{n=0}^{2} v_{m,n} x^n \tag{2.1}$$

the proof of which follows by induction on m.

When $m = 0$,

$$\sum_{n=0}^{2} v_{0,n} x^n = \sum_{n=0}^{2} P_{3-n} x^n = x^3.$$

When $m = 1$,

$$\begin{aligned}
\sum_{n=0}^{2} v_{1,n} x^n &= \sum_{n=0}^{2} \left(v_{0,n-1} + P_{3-n} v_{0,2} \right) x^n \\
&= x \sum_{n=0}^{2} v_{0,n-1} x^{n-1} + v_{0,2} \sum_{n=0}^{2} P_{3-n} x^n \\
&= x \sum_{n=0}^{1} v_{0,n} x^n + v_{0,2} x^3 \\
&= x \sum_{n=0}^{2} v_{0,n} x^n \\
&= x \sum_{n=0}^{2} P_{3-n} x^n \\
&= x^4.
\end{aligned}$$

Assume the result is true for $m = 2, 3, \ldots, s$. Then

$$\begin{aligned}
x^{s+4} &= \sum_{n=0}^{2} v_{s,n} x^{n+1} \\
&= v_{s,2} x^3 + \sum_{n=0}^{1} v_{s,n} x^{n+1} \\
&= v_{s,2} x^3 + \sum_{n=0}^{2} v_{s,n-1} x^n \\
&= v_{s,2} x^3 + \sum_{n=0}^{2} \left(v_{s+1,n} - P_{3-n} v_{s,2} \right) x^n \\
&= v_{s,2} x^3 - v_{s,2} \sum_{n=0}^{2} P_{3-n} x^n + \sum_{n=0}^{2} v_{s+1,n} x^n \\
&= v_{s,2} \left(x^3 - P_3 - P_2 x - P_1 x^2 \right) + \sum_{n=0}^{2} v_{s+1,n} x^n \\
&= \sum_{n=0}^{2} v_{s+1,n} x^n.
\end{aligned}$$

3 Application

We now apply this to the case where $P_0 = 1, P_1 = 0, P_2 = P_3 = 1$, which Hildebrand [11, p. 461] used as an example in old-fashioned numerical analysis; that is, for

$$x^6 = \sum_{n=0}^{2} v_{3,n} x^6 = 1 + 2x + x^2. \tag{3.1}$$

We also have from (2.1) that

$$f(3) = x^3 = x + 1,$$

which is, in effect, the auxiliary equation for the corresponding third order Padovan sequence, some properties of which have been developed in Jarden [12, p. 114] and Shannon *et al.* [16, 17]. If we apply (2.1), we see initially that

$$\begin{aligned}
f(3) &= x + 1, \\
f(4) &= x^2 + x, \\
f(5) &= x^2 + x + 1
\end{aligned}$$

and so on by using (2.1) again

$$f(48) = 170625x^2 + 226030x + 128801$$

and

$$f(49) = 226030x^2 + 299426x + 170625$$

which conforms with Gnanadoss [9], to find that if we use $x = 1.3$ as an approximation of the dominant root of $x^3 - x - 1 = 0$, then a better approximation with the Bernoulli iteration can be given by

$$\frac{f(49)}{f(48)} = 1.324717973\ldots,$$

which agrees with Hildebrand [11, p. 461]. This in turn suggests how this can also be achieved even more succinctly with the Padovan sequences $\{P_n\}$, such that

$$x^{n+3} = P_{n+1}x^2 + P_{n+2}x + P_n \tag{3.2}$$

as exemplified in Table 1.

Table 1. Padovan sequences

m	x^m	$A_{m+1}x^2 + A_{m+2}x + A_m$	Expansion	$P_{n+1}x^2 + P_{n+2}x + P_n$ $= x^{n+3}$	n
0	$x^0 =$	$A_1x^2 + A_2x + A_0 =$	1		
1	$x^1 =$	$A_2x^2 + A_3x + A_1 =$	x		
2	$x^2 =$	$A_3x^2 + A_4x + A_2 =$	x^2		
3	$x^3 =$	$A_4x^2 + A_5x + A_3 =$	$x + 1$	$= P_1x^2 + P_2x + P_0$	0
4	$x^4 =$	$A_5x^2 + A_6x + A_4 =$	$x^2 + x$	$= P_2x^2 + P_3x + P_1$	1
5	$x^5 =$	$A_6x^2 + A_7x + A_5 =$	$x^2 + x + 1$	$= P_3x^2 + P_4x + P_2$	2
6	$x^6 =$	$A_7x^2 + A_8x + A_6 =$	$x^2 + 2x + 1$	$= P_4x^2 + P_5x + P_3$	3
7	$x^7 =$	$A_8x^2 + A_9x + A_7 =$	$2x^2 + 2x + 1$	$= P_5x^2 + P_6x + P_4$	4
8	$x^8 =$	$A_9x^2 + A_{10}x + A_8 =$	$2x^2 + 3x + 2$	$= P_6x^2 + P_7x + P_5$	5
9	$x^9 =$	$A_{10}x^2 + A_{11}x + A_9 =$	$3x^2 + 4x + 2$	$= P_7x^2 + P_8x + P_6$	6
10	$x^{10} =$	$A_{11}x^2 + A_{12}x + A_{10} =$	$4x^2 + 5x + 3$	$= P_8x^2 + P_9x + P_7$	7
11	$x^{11} =$	$A_{12}x^2 + A_{13}x + A_{11} =$	$5x^2 + 7x + 4$	$= P_9x^2 + P_{10}x + P_8$	8
12	$x^{12} =$	$A_{13}x^2 + A_{14}x + A_{12} =$	$7x^2 + 9x + 5$	$= P_{10}x^2 + P_{11}x + P_9$	9
13	$x^{13} =$	$A_{14}x^2 + A_{15}x + A_{13} =$	$9x^2 + 12x + 7$	$= P_{11}x^2 + P_{12}x + P_{10}$	10
14	$x^{14} =$	$A_{15}x^2 + A_{16}x + A_{14} =$	$12x^2 + 16x + 9$	$= P_{12}x^2 + P_{13}x + P_{11}$	11
15	$x^{15} =$	$A_{16}x^2 + A_{17}x + A_{15} =$	$16x^2 + 21x + 12$	$= P_{13}x^2 + P_{14}x + P_{12}$	12
16	$x^{16} =$	$A_{17}x^2 + A_{18}x + A_{16} =$	$21x^2 + 28x + 16$	$= P_{14}x^2 + P_{15}x + P_{13}$	13
17	$x^{17} =$	$A_{18}x^2 + A_{19}x + A_{17} =$	$28x^2 + 37x + 21$	$= P_{15}x^2 + P_{16}x + P_{14}$	14
18	$x^{18} =$	$A_{19}x^2 + A_{20}x + A_{18} =$	$37x^2 + 49x + 28$	$= P_{16}x^2 + P_{17}x + P_{15}$	15
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
48	$x^{48} =$	$A_{49}x^2 + A_{50}x + A_{48} =$	$170625x^2 + 226030x + 128801$	$= P_{46}x^2 + P_{47}x + P_{45}$	45
49	$x^{49} =$	$A_{50}x^2 + A_{51}x + A_{49} =$	$226030x^2 + 299426x + 170625$	$= P_{47}x^2 + P_{48}x + P_{46}$	46

