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An asymptotic series with Nörlund-number coefficients for higher-order Fubini numbers with a parameter

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Abstract: For real parameters $r, \alpha > 0$, let $F_n^{(r)}(\alpha)$ denote the sequence built from Stirling numbers of the second kind by weighting with rising factorials of r and powers of α. Three familiar specializations sit inside this family: the Fubini numbers ($r = \alpha = 1$), the higher-order Fubini numbers ($\alpha = 1$, r a positive integer), and—after rescaling α in terms of p—the integer moments of a negative-binomial random variable. Our main theorem is an asymptotic expansion for $F_n^{(r)}(\alpha)$ in which the successive coefficients turn out to be the Nörlund (higher-order Bernoulli) numbers. A finite-termination phenomenon occurs at positive integer r: the algebraic singular expansion at the dominant singularity has exactly r nonzero terms, and the resulting r-term approximation agrees with $F_n^{(r)}(\alpha)$ to within an exponentially small relative error. We also prove a linear recurrence that holds for all real $r, \alpha > 0$; it powers the numerical experiments at the end of the paper.

Keywords: Higher-order Fubini numbers, Nörlund numbers, Stirling numbers of the second kind, Asymptotic expansion, Negative binomial distribution, Singularity analysis, Higher-order geometric polynomials.
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1 Introduction

Throughout, $\left\{ {n \atop m} \right\}$ stands for the Stirling number of the second kind, and the rising factorial is $(r)_m = r(r+1)\cdots(r+m-1) = \Gamma(r+m)/\Gamma(r)$. The object of study is, for real $r, \alpha > 0$,

$$F_n^{(r)}(\alpha) = \sum_{m=1}^{n} \left\{ {n \atop m} \right\} (r)_m \alpha^m \qquad (n \geq 1), \qquad F_0^{(r)}(\alpha) = 1. \tag{1}$$

A short calculation, which we postpone to Section 2, shows that the exponential generating function of $(F_n^{(r)}(\alpha))$ is

$$\mathcal{F}_\alpha^{(r)}(t) = \sum_{n=0}^{\infty} F_n^{(r)}(\alpha) \frac{t^n}{n!} = \big(1 - \alpha(e^t - 1)\big)^{-r}, \tag{2}$$

and $t^* := \log(1 + 1/\alpha)$ is the radius of convergence of this Taylor expansion at the origin.

Why bother with this two-parameter family? Three reasons. First, $r = \alpha = 1$ collapses it to the Fubini sequence $F_n^{(1)}(1) = \sum_m \left\{ {n \atop m} \right\} m!$ that counts ordered set partitions; this is A000670 in the On-Line Encyclopedia of Integer Sequences [17]. Second, fixing $\alpha = 1$ and letting r range over positive integers gives the higher-order Fubini sequence A226513, for which Ahlbach, Usatine and Pippenger [2] found a combinatorial reading in terms of barred preferential arrangements. Third—and this is the connection that ties the present paper to a much larger literature—substituting $\alpha = (1-p)/p$ (so $p = 1/(1+\alpha)$) turns the right-hand side of (2) into the moment-generating function of a negative-binomial random variable X with parameters r and p. From the standard formula $E[X(X-1)\cdots(X-m+1)] = (r)_m \alpha^m$ for factorial moments, together with the Stirling expansion of x^n in falling factorials, one gets

$$E[X^n] = F_n^{(r)}(\alpha) \qquad (\alpha = (1-p)/p). \tag{3}$$

We maintain the relation $p = 1/(1+\alpha)$ systematically throughout; it will appear again in Lemma 2.1 and Section 4.

1.1 Relation to the existing literature

The identity (3) is implicit in standard references on discrete distributions, including Johnson, Kemp and Kotz [10] and Charalambides [6], and it has resurfaced several times in recent years [1, 3, 8, 16]. The case $\alpha = 1$, without the negative-binomial framing, has also been studied; see Mező [14] and Komatsu–Ramírez [13].