This seems to follow because Gnanadoss and Hildebrand both chose Padovan polynomials in their examples, and the initial Padovan values are $P_0 = 1, P_1 = 0, P_2 = x$, respectively so that (3.2) can be proved by induction on n. Thus, we can obtain:

Theorem 3.1. *Let* $\{P_n\}_{n\geq 0}$ *with* $P_0 = 1$, $P_1 = 0$, $P_2 = 1$, *and* $P_{n+3} = P_{n+1} + P_n (n \geq 0)$ *be the Padovan sequence. Then*

$$x^{n+3} = P_{n+1}x^2 + P_{n+2}x + P_n \text{ for } n \geq 0,$$

where x *is a root of* $x^3 - x - 1$.

Corollary 3.1. *Following N. J. A. Sloane* [20], *let* $\{A_m\}_{m\geq 0}$ *with* $A_0 = 1$, $A_1 = A_2 = 0$, *and* $A_{m+3} = A_{m+1} + A_m$ $(m \geq 0)$. *Then*

$$x^m = A_{m+1}x^2 + A_{m+2}x + A_m \text{ for } n \geq 0,$$

where x *is a root of* $x^3 - x - 1$.

We provide the proofs of Theorem 3.1 and Corollary 3.1 later on. They are based on the following Lemma 3.1 and noting that $P_n = A_{n+3}$.

Lemma 3.1. $x^n = (x^3 - x - 1)\, g_n(x) + A_{n+1}x^2 + A_{n+2}x + A_n$.

Proof. Consider

$$x^n = \left(x^3 - x - 1\right) g_n(x) + a_n x^2 + b_n x + c_n. \tag{3.3}$$

Then $x^{n-1} = (x^3 - x - 1)\, g_{n-1}(x) + a_{n-1}x^2 + b_{n-1}x + c_{n-1}$. Hence,

$$\begin{aligned} x^n = xx^{n-1} &= x\left(x^3 - x - 1\right) g_{n-1}(x) + a_{n-1}x^3 + b_{n-1}x^2 + c_{n-1}x \\ &= \left(x^3 - x - 1\right)\left[xg_{n-1}(x) + a_{n-1}\right] + a_{n-1}x + a_{n-1} + b_{n-1}x^2 + c_{n-1}x \\ &= \left(x^3 - x - 1\right)\left[xg_{n-1}(x) + a_{n-1}\right] + b_{n-1}x^2 + \left(a_{n-1} + c_{n-1}\right)x + a_{n-1}. \end{aligned} \tag{3.4}$$

By comparing (3.3) and (3.4), we obtain

$$a_n = b_{n-1}, \tag{3.5}$$

$$b_n = a_{n-1} + c_{n-1}, \tag{3.6}$$

$$c_n = a_{n-1}. \tag{3.7}$$

Then $c_{n+1} = a_n = b_{n-1}$ by (3.7) and (3.5), respectively. Further, $b_{n-1} = a_{n-2} + c_{n-2} = c_{n-1} + c_{n-2}$ by (3.5), (3.6), and (3.7), respectively. Thus,

$$c_{n+1} = c_{n-1} + c_{n-2}. \tag{3.8}$$

Now, for $n = 0$, let $a_0 = b_0 = 0$, $c_0 = 1$. Let $n = 1$; then $a_1 = 0$, $b_1 = 1$, $c_1 = 0$. Let $n = 2$; then $a_2 = 1$, $b_2 = 0$, $c_2 = 0$. Hence, we have $c_{n+1} = c_{n-1} + c_{n-2}$ with $c_0 = 1$ and $c_1 = c_2 = 0$, which implies that $c_n = A_n$. Additionally, we have $a_n = c_{n+1} = A_{n+1}$, $b_n = a_{n-1} + c_{n-1} = c_n + c_{n-1} = A_n + A_{n-1} = A_{n+2}$ by (3.6) and (3.7). Therefore, we have

$$x^n = \left(x^3 - x - 1\right) g(x) + A_{n+1}x^2 + A_{n+2}x + A_n. \qquad \square$$

Corollary 3.2. $x^{n+3} = (x^3 - x - 1)\, g_n(x) + P_{n+1}x^2 + P_{n+2}x + P_n$.

Proof. We have $x^{n+3} = (x^3 - x - 1)\, g_n(x) + A_{n+4}x^2 + A_{n+5}x + A_{n+3}$ by Lemma 3.1. Since $A_{n+3} = P_n$, it follows that $x^{n+3} = (x^3 - x - 1)\, g_n(x) + P_{n+1}x^2 + P_{n+2}x + P_n$. $\square$

We include the next result to demonstrate the similarity between Padovan sequences and Fibonacci sequences. Further results are given by Roettger and Williams [14].

Corollary 3.3. $x^n = (x^2 - x - 1)\, g_n(x) + F_n x + F_{n-1}$ *for* $n \geq 1$.

Proof. The proof is similar to that of Lemma 3.1. □

Example 1. See Table 2 below.

Table 2. Powers of a root of $x^2 - x - 1 = 0$ expressed in terms of Fibonacci numbers.

n	$x^n = F_n x + F_{n-1}$
0	$x^0 = 1$
1	$x^1 = F_1 x + F_0 = x$
2	$x^2 = F_2 x + F_1 = x + 1$
3	$x^3 = F_3 x + F_2 = 2x + 1$
4	$x^4 = F_4 x + F_3 = 3x + 2$
5	$x^5 = F_5 x + F_4 = 5x + 3$
⋮	⋮

We can now readily and simply prove Theorem 3.1 and Corollary 3.1.

Proof of Theorem 3.1 and Corollary 3.1. By Corollary 3.2, we have

$$x^{n+3} = (x^3 - x - 1)g(x) + P_{n+1}x^2 + P_{n+2}x + P_n.$$

Thus, because $x^3 = x + 1$,

$$x^{n+3} = P_{n+1}x^2 + P_{n+2}x + P_n,$$

which proves Theorem 3.1.

Furthermore, since $P_n = A_{n+3}$, we have

$$x^{n+3} = P_{n+1}x^2 + P_{n+2}x + P_n = A_{n+4}x^2 + A_{n+5}x + A_{n+3}.$$

Let $n + 3 = m$, so that $x^m = A_{m+1}x^2 + A_{m+2}x + A_m$ for $m \geq 3$. The cases $m = 0, 1, 2$ follow directly from the initial values and the recurrence for $\{A_m\}_{m\geq 0}$.

This proves Corollary 3.1. □

Corollary 3.4. *Let* $x^{n+3} = P_{n+1}x^2 + P_{n+2}x + P_n$ *and* $x^{n+4} = P_{n+2}x^2 + P_{n+3}x + P_{n+1}$. *Then* $x^{n+6} = (P_{n+1} + P_{n+2})x^2 + (P_{n+2} + P_{n+3})x + (P_n + P_{n+1})$. *Furthermore,* $\{x^{n+3}\}_{n\geq 0} \bigcup \{1, x, x^2\}$ *is a Padovan sequence with initial values,* $1, x, x^2$, *where* $x \in \mathbb{N}$.

Proof. Since $P_{n+1} + P_{n+2} = P_{n+4}$, $P_{n+2} + P_{n+3} = P_{n+5}$, and $P_n + P_{n+1} = P_{n+3}$, we obtain $x^{n+6} = P_{n+4}x^2 + P_{n+5}x + P_{n+3}$. □

Corollary 3.5. *Let* $x^m = A_{m+1}x^2 + A_{m+2}x + A_m$ *and* $x^{m+1} = A_{m+2}x^2 + A_{m+3}x + A_{m+1}$.

Then $x^{n+3} = (A_{m+1} + A_{m+2})x^2 + (A_{m+2} + A_{m+3})x + (A_m + A_{m+1})$. Furthermore, $\{x^{n+3}\}_{n\geq 0} \bigcup \{1, x, x^2\}$ is a Padovan sequence with initial values, $1, x, x^2$, where $x \in \mathbb{N}$.

Corollary 3.6. *Let* $\{Q_n\}_{n\geq 0}$ *with* $Q_0 = Q_1 = Q_2 = 1$ *and* $Q_{n+3} = Q_{n+1} + Q_n (n \geq 0)$. *Let* $x^{n+5} = Q_{n+1}x^2 + Q_{n+2}x + Q_n$ *and* $x^{n+6} = Q_{n+2}x^2 + Q_{n+3}x + Q_{n+1}$. *Then* $x^{n+8} = (Q_{m+1} + Q_{m+2})x^2 + (Q_{m+2} + Q_{m+3})x + (Q_m + Q_{m+1})$. *Furthermore,* $\{x^{n+3}\}_{n\geq 0} \bigcup \{1, x, x^2\}$ *is a Padovan sequence with initial values,* $1, x, x^2$, *where* $x \in \mathbb{N}$.

The sequence $\{Q_n\}_{n\geq 0}$, also called the Cordonnier sequence, was discussed in Shannon *et al.* [17, 18].

Corollary 3.4 can be used to compute x^{n+3} without $\{P_n\}_{n\geq 0}$ by adding the corresponding coefficients of x^2, x, and x^0, respectively.

Example 2. We compute x^7.

$$\begin{aligned}
x^0 &= 1 = 0x^2 + 0x + 1 \\
x^1 &= x = 0x^2 + 1x + 0 \\
x^2 &= x^2 = 1x^2 + 0x + 0 \\
x^3 &= x + 1 && \text{(by adding the coefficients of } x^1 \text{ and } x^0\text{)}, \\
x^4 &= x^2 + x && \text{(by adding the coefficients of } x^2 \text{ and } x^1\text{)}, \\
x^5 &= x^2 + x + 1 && \text{(by adding the coefficients of } x^3 \text{ and } x^2\text{)}, \\
x^6 &= x^2 + 2x + 1 && \text{(by adding the coefficients of } x^4 \text{ and } x^3\text{)}, \\
x^7 &= 2x^2 + 2x + 1 && \text{(by adding the coefficients of } x^5 \text{ and } x^4\text{)}.
\end{aligned}$$

We have similar results and applications for $\{A_n\}_{n\geq 0}$ and $\{F_n\}_{n\geq 0}$.

Corollary 3.7. *Let* $x^{n+1} = F_{n+1}x + F_n$ *and* $x^{n+2} = F_{n+2}x + F_{n+1}$. *Then* $x^{n+3} = (F_{n+1} + F_{n+2})x + (F_n + F_{n+1})$.

Example 3.

$$\begin{aligned}
x^0 &= 1 = 0x + 1 \\
x^1 &= x = x + 0 \\
x^2 &= x + 1 && \text{(by adding the coefficients of } x^1 \text{ and } x^0\text{,} \\
x^3 &= 2x + 1 && \text{(by adding the coefficients of } x^2 \text{ and } x^1\text{,} \\
x^4 &= 3x + 2 && \text{(by adding the coefficients of } x^3 \text{ and } x^2\text{,} \\
x^5 &= 5x + 3 && \text{(by adding the coefficients of } x^4 \text{ and } x^3
\end{aligned}$$

Theorem 3.2. *Let* $\{P_n\}$ *be the Padovan sequence with characteristic polynomial* $x^3 - x - 1 = 0$. *If* $b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0 \equiv 0 (mod x^3 - x - 1)$, *then* $b_n P_n + b_{n-1}P_{n-1} + \cdots + b_1 P_1 + P_0 = 0$.

Proof. This is a special case of Lucas's [13] results. □

Example 4. Prove $P_{n-2} = 3P_{n-5} - 2P_{n-8} + P_{n-11}$ for $n \geq 11$, and $A_{n+1} = 3A_{n-2} + A_{n-8}$ for $n \geq 8$ (see Tedford [22]).