The two-parameter family (1) is closely related to several established objects in the literature. Setting $x = r$ and $y = \alpha$, the generating function $\big(1 - y(e^t - 1)\big)^{-x}$ is, up to renaming of variables, the exponential generating function of the *two-variable higher-order Fubini polynomials* studied by Kim, Kim, Kwon and Park [12]. The special case $\alpha = 1$ gives the *higher-order geometric polynomials* of Kargın and Çekim [11], while connections between generalized Bernoulli (Nörlund) polynomials, Stirling numbers and geometric polynomial families have been investigated by Boutiche, Rahmani and Srivastava [4]. The recurrence of Theorem 3.1 below can also be derived from the differential equation for higher-order geometric polynomials; we include it here for completeness and because it is essential for the numerical experiments.

Given these antecedents, the novelty of the present paper lies not in the definition (1) or its generating function, but specifically in:

(i) the identification of the Nörlund numbers $B_k^{(r)}$ as the explicit coefficients in the large-n asymptotic expansion of $F_n^{(r)}(\alpha)$ (Theorem 2.2);

(ii) the termination of the algebraic singular expansion at positive integer r, yielding an approximation with exponentially small remainder (Corollary 2.3).

1.2 Outline

The dominant singularity of $\mathcal{F}_\alpha^{(r)}$ lies at $t = t^*$, and singularity analysis gives the leading-order asymptotic; the corrections, however, are more interesting. After dividing out the algebraic singularity, one is left with the analytic function

$$H_r(u) = \left(\frac{u}{1-e^{-u}}\right)^r,$$

and a classical identity of Nörlund [15] (see also Carlitz [5]) says that H_r is, up to a sign per coefficient, the generating function of the Nörlund numbers $B_k^{(r)}$. Plugging the Nörlund expansion of H_r into the transfer theorem produces the asymptotic series of Theorem 2.2, term by term.

The k-th correction acquires a factor of $(r-1)(r-2)\cdots(r-k)$, which causes no trouble while r stays generic. But once r is a positive integer, this factor vanishes the moment k reaches r, and so does every subsequent term. In other words, at positive integer r the *algebraic singular expansion* at the dominant singularity has exactly r nonzero terms. We show in Corollary 2.3, via a separate meromorphic argument, that the truncation at this point agrees with $F_n^{(r)}(\alpha)$ except for a contribution from the next-nearest pair of singularities at $t^* \pm 2\pi i$, which is exponentially small. We write out the cases $r = 1, 2, 3$. The $r = 1$ formula is the textbook asymptotic for the Fubini sequence A000670.*

Section 3 gives a linear recurrence for $F_n^{(r)}(\alpha)$ that works for any real $r, \alpha > 0$. It is the consequence of the ODE that $\mathcal{F}_\alpha^{(r)}$ satisfies, and it is essential for the numerical experiments in Section 5, because evaluating the direct sum (1) for large n requires Stirling numbers of growing size. The tables in Section 5 bear out both halves of the main theorem: at non-integer r we see the predicted power-law convergence, and at integer r the error collapses exponentially.

2 The asymptotic expansion

We first verify (2). Using the Stirling identity $x^n = \sum_{m=0}^n \left\{ {n \atop m} \right\} x(x-1)\cdots(x-m+1)$ [9, §6.1],

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{m=0}^{n} \left\{ {n \atop m} \right\} (r)_m \alpha^m = \sum_{m=0}^{\infty} (r)_m \alpha^m \frac{(e^t-1)^m}{m!} = \big(1-\alpha(e^t-1)\big)^{-r}, \tag{4}$$

where the last step is the negative-binomial series applied at $\alpha(e^t-1)$. Comparing the constant term forces $F_0^{(r)}(\alpha) = 1$.

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* It is a refinement of $F_n \approx n!/(2\log^{n+1} 2)$, which appears throughout the combinatorics literature.