Proof. The characteristic polynomial of $P_{n-2} = 3P_{n-5} - 2P_{n-8} + P_{n-11}$ is $x^9 - 3x^6 + 2x^3 - 1$. Now, from Table 1,

$$x^9 = 3x^2 + 4x + 2,$$
$$x^6 = x^2 + 2x + 1,$$
$$x^3 = x + 1.$$

Hence,

$$x^9 - 3x^6 + 2x^3 - 1 = (3x^2 + 4x + 2) - 3(x^2 + 2x + 1) + 2(x + 1) - 1 = 0.$$

This means that $(x^3 - x - 1)|(x^9 - 3x^6 + 2x^3 - 1)$. Thus, $P_{n-2} = 3P_{n-5} - 2P_{n-8} + P_{n-11}$. Furthermore, since $P_n = A_{n+3}$, it follows that $A_{n+1} = 3A_{n-2} - 2A_{n-5} + A_{n-8}$ for $n \geq 8$. □

Remark 1. One may use mathematical induction or long division to prove this identity, but by using Table 1 is the best one.

There is another advantage by using Table 1 to create a new identity. Suppose we want to find a, b, and c so that $P_n = aP_{n-4} + bP_{n-8} + cP_{n-12}$ is an identity for a Padovan sequence. The characteristic polynomial is $x^{12} - ax^8 - bx^4 - c = 0$. Since

$$x^{12} = 7x^2 + 9x + 5,$$
$$x^8 = 2x^2 + 3x + 2,$$
$$x^4 = x^2 + x,$$

we have

$$(7x^2 + 9x + 5) - a(2x^2 + 3x + 2) - b(x^2 + x) - c = 0.$$

Then $7 - 2a - b = 0$, $9 - 3a - b = 0$, and $5 - 2a - c = 0$, and so $a = 2$, $b = 3$, and $c = 1$. Thus, $P_n = 2P_{n-4} + 3P_{n-8} + P_{n-12}$. Similarly, $P_n = 3P_{n-3} - 2P_{n-6} + P_{n-9}$. (See Cerda-Morales [6, Theorem 3] for another approach.)

Using Theorem 3.2, we have the following algorithm for establishing $b_nP_n + b_{n-1}P_{n-1} + \cdots + b_0P_0 = 0$, $b_nA_n + b_{n-1}A_{n-1} + \cdots + b_0A_0 = 0$, and $b_nQ_n + b_{n-1}Q_{n-1} + \cdots + b_0Q_0 = 0$, where $b_i \in \mathbb{Z}(i = 0, 1, 2, \ldots, n)$.

Algorithm

Step 1. Pick an arbitrary polynomial $b_nx^n + b_{n-1}x^{n-1} + \cdots + b_1x + b_0 = 0, b_i \in \mathbb{Z}$.

Step 2. Using Table 1, compute $b_nx^n + b_{n-1}x^{n-1} + \cdots + b_1x + b_0 = px^2 + qx + r$.

Step 3. Compute $b_nx^n + b_{n-1}x^{n-1} + \cdots + b_1x + b_0 - px^2 - qx - r = 0$, i.e., $b_nx^n + b_{n-1}x^{n-1} + \cdots + b_3x^3 + (b_2 - p)x^2 + (b_1 - q)x + (b_0 - r) = 0$.

Step 4. Applying Theorem 3.2, we have

$$b_nP_n + b_{n-1}P_{n-1} + \cdots + b_3P_3 + (b_2 - p)P_2 + (b_1 - q)P_1 + (b_0 - r) = 0,$$

and since $P_n = A_{n+3}$ and $Q_n = A_{n+5}$, we have further that

$$b_nA_{n+3} + b_{n-1}A_{n+2} + \cdots + b_3A_6 + (b_2 - p)A_5 + (b_1 - q)A_4 + (b_0 - r)A_3 = 0,$$

and

$$b_nQ_n + b_{n-1}Q_{n-1} + \cdots + b_3Q_3 + (b_2 - p) + (b_1 - q) + (b_0 - r) = 0.$$

Example 5. We pick $x^8-x^5+2x^4+x+1=0$. From Table 1, $x^8=2x^2+3x+2$, $x^5=x^2+x+1$, and $x^4=x^2+x$, and so $x^8-x^5+2x^4+x+1=3x^2+5x+2$. Then
$x^8-x^5+2x^4-3x^2-4x-1=0$. Equivalently,

$$x^{n+8}-x^{n+5}+2x^{n+4}-3x^{n+2}-4x^{n+1}-x^n=0$$

By Theorem 3.2, we have

$$\begin{gathered}P_{n+8}-P_{n+5}+2P_{n+4}-3P_{n+2}-4P_{n+1}-P_n=0,\\ A_{n+11}-A_{n+8}+2A_{n+7}-3A_{n+5}-4A_{n+4}-A_{n+3}=0, \text{ and}\\ Q_{n+8}-Q_{n+5}+2Q_{n+4}-3Q_{n+2}-4Q_{n+1}-Q_n=0.\end{gathered}$$

4 More identities

In this section, we consider the same sequences P_n and A_n. However, we explore the various ways in which we may sum the terms of each sequence.

Theorem 4.1. $A_0+A_1+A_2+\cdots+A_n=A_{n+5}$.

Proof. Consider $1+x+x^2+\cdots+x^n=ax^2+bx+c$. Then

$$\frac{1-x^{n+1}}{1-x}=ax^2+bx+c,$$

and so

$$1-x^{n+1}=(1-x)(ax^2+bx+c).$$

Since

$$x^{n+1}=A_{n+2}x^2+A_{n+3}x+A_{n+1},$$

we have

$$\begin{aligned}1-(A_{n+2}x^2+A_{n+3}x+A_{n+1})&=(ax^2+bx+c)-(ax^3+bx^2+cx)\\&=ax^2+bx+c-[a(x+1)+bx^2+cx]\\&=(a-b)x^2+(b-a-c)x+(c-a).\end{aligned}$$

This implies

$$\begin{aligned}-A_{n+2}&=a-b,\\-A_{n+3}&=b-a-c,\\1-A_{n+1}&=c-a;\end{aligned}$$

equivalently,

$$\begin{aligned}A_{n+2}&=b-a,\\A_{n+3}&=a+c-b,\\A_{n+1}&=a-c+1.\end{aligned}$$

Thus, $c=A_{n+2}+A_{n+3}$, $a=(A_{n+1}+A_{n+2}+A_{n+3})-1$, $b=2A_{n+2}+A_{n+1}+A_{n+3}-1$.