For the next ingredient, recall that the Nörlund numbers $B_k^{(r)}$ (also called higher-order Bernoulli numbers) are defined, for real r and integer $k \geq 0$, by

$$\left(\frac{t}{e^t - 1}\right)^r = \sum_{k=0}^{\infty} B_k^{(r)} \frac{t^k}{k!}; \tag{5}$$

this is Nörlund's definition in [15, Chap. 6]. We use the term "Nörlund numbers" and "higher-order Bernoulli numbers" interchangeably; these should be distinguished from the Nörlund *polynomials* $B_k^{(r)}(x)$, which we do not require here. At $r = 1$ they are the usual Bernoulli numbers with the convention $B_1 = -1/2$. For arbitrary real r, each $B_k^{(r)}$ is a polynomial of degree k in r; the first four are

$$B_0^{(r)} = 1, \quad B_1^{(r)} = -\tfrac{r}{2}, \quad B_2^{(r)} = \tfrac{r(3r-1)}{12}, \quad B_3^{(r)} = -\tfrac{r^2(r-1)}{8}. \tag{6}$$

Everything else flows from the local behavior of $\mathcal{F}_\alpha^{(r)}$ near its dominant singularity. We isolate this as a lemma.

Lemma 2.1. *Fix $r, \alpha > 0$ and write $t^* = \log(1 + 1/\alpha)$, $p = 1/(1 + \alpha)$, $u = t^* - t$. Then*

$$\mathcal{F}_\alpha^{(r)}(t) = p^r (t^* - t)^{-r} H_r(t^* - t), \tag{7}$$

with

$$H_r(u) = \left(\frac{u}{1 - e^{-u}}\right)^r = \sum_{k=0}^{\infty} (-1)^k B_k^{(r)} \frac{u^k}{k!}, \tag{8}$$

and H_r is analytic on the disk $\{|u| < 2\pi\}$. For non-integer r, we take the principal branch of $(t^* - t)^{-r} = |t^* - t|^{-r} e^{-ir \arg(t^* - t)}$ with the branch cut along $[t^*, \infty)$.

Proof. Setting $u = t^* - t$ yields $e^t = (1 + 1/\alpha)e^{-u}$, hence $\alpha e^t = (1 + \alpha)e^{-u}$, and therefore

$$1 - \alpha(e^t - 1) = (1 + \alpha)(1 - e^{-u}).$$

Substituting this back,

$$\mathcal{F}_\alpha^{(r)}(t) = (1 + \alpha)^{-r}(1 - e^{-u})^{-r} = p^r u^{-r} \left(\frac{u}{1 - e^{-u}}\right)^r,$$

which is (7). For (8), send $t \mapsto -u$ in (5): this maps $t/(e^t - 1)$ to $u/(1 - e^{-u})$ and multiplies the coefficient at k by $(-1)^k$. The radius of convergence is 2π because the nearest zeros of $1 - e^{-u}$ are at $u = \pm 2\pi i$. □

Theorem 2.2. *Let $r, \alpha > 0$ be real with r not a positive integer. Set $t^* = \log(1 + 1/\alpha)$, $p = 1/(1 + \alpha)$, and let $B_k^{(r)}$ denote the Nörlund numbers* (5)*. Then, for every fixed integer $K \geq 1$,*

$$F_n^{(r)}(\alpha) = \frac{p^r (r)_n}{(t^*)^{n+r}} \sum_{k=0}^{K-1} \frac{(-1)^k B_k^{(r)} (t^*)^k}{k!} \prod_{j=1}^{k} \frac{r - j}{n + r - j} + R_{n,K}, \tag{9}$$

where the remainder satisfies

$$|R_{n,K}| \leq C_{r,\alpha,K} \, \frac{(r)_n}{(t^*)^{n+r}} \, n^{-K} \qquad (n \to \infty), \tag{10}$$

for some constant $C_{r,\alpha,K} > 0$ that depends only on r, α and K. The contributions to $F_n^{(r)}(\alpha)$ from the secondary singularities at $t^* \pm 2\pi i k$ ($k \in \mathbb{Z} \setminus \{0\}$) are smaller than the leading term by a factor of order $(t^*/\sqrt{(t^*)^2 + 4\pi^2})^n$, and are absorbed into $R_{n,K}$.

Proof. By Lemma 2.1, the function $\mathcal{F}_\alpha^{(r)}(t) = p^r(t^*-t)^{-r}H_r(t^*-t)$ has an algebraic singularity of order r at $t = t^*$, modulated by the analytic function H_r, which is analytic on $\{|u| < 2\pi\}$. We work in a Δ-domain at t^* in the sense of Flajolet and Sedgewick [7, Def. VI.1]: fix $0 < \phi < \pi/2$ and let $\Delta = \Delta(\phi, \eta)$ be the indented disk at t^* with opening angle ϕ and small exclusion radius η. Since the nearest singularity other than t^* is at distance 2π (at $t^* \pm 2\pi i$), the function $\mathcal{F}_\alpha^{(r)}$ is analytic on such a Δ-domain for any $\eta < 2\pi$. The principal branch of $(t^*-t)^{-r}$ with branch cut along $\{t>t^*\}$ (see Lemma 2.1) ensures that $(t^*-t)^{-r}$ has the required analyticity properties in Δ.