Hence,

$$\begin{aligned}1+x+x^2+\ldots+x^n &= \left[\left(A_{n+1}+A_{n+2}+A_{n+3}\right)-1\right]x^2+\\ &\quad [2A_{n+2}+A_{n+1}+A_{n+3}-1]x+[A_{n+2}+A_{n+3}]\\ &= (A_{n+2}+A_{n+3})(1+x+x^2)+(A_{n+1}-1)(x^2+x)+A_{n+2}x\\ &= A_{n+5}x^5+(A_{n+1}-1)x^4+A_{n+2}x.\end{aligned}$$

Thus, since $A_2 = A_1 = 0$,

$$\begin{aligned}A_0+A_1+A_2+\ldots+A_n &= A_{n+5}*A_5+\left(A_{n+1}-1\right)A_4+A_{n+1}A_1\\ &= A_{n+5}.\end{aligned}$$

□

Example 6. $A_0+A_1+\cdots+A_{10} = 1+0+0+1+0+1+1+1+2+2+3 = 12 = A_{15}$.

Corollary 4.1. *Let $\{P_n\}$ be the Padovan sequence given by the third order linear recurrence relation $P_{n+3} = P_{n+1}+P_n$ for $n \geq 0$ with initial conditions $P_0 = 1, P_1 = 0, P_2 = 1$. Then $P_0+P_1+\cdots+P_n = P_{n+5}-1$.*

Proof. Note that $P_n = A_{n+3}$. By Theorem 4.1, we have

$$\begin{aligned}P_0+P_1+\cdots+P_n &= A_3+A_4+\cdots+A_{n+3}\\ &= (A_0+A_1+A_2+A_3+A_4+\cdots+A_{n+3})-(A_0+A_1+A_2)\\ &= A_{n+8}-(1+0+0)\\ &= P_{n+5}-1.\end{aligned}$$

□

Corollary 4.2. *Let $\{Q_n\}$ be the Padovan sequence given by the third order linear recurrence relation $Q_{n+3} = Q_{n+1}+Q_n$ for $n \geq 0$ with initial conditions $Q_0 = Q_1 = Q_2 = 1$. Then $Q_0+Q_1+\cdots+Q_n = Q_{n+5}-2$.*

Proof. Note that $Q_n = A_{n+5}$. By Theorem 4.1, we have

$$\begin{aligned}Q_0+Q_1+\cdots+Q_n &= A_5+A_6+\cdots+A_{n+5}\\ &= (A_0+A_1+A_2+A_3+A_4+A_5+A_6+\cdots+A_{n+5})\\ &\quad -(A_0+A_1+A_2+A_3+A_4)\\ &= A_{n+8}-(1+0+0+1+0)\\ &= A_{n+8}-2\\ &= Q_{n+5}-2.\end{aligned}$$

□

Example 7.

1. $$\begin{aligned}P_0+P_1+\cdots+P_{10} &= 1+0+1+1+1+2+2+3+4+5+7\\ &= 26\\ &= P_{15}-1;\end{aligned}$$

2. $$\begin{aligned}Q_0+Q_1+\cdots+Q_{10} &= 1+1+1+2+2+3+4+5+7+9+12\\ &= 47\\ &= Q_{15}-2.\end{aligned}$$

Theorem 4.2. $A_0 + A_2 + \cdots + A_{2n} = A_{2n+3}$.

Proof. Consider

$$1 + x^2 + x^4 + \cdots + x^{2n} = ax^2 + bx + c.$$

Then

$$1 - x^{2n+2} = (1 - x^2)(ax^2 + bx + c).$$

Since

$$x^{2n+2} = A_{2n+3}x^2 + A_{2n+4}x + A_{2n+2},$$

we have

$$\begin{aligned} 1 - (A_{2n+3}x^2 + A_{2n+4}x + A_{2n+2}) &= (ax^2 + bx + c) - (ax^4 + bx^3 + cx^2) \\ &= ax^2 + bx + c - [a(x^2 + x) + b(x + 1) + cx^2] \\ &= -cx^2 - ax + (c - b). \end{aligned}$$

This implies

$$-A_{2n+3} = -c, -A_{2n+4} = -a, \text{ and } 1 - A_{2n+2} = c - b,$$

and so

$$A_{2n+3} = c, A_{2n+4} = a, \text{ and } b = A_{2n+3} + A_{2n+2} - 1.$$

Hence,

$$1 + x^2 + x^4 + \cdots + x^{2n} = A_{2n+4}x^2 + (A_{2n+3} + A_{2n+2} - 1)x + A_{2n+3}.$$

Thus, since $A_2 = A_1 = 0$,

$$\begin{aligned} A_0 + A_2 + \ldots + A_{2n} &= A_{2n+4}A_2 + (A_{2n+5} - 1)\, A_1 + A_{2n+3} \\ &= A_{2n+3}. \end{aligned}$$

□

Example 8. $A_0 + A_2 + A_4 + \cdots + A_{18} = 1 + 0 + 0 + 1 + 2 + 3 + 5 + 9 + 16 + 28 = 65 = A_{21}$.

Theorem 4.3. $A_1 + A_3 + \cdots + A_{2n+1} = A_{2n+4}$.

Proof. Consider

$$x + x^3 + \cdots + x^{2n+1} = ax^2 + bx + c;$$

equivalently,

$$x(1 + x^2 + x^4 + \cdots + x^{2n}) = ax^2 + bx + c.$$

Then, from the proof of Theorem 4.2, we have

$$x - x^{2n+3} = (1 - x^2)(ax^2 + bx + c) = -cx^2 - ax + (c - b)$$

Since

$$x^{2n+3} = A_{2n+4}x^2 + A_{2n+5}x + A_{2n+3},$$

we have

$$x - (A_{2n+4}x^2 + A_{2n+5}x + A_{2n+3}) = -cx^2 - ax + (c - b).$$

Then $c = A_{2n+4}$, $a = A_{2n+5} - 1$, $b = A_{2n+6}$, and hence,

$$x + x^3 + \cdots + x^{2n+1} = (A_{2n+5} - 1)x^2 + A_{2n+6}x + A_{2n+4}.$$

Thus, since $A_2 = A_1 = 0$,

$$A_1 + A_3 + \cdots + A_{2n+1} = A_{2n+4}. \qquad \square$$

Example 9. $A_1 + A_3 + A_5 + \cdots + A_{15} = 0 + 1 + 1 + 1 + 2 + 4 + 7 + 12 = 28 = A_{18}$.