Expanding $H_r(t^*-t) = \sum_{k=0}^{K-1} \frac{(-1)^k B_k^{(r)}}{k!}(t^*-t)^k + O(|t^*-t|^K)$ and multiplying by $(t^*-t)^{-r}$, we obtain the singular expansion

$$\mathcal{F}_\alpha^{(r)}(t) = p^r \sum_{k=0}^{K-1} \frac{(-1)^k B_k^{(r)}}{k!}(t^*-t)^{k-r} + O\big(|t^*-t|^{K-r}\big)$$

in Δ. Since r is not a positive integer, each exponent $k-r$ is non-integer, and we may apply the Flajolet–Sedgewick transfer theorem [7, Cor. VI.1] term by term. The transfer theorem gives, for non-integer β,

$$n!\,[t^n]\,(t^*-t)^{-\beta} = \frac{n^{\beta-1}}{\Gamma(\beta)\,(t^*)^{n+\beta}}\left(n! + O\left(\frac{n!}{n}\right)\right). \tag{11}$$

More precisely, for $\beta = r-k$ (which is non-integer when $r \notin \mathbb{Z}$),

$$n!\,[t^n]\,(t^*-t)^{k-r} = \frac{(r-k)_n}{(t^*)^{n+r-k}}, \tag{12}$$

where $(r-k)_n = \Gamma(n+r-k)/\Gamma(r-k)$ is the rising factorial. Under our hypothesis that $r \notin \mathbb{Z}_{>0}$, the quantity $r-k$ is never a nonpositive integer for $k \geq 0$, so $\Gamma(r-k)$ is finite and nonzero and the identity holds unambiguously for all $n \geq 1$. It follows from the binomial series $(1-t/t^*)^{k-r} = \sum_{n\geq 0}\binom{n+r-k-1}{n}(t/t^*)^n$ combined with $\binom{n+r-k-1}{n} = (r-k)_n/n!$. (The exceptional cases in which r is a positive integer, where the formal expression $\Gamma(r)/\Gamma(r-k)$ with $k \geq r$ requires separate treatment as a polynomial coefficient, are handled directly in Corollary 2.3 via the meromorphic decomposition, not through (12).)

The ratio

$$\frac{(r-k)_n}{(r)_n} = \frac{\Gamma(n+r-k)/\Gamma(r-k)}{\Gamma(n+r)/\Gamma(r)} = \frac{\prod_{j=1}^{k}(r-j)}{\prod_{j=1}^{k}(n+r-j)}$$

lets us factor $(r)_n$ out, producing the bracket in (9). The bound (10) is the standard residual estimate of [7, Cor. VI.1], and the secondary singularity contribution follows from comparing moduli: $|t^* \pm 2\pi i|^{-1} = (t^{*2} + 4\pi^2)^{-1/2} < (t^*)^{-1}$. □

We now treat the case of positive integer r separately, which requires a different argument.

Corollary 2.3 (Finite termination of the algebraic singular expansion at integer r)**.** *Let r be a positive integer and let $\alpha > 0$. Define $A_n^{(r)}(\alpha)$ to be the r-term approximation*

$$A_n^{(r)}(\alpha) = \frac{p^r(r)_n}{(t^*)^{n+r}} \sum_{k=0}^{r-1} \frac{(-1)^k B_k^{(r)}(t^*)^k}{k!} \prod_{j=1}^{k} \frac{r-j}{n+r-j}, \tag{13}$$

and set $E_n = F_n^{(r)}(\alpha) - A_n^{(r)}(\alpha)$. Then there exist constants $C, \gamma > 0$ depending only on r and α such that

$$|E_n| \le C \cdot \frac{(r)_n}{(t^{*2}+4\pi^2)^{(n+r)/2}} \qquad \textit{for all } n \ge 1. \tag{14}$$

In particular, the relative error satisfies $E_n/A_n^{(r)}(\alpha) = O(\rho^n)$ *as* $n \to \infty$*, where* $\rho = t^*/\sqrt{t^{*2}+4\pi^2} < 1$.

For the first three positive integers:

$$r = 1: \quad F_n^{(1)}(\alpha) = \frac{p\, n!}{(t^*)^{n+1}} + E_n^{(1)}, \tag{15}$$

$$r = 2: \quad F_n^{(2)}(\alpha) = \frac{p^2 (2)_n}{(t^*)^{n+2}} \left[1 + \frac{t^*}{n+1}\right] + E_n^{(2)}, \tag{16}$$

$$r = 3: \quad F_n^{(3)}(\alpha) = \frac{p^3 (3)_n}{(t^*)^{n+3}} \left[1 + \frac{3t^*}{n+2} + \frac{2(t^*)^2}{(n+2)(n+1)}\right] + E_n^{(3)}. \tag{17}$$

When $\alpha = 1$*, we have* $t^* = \log 2$*, and* (15) *reduces to the familiar asymptotic* $F_n^{(1)}(1) \approx n!/(2\log^{n+1} 2)$ *for the Fubini sequence A000670.*

Proof. When r is a positive integer, the generating function $\mathcal{F}_\alpha^{(r)}(t) = (1 - \alpha(e^t - 1))^{-r}$ is a rational function of e^t and hence meromorphic in $\mathbb{C}$. Its poles are located at the solutions of $1 - \alpha(e^t - 1) = 0$, that is, at $t_\ell = t^* + 2\pi i\ell$ for $\ell \in \mathbb{Z}$. Each pole has order r.

Step 1: Principal part at t^*. From Lemma 2.1, near $t = t^*$ we have

$$\mathcal{F}_\alpha^{(r)}(t) = p^r (t^* - t)^{-r} H_r(t^* - t),$$

where $H_r(u) = (u/(1 - e^{-u}))^r$ is analytic for $|u| < 2\pi$. Since r is a positive integer, $(t^* - t)^{-r}$ is a genuine pole of order r, and the Laurent expansion at t^* has the form

$$\mathcal{F}_\alpha^{(r)}(t) = \sum_{j=-r}^{\infty} c_j (t - t^*)^j.$$

The principal part (terms with $j < 0$) is exactly

$$P(t) = p^r \sum_{k=0}^{r-1} \frac{(-1)^k B_k^{(r)}}{k!} (t^* - t)^{k-r},$$

which is a finite sum with r terms. For $k \ge r$, the product $\prod_{j=1}^{k}(r - j)$ contains the factor $r - r = 0$, so the corresponding terms in the singular expansion vanish. However, note that this vanishing reflects the fact that $(t^* - t)^{k-r}$ for $k \ge r$ is a nonnegative integer power of $(t^* - t)$, i.e., a regular (analytic) term, not a singular one. The principal part therefore has exactly r terms, corresponding to $k = 0, 1, \ldots, r - 1$.

Step 2: Residual function. Define $G(t) = \mathcal{F}_\alpha^{(r)}(t) - P(t)$. The function $P(t)$ is a rational function with its only pole at t^*, so G is meromorphic on $\mathbb{C}$ with the same poles as $\mathcal{F}_\alpha^{(r)}$ *except* at t^*, where the subtraction of P cancels the principal part and leaves a removable singularity. The nearest poles of G are therefore $t_{\pm 1} = t^* \pm 2\pi i$, each at distance $\sqrt{(t^*)^2 + 4\pi^2}$ from the origin. Hence G is analytic on the open disk $\{|t| < \sqrt{(t^*)^2 + 4\pi^2}\}$, which contains the origin.