Remark 2. Theorem 4.3 can be proved by Theorem 4.1 and Theorem 4.2 as follows.
From Theorem 4.1, we have

$$A_0 + A_1 + A_2 + \cdots + A_{2n+1} = A_{2n+6};$$

equivalently,

$$(A_0 + A_2 + \cdots + A_{2n}) + (A_1 + A_3 + \cdots + A_{2n+1}) = A_{2n+6}.$$

From Theorem 4.2, we have

$$A_0 + A_2 + \cdots + A_{2n} = A_{2n+3}.$$

Combining these facts, we obtain

$$A_1 + A_3 + \cdots + A_{2n+1} = A_{2n+6} - A_{2n+3} = A_{2n+4},$$

i.e., Theorem 4.3.

Corollary 4.3. *Let $\{P_n\}$ be the Padovan sequence given by the third order linear recurrence relation $P_{n+3} = P_{n+1} + P_n$ for $n \geq 0$ with initial conditions $P_0 = 1, P_1 = 0, P_2 = 1$. Then*

1. $P_0 + P_2 + \cdots + P_{2n} = P_{2n+3}$
2. $P_1 + P_3 + \cdots + P_{2n+1} = P_{2n+4} - 1$

Proof. Note that $P_n = A_{n+3}$ so that

$$\begin{aligned} P_0 + P_2 + \cdots + P_{2n} &= A_3 + A_5 + \cdots + A_{2n+3} \\ &= (A_1 + A_3 + A_5 + \cdots + A_{2n+3}) - A_1 \\ &= A_{2n+6} \qquad \text{(by Theorem 4.3)} \\ &= P_{2n+3}, \end{aligned}$$

and

$$\begin{aligned} P_1 + P_3 + \cdots + P_{2n+1} &= A_4 + A_6 + \cdots + A_{2n+4} \\ &= (A_0 + A_2 + A_4 + A_6 + \cdots + A_{2n+4}) - (A_0 + A_2) \\ &= A_{2n+7} - 1 \qquad \text{(by Theorem 4.2)} \\ &= P_{2n+4} - 1. \end{aligned}$$

$\square$

Example 10. 1. $P_0 + P_2 + P_4 + \cdots + P_{20} = 1 + 1 + 1 + \cdots + 114 = 265 = P_{23}$.

2. $P_1 + P_3 + P_5 + \cdots + P_{19} = 0 + 1 + 2 + \cdots + 86 = 200 - 1 = 199$.

Corollary 4.4. *Let $\{Q_n\}$ be the Padovan sequence given by the third order linear recurrence relation $Q_{n+3} = Q_{n+1} + Q_n$ for $n \geq 0$ with initial conditions $Q_0 = Q_1 = Q_2 = 1$. Then*

1. $Q_0 + Q_2 + \cdots + Q_{2n} = Q_{2n+3} - 1$
2. $Q_1 + Q_3 + \cdots + Q_{2n+1} = Q_{2n+4} - 1$

Proof. Recall that $Q_n = A_{n+5}$ so that

$$\begin{aligned} Q_0 + Q_2 + \cdots + Q_{2n} &= A_5 + A_7 + \cdots + A_{2n+5} \\ &= (A_1 + A_3 + A_5 + A_7 + \cdots + A_{2n+5}) - (A_1 + A_3) \\ &= A_{2n+8} - 1 \text{ (by Theorem 4.3)} \\ &= Q_{2n+3} - 1, \end{aligned}$$

and

$$\begin{aligned} Q_1 + Q_3 + \cdots + Q_{2n+1} &= A_6 + A_8 + \cdots + A_{2n+6} \\ &= (A_0 + A_2 + A_4 + A_6 + A_8 + \cdots + A_{2n+6}) - (A_0 + A_2 + A_4) \\ &= A_{2n+9} - 1 \qquad \text{(by Theorem 4.3)} \\ &= Q_{2n+4} - 1. \end{aligned}$$

□

Theorem 4.4. $A_0 + A_3 + A_6 + \cdots + A_{3n} = A_{3n+2} + 1$.

Proof. Consider $1 + x^3 + x^6 + x^9 + \cdots + x^{3n} = ax^2 + bx + c$. Then

$$\frac{1 - x^{3n+3}}{1 - x^3} = ax^2 + bx + c.$$

Equivalently,

$$1 - x^{3n+3} = (1 - x^3)(ax^2 + bx + c).$$

Since $x^{3n+3} = A_{3n+4}x^2 + A_{3n+5}x + A_{3n+3}$, we have

$$\begin{aligned} 1 - (A_{3n+4}x^2 + A_{3n+5}x + A_{3n+3}) &= (ax^2 + bx + c) - (ax^5 + bx^4 + cx^3) \\ &= (ax^2 + bx + c) - [a(x^2 + x + 1) + b(x^2 + x) + c(x + 1)] \\ &= -bx^2 - (a + c)x - a. \end{aligned}$$

This implies

$$A_{3n+4} = b, A_{3n+5} = a + c, \text{ and } A_{3n+3} = a + 1.$$

Then $1 + x^3 + x^6 + \cdots + x^{3n} = (A_{3n+3} - 1)x^2 + A_{3n+4}x + (A_{3n+5} - A_{3n+3} + 1)$, and so

$$A_0 + A_3 + A_6 + \cdots + A_{3n} = (A_{3n+3} - 1)A_2 + A_{3n+4}A_1 + A_{3n+2} + 1.$$

Since $A_2 = A_1 = 0$, we have

$$A_0 + A_3 + A_6 + \cdots + A_{3n} = A_{3n+2} + 1.$$

□

Corollary 4.5. *Let $\{P_n\}$ be the Padovan sequence given by the third order linear recurrence relation $P_{n+3} = P_{n+1} + P_n$ for $n \geq 0$ with initial conditions $P_0 = 1, P_1 = 0, P_2 = 1$. Then*

$$P_0 + P_3 + \cdots + P_{3n} = P_{3n+2}.$$

Proof. Note that $P_n = A_{n+3}$ so that

$$\begin{aligned} P_0 + P_3 + \cdots + P_{3n} &= A_3 + A_6 + \cdots + A_{3n+3} \\ &= (A_0 + A_3 + A_6 + \cdots + A_{3n+3}) - A_0 \\ &= (A_{3n+5} + 1) - 1 \qquad \text{(by Theorem 4.4)} \\ &= A_{3n+5} \\ &= P_{3n+2}. \end{aligned}$$

□

Theorem 4.5. $A_0 + A_5 + A_{10} + \cdots + A_{5n} = A_{5n+1} + 1$.