Step 3: Coefficient bound for G. By the Cauchy estimate, for any $0 < R < \sqrt{t^{*2} + 4\pi^2}$ and $M = \max_{|t|=R} |G(t)|$,

$$|n!\,[t^n]\,G(t)| = |G^{(n)}(0)| \le n!\,M/R^n.$$

To obtain the bound at the radius $D = \sqrt{(t^*)^2 + 4\pi^2}$, subtract from G the principal parts at its two nearest poles $t_{\pm 1} = t^* \pm 2\pi i$. Since these poles have order r, their coefficients are bounded by $C_1 n^{r-1} D^{-n}$. The resulting function has its nearest poles at $t^* \pm 4\pi i$ and is therefore analytic in a disk of radius strictly larger than D; its coefficients are exponentially smaller by the Cauchy estimate. It follows that

$$|n![t^n]G(t)| \le C' n! n^{r-1} D^{-n}$$

for some constant C' depending only on r and α.

Step 4: Coefficients of the principal part. Each term $p^r \frac{(-1)^k B_k^{(r)}}{k!}(t^* - t)^{k-r}$ in the principal part, for $0 \le k \le r-1$, has $k - r < 0$, so $(t^* - t)^{k-r}$ is a genuine negative power. Using the binomial series as in the proof of Theorem 2.2:

$$n!\,[t^n]\,(t^* - t)^{k-r} = \frac{(r-k)_n}{(t^*)^{n+r-k}}.$$

This is valid since $r - k \ge 1$ is a positive integer for $0 \le k \le r-1$, and in this range the binomial expansion is valid and yields a well-defined rising factorial.

Step 5: Conclusion. Since $F_n^{(r)}(\alpha) = n!\,[t^n]\,\mathcal{F}_\alpha^{(r)}(t) = n!\,[t^n]\,P(t) + n!\,[t^n]\,G(t)$, we have $E_n = n!\,[t^n]\,G(t)$, and the bound (14) follows from Step 3 since $(r)_n/n! \asymp n^{r-1}$. The relative error estimate follows because the leading term of $A_n^{(r)}(\alpha)$ grows as $(t^*)^{-n}$, while $|E_n|$ grows at most as R^{-n} with $R > t^*$. The three displayed formulas come from substituting $B_0^{(r)} = 1$, $B_1^{(r)} = -r/2$ and $B_2^{(r)} = r(3r-1)/12$. □

Remark 2.4. The $k = 0$ term alone, combined with the asymptotic $(r)_n/n! = \Gamma(n+r)/[\Gamma(r)n!] \sim n^{r-1}/\Gamma(r)$, gives

$$F_n^{(r)}(\alpha) \sim \frac{p^r}{\Gamma(r)} n^{r-1} \frac{n!}{(t^*)^{n+r}} \qquad (n \to \infty). \tag{18}$$

Remark 2.5. Writing out the first three terms of (9):

$$F_n^{(r)}(\alpha) = \frac{p^r (r)_n}{(t^*)^{n+r}} \left[1 + \frac{r(r-1)t^*}{2(n+r-1)} + \frac{r(r-1)(r-2)(3r-1)(t^*)^2}{24(n+r-1)(n+r-2)} + O(n^{-3})\right]. \tag{19}$$

The pattern is now visible. At step k, the coefficient acquires another factor $r - k$ from $\prod_{j=1}^{k}(r-j)$; for positive integer r, once k reaches r, this product contains the factor zero, and the algebraic singular expansion at t^* terminates. For non-integer r, no such cancellation occurs.

Remark 2.6. For non-integer $r > 0$, the function $H_r(u)$ has its nearest singularities at $u = \pm 2\pi i$, so its Taylor coefficients $(-1)^k B_k^{(r)}/k!$ have radius of convergence exactly 2π. Consequently $|B_k^{(r)}|/k!$ grows essentially as $(2\pi)^{-k}$ up to algebraic factors in k (see, e.g., [15, Chap. 6]). The k-th term of (9) is therefore of order

$$|B_k^{(r)}|(t^*)^k/k! \cdot \prod_{j=1}^{k}(r-j)/(n+r-j) \sim (t^*/2\pi)^k \cdot k^{|r|}/n^k$$

(up to constants), but this coefficient estimate alone does not determine the convergence of the formal infinite sum for fixed n . Nevertheless, for each fixed K the expansion (9) is asymptotic in the sense of (10). At positive integer r, this issue does not arise, since the sum is then finite.