Proof. Consider $1 + x^5 + x^{10} + \cdots + x^{5n} = \frac{1-x^{5n+5}}{1-x^5} = ax^2 + bx + c$. Then

$$\begin{aligned} 1 - x^{5n+5} &= (1 - x^5)(ax^2 + bx + c) = ax^2 + bx + c - ax^7 - bx^6 - cx^5 \\ &= ax^2 + bx + c - a(2x^2 + 2x + 1) - b(x^2 + 2x + 1) - c(x^2 + x + 1). \end{aligned}$$

Since $x^{5n+5} = A_{5n+6}x^2 + A_{5n+7}x + A_{5n+5}$, we have

$$1 - x^{5n+5} = 1 - A_{5n+6}x^2 - A_{5n+7}x - A_{5n+5}$$

so that

$$1 - A_{5n+6}x^2 - A_{5n+7}x - A_{5n+5} = (a - 2a - b - c)x^2 + (b - 2a - 2b - c)x + (c - a - b - c).$$

Equivalently,

$$1 - A_{5n+6}x^2 - A_{5n+7}x - A_{5n+5} = -(a + b + c)x^2 - (2a + b + c)x - (a + b).$$

This implies

$$A_{5n+6} = a + b + c, \tag{4.1}$$

$$A_{5n+7} = 2a + b + c, \tag{4.2}$$

$$A_{5n+5} = a + b + 1. \tag{4.3}$$

From Equations (4.1) and (4.3), we have

$$c = A_{5n+6} - A_{5n+5} + 1. \tag{4.4}$$

From Equations (4.1) and (4.4), we have

$$A_{5n+5} + 1 = a + b, \tag{4.5}$$

and from Equations (4.2) and (4.4), we have

$$A_{5n+7} - A_{5n+6} + A_{5n+5} - 1 = 2a + b. \tag{4.6}$$

From Equations (4.5) and (4.6), we have

$$a = A_{5n+7} - A_{5n+6}. \tag{4.7}$$

From equations (4.3) and (4.7), we have $b = A_{5n+5} - A_{5n+7} + A_{5n+6} - 1$. Hence,

$$1 + x^5 + x^{10} + \cdots + x^{5n} = (A_{5n+7} - A_{5n+6})x^2 + (A_{5n+5} - A_{5n+7} + A_{5n+6} - 1)x + A_{5n+6} - A_{5n+5} + 1;$$

equivalently,

$$A_0+A_5+A_{10}+\cdots+A_{5n}=(A_{5n+7}-A_{5n+6})A_2+(A_{5n+5}-A_{5n+7}+A_{5n+6}-1)A_1+A_{5n+6}-A_{5n+5}+1.$$

Since $A_1 = A_2 = 0$, we have $A_0 + A_5 + A_{10} + \cdots + A_{5n} = A_{5n+6} - A_{5n+5} + 1 = A_{5n+1} + 1$. □

Example 11. $A_0+A_5+A_{10}+\cdots+A_{35} = 1+1+3+12+49+200+816+3329 = 4411 = A_{36}+1$.

Corollary 4.6. *Let $\{Q_n\}$ be the Padovan sequence given by the third order linear recurrence relation $Q_{n+3} = Q_{n+1} + Q_n$ for $n \geq 0$ with initial conditions $Q_0 = Q_1 = Q_2 = 1$. Then*

$$Q_0 + Q_5 + \cdots + Q_{5n} = Q_{5n+6}.$$

Proof. Proof. Note that $Q_n = A_{n+5}$ so that

$$\begin{aligned} Q_0 + Q_5 + \cdots + Q_{5n} &= A_5 + A_{10} + \cdots + A_{5n+5} \\ &= (A_0 + A_5 + \cdots + A_{5n+5}) - A_0 \\ &= (A_{5n+1} + 1) - 1 \qquad \text{(by Theorem 4.5)} \\ &= Q_{5n+6}. \end{aligned}$$

□

Theorem 4.6. $\displaystyle\sum_{k=0}^{n} \binom{n}{k} P_k = P_{3n}$.

Proof. Consider the characteristic polynomial

$$f(x) = x^{3n} - (x+1)^n = (x^3)^n - (x+1)^n = [x^3 - (x+1)]g(x) = 0.$$

Then $(x^3 - x - 1)|f(x)$. Further, note that

$$f(x) = (x^3)^n - (x+1)^n = x^{3n} - [\binom{n}{0}x^n + \binom{n}{1}x^{n-1} + \cdots + \binom{n}{n}x^0] = 0.$$

Then

$$x^{3n} = \binom{n}{0}x^n + \binom{n}{1}x^{n-1} + \cdots + \binom{n}{n}x^0,$$

i.e., $P_{3n} = \sum_{k=0}^{n} \binom{n}{k} P_k$ by Theorem 3.2 and $x^3 = x + 1$. □

Corollary 4.7. $\displaystyle P_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} P_{3k}$.

Proof. Recall the binomial inversion formula: $b_n = \sum_{k=0}^{n} \binom{n}{k} a_k \Leftrightarrow a_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} b_k$ (Sloane [20]). Then

$$P_{3n} = \sum_{k=0}^{n} \binom{n}{k} P_k \Leftrightarrow P_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} P_{3k}.$$