3 A linear recurrence

Theorem 3.1. *For real $r, \alpha > 0$, the sequence $(F_n^{(r)}(\alpha))_{n\geq 0}$ obeys $F_0^{(r)}(\alpha) = 1$ together with the recurrence*

$$F_n^{(r)}(\alpha) = \alpha\Big[r\sum_{j=0}^{n-1}\binom{n-1}{j}F_j^{(r)}(\alpha) + \sum_{j=1}^{n-1}\binom{n-1}{j-1}F_j^{(r)}(\alpha)\Big], \qquad n \geq 1. \tag{20}$$

Proof. Differentiate $\mathcal{F}_\alpha^{(r)}(t) = (1-\alpha(e^t-1))^{-r}$:

$$\big(\mathcal{F}_\alpha^{(r)}\big)'(t) = \frac{r\alpha e^t}{1-\alpha(e^t-1)}\mathcal{F}_\alpha^{(r)}(t).$$

Clearing the denominator gives the linear ODE

$$\big(\mathcal{F}_\alpha^{(r)}\big)'(t) = \alpha\Big[re^t\mathcal{F}_\alpha^{(r)}(t) + (e^t-1)\big(\mathcal{F}_\alpha^{(r)}\big)'(t)\Big]. \tag{21}$$

Matching coefficients of $t^{n-1}/(n-1)!$ in (21): the left-hand side contributes $F_n^{(r)}(\alpha)$, and applying the binomial convolution rule to the two products on the right reproduces the two sums in (20). □

Remark 3.2. Specializing to $r = \alpha = 1$, the recurrence (20) reduces to the classical $F_n = \sum_{k=1}^n \binom{n}{k}F_{n-k}$ for A000670. Equivalent recurrences for higher-order geometric polynomials can also be derived from their differential equations; see, e.g., [11]. In general, computing $F_n^{(r)}(\alpha)$ from (20) takes $O(n^2)$ arithmetic operations once the earlier values are in hand; the Stirling-number sum (1) has a comparable cost if Stirling numbers are precomputed recursively, but the recurrence is convenient for high-precision numerical checks with growing n.

4 Moments of the negative binomial

The identification $F_n^{(r)}(\alpha) = E[X^n]$ at $\alpha = (1-p)/p$, established in (3), allows us to translate Theorem 2.2 and Corollary 2.3 directly into statements about negative-binomial moments.

Corollary 4.1. *Let X be negative-binomial with parameters $r > 0$ and $p \in (0,1)$, and put $t^* = \log(1/(1-p))$. Then, for each fixed integer $K \geq 1$ and $n \to \infty$ (with $r \notin \mathbb{Z}_{>0}$),*

$$E[X^n] = \frac{p^r(r)_n}{(t^*)^{n+r}}\sum_{k=0}^{K-1}\frac{(-1)^k B_k^{(r)}(t^*)^k}{k!}\prod_{j=1}^{k}\frac{r-j}{n+r-j}\cdot\big(1+O(n^{-K})\big). \tag{22}$$

When r is a positive integer, the same formula holds with $K = r$ and an exponentially small relative error:

$$E[X^n] = A_n^{(r)}\left(\frac{1-p}{p}\right)\big(1+O(\rho^n)\big), \tag{23}$$

where $A_n^{(r)}$ is defined in (13) *and $\rho = t^*/\sqrt{t^{*2}+4\pi^2} < 1$.*

The exponential accuracy at integer r is what makes the r-term approximation genuinely useful in practice. As a concrete illustration, take $r = 3$ and $p = 1/2$, so that $\alpha = 1$ and $t^* = \log 2$. Corollary 4.1 then specializes to

$$E[X^n] = \frac{6\,n!\,(n+2)(n+1)}{8\,(\log 2)^{n+3}} \left[1 + \frac{3\log 2}{n+2} + \frac{2(\log 2)^2}{(n+2)(n+1)}\right] \cdot \big(1 + O(0.110^n)\big). \tag{24}$$

By $n = 10$, this approximation already achieves a relative error of approximately 4.59×10^{-13}, which corresponds to more than twelve correct significant digits; see Table 1.

5 Numerical verification

We computed the values $F_n^{(r)}(\alpha)$ in the tables below in two independent ways: directly from (1) using the `mpmath` library at fifty decimal digits of working precision, and via the recurrence (20) at the same precision. The two methods agreed in every displayed digit. The truncation $A_n^{(r)}(\alpha)$ was evaluated from formula (13) (or its generalization for non-integer r) at the same precision, and the signed relative error $\varepsilon_n = F_n^{(r)}(\alpha)/A_n^{(r)}(\alpha) - 1$ was then computed.

The first case is the one promised by Corollary 2.3. Taking $r = 3$, the three-term truncation should agree with $F_n^{(3)}(\alpha)$ up to a relative error of order $\rho^n \approx 0.1097^n$, and Table 1 confirms this: by $n = 30$ the relative error has fallen below 10^{-31}.

Table 1. $F_n^{(3)}(1)$ versus the three-term approximation $A_n^{(3)}(1)$ from (13). Predicted relative error: order $\rho^n \approx 0.1097^n$ where $\rho = \log 2/\sqrt{(\log 2)^2 + 4\pi^2}$.

n	$F_n^{(3)}(1)$	$\varepsilon_n = F_n^{(3)}(1)/A_n^{(3)}(1) - 1$
5	7 803	-8.83×10^{-8}
10	4.1455×10^{9}	4.59×10^{-13}
15	1.83473×10^{16}	1.31×10^{-17}
20	3.52947×10^{23}	2.99×10^{-23}
30	3.13725×10^{39}	-3.91×10^{-32}

Table 2 reports the corresponding numbers for $r = 1$, where the one-term approximation is the classical Fubini asymptotic $n!/(2(\log 2)^{n+1})$.

Table 2. The Fubini sequence A000670 versus the single-term approximation $A_n^{(1)}(1) = n!/(2(\log 2)^{n+1})$.

n	$F_n^{(1)}(1)$	$\varepsilon_n = F_n^{(1)}(1)/A_n^{(1)}(1) - 1$
5	541	-2.81×10^{-6}
10	102 247 563	-5.16×10^{-11}
15	2.30283×10^{14}	-1.63×10^{-16}
20	2.67769×10^{21}	1.03×10^{-20}
30	1.14036×10^{37}	9.10×10^{-31}

To round out the picture, Table 3 shows what the expansion looks like when r is not an integer. Adding one more term shrinks the relative error by a factor of order $1/n$, in line with (10), but the decay is now polynomial rather than exponential.

Table 3. Absolute relative error $|\varepsilon_n|$ for $r = 2.5$, $\alpha = 1.5$ (so $p = 0.4$).
Here r is non-integer, so the algebraic singular expansion does not terminate.

n	$K = 1$	$K = 2$	$K = 3$	$K = 4$	$K = 5$
10	7.78×10^{-2}	1.00×10^{-3}	7.75×10^{-6}	9.68×10^{-8}	2.64×10^{-10}
20	4.29×10^{-2}	2.87×10^{-4}	1.08×10^{-6}	6.15×10^{-9}	9.05×10^{-12}
40	2.26×10^{-2}	7.69×10^{-5}	1.43×10^{-7}	3.90×10^{-10}	2.88×10^{-13}
80	1.16×10^{-2}	1.99×10^{-5}	1.85×10^{-8}	2.46×10^{-11}	9.06×10^{-15}

6 Conclusion

The main result of this paper is an asymptotic expansion of $F_n^{(r)}(\alpha)$—a two-parameter family that encompasses higher-order Fubini numbers, higher-order geometric polynomials, and negative-binomial moments—whose successive coefficients turn out to be the classical Nörlund (higher-order Bernoulli) numbers. Specializing to a positive integer r produces a finite-termination phenomenon for the algebraic singular expansion at the dominant singularity: only r terms of the expansion are nonzero, and we proved via an explicit meromorphic/Cauchy-estimate argument that the resulting r-term approximation agrees with $F_n^{(r)}(\alpha)$ to within an exponentially small relative error. A linear recurrence valid for arbitrary real $r, \alpha > 0$ was given as well, and we used it to check the asymptotic predictions numerically. Through the substitution $\alpha = (1 - p)/p$, the same analysis yields a sharp asymptotic for the integer moments of negative-binomial random variables. We see several directions worth pursuing further: extending the expansion to other Sheffer-type sequences whose generating functions have algebraic singularities of similar form, and asking whether the termination phenomenon at integer r has a higher-dimensional analogue.

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