□

5 Generalized continued fractions

Fibonacci numbers have many connections with ordinary continued fractions both pure (complex analysis and number theory) Anderson, Bernstein [2] and [4, Ch. 2], and applied (linguistics and numerical analysis) Anderson *et al.*, Shannon and Crothers [1, 18]. Here we consider a plausible case for how a generalized continued fraction can be related to arbitrary order sequences. Previous work (for example Shannon, Wong, and Crothers [18, 19]) has tended to focus on going from the continued fraction algorithm to the recurrence relation or difference equation, whereas we are attempting to go in the opposite direction. From (1.2) we can write, both formally and more generally, that for $j = 0, 1, 2,$ ($j = 1$ is the case in (1.2)):

$$\begin{aligned} v_{3-j,n} &= v_{2-j,n-1} + P_{j+1} v_{m-1,2}, n > 0, \\ v^{(r)}_{r-j,n+1} &= v^{(r)}_{r-j-1,n} - (-1)^j P_j v^{(r)}_{r-1,n} \end{aligned}$$

and

$$\begin{aligned} v_{2,,n} &= v_{1,n-1} + P_1 v_{2,2} \\ v^{(r)}_{r-1,n+1} &= v^{(r)}_{r-2,n} + (-1)^1 P_1 v^{(r)}_{r-1,n} \end{aligned}$$

so that if we put

$$a^{(n)}_{j-1} = \frac{v^{(r)}_{r-j,n}}{v^{(r)}_{r-1,n}}$$

and $b^{(n)}_j = (-1)^j P_j$. Then

$$\frac{\dfrac{v^{(r)}_{r-j,n+1}}{v^{(r)}_{r-1,n}}}{\dfrac{v^{(r)}_{r-1,n+1}}{v^{(r)}_{r-1,n}}} = \frac{\dfrac{v^{(r)}_{r-j-1,n}}{v^{(r)}_{r-1,n} - (-1)^j P_j}}{\dfrac{v^{(r)}_{r-2,n}}{v^{(r)}_{r-1,n}} - (-1) P_1}$$

can be re-written, if we set $a^{(n)}_0 = 1, \forall n$, as

$$a^{(n+1)}_{j-1} = \frac{a^{(n)}_j - b^{(n)}_j}{a^{(n)}_1 - b^{(n)}_1}$$

.

An ordinary periodic continued fraction is an irrational root of a quadratic equation with irrational roots; in particular, the convergents satisfy second order linear recurrence relations with suitable initial conditions with the n-th convergent

$$\frac{p_n}{q_n} = \frac{x_1^{n+2} - x_2^{n+2}}{x_1^{n+1} - x_2^{n+1}}$$

where $p_{n+1} = a_n p_n + p_{n-1}$, and $q_{n+1} = a_n q_n + q_{n-1}$.

For k a non-negative integer and

$$a^{(k)}_i \in R,\ i = 1, 2, \ldots, n-1,$$

let [7]

$$a^{(k)} = \left[a^{(k)}_1, a^{(k)}_2, \ldots, a^{(k)}_{n-1}\right] = a^{(k)} \in E_{n-1}$$

and let

$$f(a^{(k)}) = \left[b_1^{(k)}, b_2^{(k)}, \ldots, b_{n-1}^{(k)}\right] = b^{(k)} \in E_{n-1}$$

with $a_1^{(k)} \neq b_1^{(k)}$, so that for $T : E_{n-1} \longrightarrow E_{n-1}$

$$a^{(k)}T = \left(a_1^{(k)} - b_1^{(k)}\right)^{-1}\left(a_2^{(k)} - b_2^{(k)}, a_3^{(k)} - b_3^{(k)}, \ldots, a_n^{(k)} - b_n^{(k)}\right) = a^{(k+1)},$$

which is a Jacobi–Perron Algorithm (JPA) of the vector $a^{(0)}$ as a sequence $\{a^{(k)}\}$ of vectors in E_{r-1} and Bernstein has demonstrated how this is a generalization of the continued fraction algorithm.

6 Concluding suggestions

This raises the related question that given the connection between the ordinary continued fraction and the Euclidean Algorithm, can the latter also be generalized along the following lines? Instead of relating two integers, suppose we connect two pairs of integers (a, b) and (c, d) in steps analogous to the ordinary Euclidean Algorithm. Suppose $c > a > b > d$, so that

$$\begin{aligned}
(c, d) &= x_1(a, b) + (c - x_1 a, x_a - d) \\
(a, b) &= x_2(c - x_1 a, x_1 - d) + (a - x_2(c - x_1 a), x_2 b - (x_1 - d)) \\
\ldots &= x_n(u, v) + (-b, -a).
\end{aligned} \tag{6.1}$$

For example, consider the vector pair $(17, 1)$ and $(65, 0)$:

$$\begin{aligned}
(65, 0) &= 3(17, 1) + (14, -3) \\
(17, 1) &= 1(14, -3) + (3, 4) \\
(14, -3) &= 4(3, 4) + (2, -19) \\
(3, 4) &= 1(2, -19) + (1, 23) \\
(2, -19) &= 2(1, 23) + (0, -65)
\end{aligned} \tag{6.2}$$

in which from (6.1), the partial quotients are:

$$x_1 = 3,\ x_2 = 1,\ x_3 = 4,\ x_4 = 1,\ x_5 = 2.$$

In a sense, (6.2) contains two Euclidean Algorithms [19], namely

$$\begin{aligned}
65 &= 3 \times 17 + 14 \\
17 &= 1 \times 14 + 3 \\
14 &= 4 \times 3 + 2 \\
3 &= 1 \times 2 + 1 \\
2 &= 2 \times 1 + 0
\end{aligned} \tag{6.3}$$

and

$$
\begin{aligned}
65 &= 2 \times 23 + 19 \\
23 &= 1 \times 19 + 4 \\
19 &= 4 \times 4 + 3 \\
4 &= 1 \times 3 + 1 \\
3 &= 3 \times 1 + 0.
\end{aligned} \tag{6.4}
$$

From (6.3) and (6.4), we have two finite continued fractions

$$65/17 = [3, 1, 4, 1, 2] \text{ and } 65/23 = [2, 1, 4, 1, 3].$$

In general, if $p_n/q_n = [a_0, a_1, a_2, \ldots, a_n]$, then $p_n/p_{n-1} = [a_n, \ldots, a_1, a_0]$ (see Borwein *et al.* [5, §2.4]). For $65/17$, we have

$$a_n : 3, 1, 4, 1, 2, \text{ and } p_n : 0, 1, 3, 4, 19, 23, 65, \text{ and } q_n : 1, 0, 1, 1, 5, 6, 17.$$

Then

$$p_4 = 65, p_3 = 23, q_4 = 17, q_3 = 6.$$

Thus,

$$\frac{p_n}{p_{n-1}} = \frac{p_4}{p_3} = \frac{65}{23} = [2, 1, 4, 1, 3].$$

The partial quotients in (6.3) are the same as those in (6.4) but in different orders. Further developments of Padovan sequences Deveci and Shannon [7] could be related to the JPA considered as a generalization of the ordinary continued fraction algorithm of Szekeres, Van der Cruyssen [21, 23].

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