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Some properties of Padovan matrices and bi-periodic Padovan matrices

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Abstract: Let $(P_n)_{n\geq 0}$ be the sequence of bi-periodic Padovan numbers and let $(M_{p_n})_{n\geq 0}$ be the sequence of bi-periodic Padovan matrices. In this article we study when these matrices are diagonalizable and we obtain a certain connection with the Lucas number sequence. We also obtain some connections of these matrices with the generating matrix Q for the Padovan numbers. At the end, we obtain an infinite cyclic subgroup of the real Heisenberg group.
Keywords: Difference equations, Padovan numbers, Padovan matrices, Bi-periodic Padovan numbers, Bi-periodic Padovan matrices, Diagonalizable matrices.
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1 Introduction and preliminaries

The Fibonacci sequence $(F_n)_{n\geq 0}$ is defined by second-order linear recurrence relation

$$F_n = F_{n-1} + F_{n-2},\ n \geq 2,$$

with the first two terms $F_0 = 0, F_1 = 1$.

Closely related to the Fibonacci sequence was the Lucas sequence $(L_n)_{n\geq 0}$, also defined as a second-order linear recurrence



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$$L_n = L_{n-1} + L_{n-2},\ n \geq 2,$$

with the first two terms $L_0 = 2, L_1 = 1$.

If we consider the characteristic equation $t^2 - t - 1 = 0$ associated with the recurrence satisfied by the Fibonacci sequence and the Lucas sequence and denote by $t_1 = \alpha$ and $t_2 = \beta$ the solutions of this equation, it immediately follows:

- **Binet's formula for the Fibonacci numbers**: $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$
- **Binet's formula for the Lucas numbers**: $L_n = \alpha^n + \beta^n$.

The Padovan sequence $(p_n)_{n\geq 0}$ is defined by a third-order linear recurrence relation

$$p_n = p_{n-2} + p_{n-3},\ n \geq 3,$$

with the first three terms $p_0 = 1, p_1 = 1$ and $p_2 = 1$. This number sequence was introduced by Richard Padovan in [7]. Many mathematicians work with the shifted Padovan sequence, presenting as initial values $p_0 = 1, p_1 = 0$ and $p_2 = 1$.

Many papers have been written about the study of the properties of Fibonacci sequence, Lucas sequence, Padovan sequence or others particular difference equations of degree 2 or of degree 3 with integer coefficients (see [1, 3–5, 8–11, 13]).

In the paper [12], the authors considered the generating matrix for Padovan numbers:

$$Q = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

They proved that, with $p_0 = p_1 = p_2 = 1$ and $n \geq 5$,

$$Q^n = \begin{pmatrix} p_{n-2} & p_{n-1} & p_{n-3} \\ p_{n-3} & p_{n-2} & p_{n-4} \\ p_{n-4} & p_{n-3} & p_{n-5} \end{pmatrix}.$$

The matrix Q has one real eigenvalue and two non-real complex eigenvalues, so it is not diagonalizable over $\mathbb{R}$ (see [14]).

In the paper [2], Dişkaya and Menken introduced the sequence $(P_n)_{n\geq 0}$ of bi-periodic Padovan numbers like this

$$P_n = \begin{cases} aP_{n-2} + P_{n-3}, & \text{if } n \text{ is even}, \\ bP_{n-2} + P_{n-3}, & \text{if } n \text{ is odd}, \end{cases} \quad \text{for } n \geq 3,$$

with the initial values $P_0 = 1,\ P_1 = 0,$ and $P_2 = a.$ where a and b are nonzero real numbers.

In the same paper [2], the authors gave the Binet-like formula of the bi-periodic Padovan numbers. From the definition of bi-periodic Padovan numbers, the equation

$$t^3 - a \cdot b \cdot t - a \cdot b = 0$$

is derived for bi-periodic Padovan sequences. We denote the roots of this equation by λ, μ, and δ. In Theorem 1, from the paper [2] the **Binet-like formula of the bi-periodic Padovan numbers** was given:

$$P_n = \begin{cases} x \cdot \lambda^n + y \cdot \mu^n + z \cdot \delta^n, & \text{if } n \text{ is even,} \\ u \cdot \lambda^n + v \cdot \mu^n + w \cdot \delta^n, & \text{if } n \text{ is odd,} \end{cases} \quad \text{for } n \geq 0,$$

with

$$x = \frac{(\mu - a) \cdot (\delta - a)}{(\lambda - \mu) \cdot (\lambda - \delta)}, y = \frac{(\lambda - a) \cdot (\delta - a)}{(\mu - \lambda) \cdot (\mu - \delta)}, z = \frac{(\lambda - a) \cdot (\mu - a)}{(\delta - \lambda) \cdot (\delta - \mu)},$$
$$u = \frac{a + b - \mu - \delta}{(\lambda - \mu) \cdot (\lambda - \delta)}, v = \frac{a + b - \lambda - \delta}{(\mu - \lambda) \cdot (\mu - \delta)}, w = \frac{a + b - \lambda - \mu}{(\delta - \lambda) \cdot (\delta - \mu)}$$

Also, in the paper [2], Dişkaya and Menken introduced the sequence $(M_{p_n})_{n \geq 0}$ of bi-periodic Padovan matrices like this

$$M_{p_n} = \begin{cases} aM_{p_{n-2}} + M_{p_{n-3}}, & \text{if } n \text{ is even,} \\ bM_{p_{n-2}} + M_{p_{n-3}}, & \text{if } n \text{ is odd,} \end{cases} \quad \text{for } n \geq 3, \tag{1.1}$$

where a and b are nonzero real numbers, with the initial terms

$$M_{p_0} = I_3, \quad M_{p_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & a & 0 \end{pmatrix}, \quad M_{p_2} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & a & 0 \\ 0 & 1 & a \end{pmatrix}.$$

In this article we obtain some properties of matrices Q and M_{p_n}, by studying whether they are diagonalizable or not. In the article [11], starting from the diagonalizable matrix

$$M = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

we obtained a set of diagonalizable matrices, in connection with the matrix M. In this article we find out for which real numbers a and b all matrices in the sequence $(M_{p_n})_{n \geq 0}$ are diagonalizable. Throughout the article, diagonalizability and simultaneous diagonalizability are understood over $\mathbb{R}$ unless otherwise stated. For this we recall what it means that two matrices are simultaneously diagonalizable.

Let $A, B \in M_n(\mathbb{R})$ be two diagonalizable matrices and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of the matrix A, and $\lambda_1', \lambda_2', \ldots, \lambda_n'$ be the eigenvalues of the matrix B. A and B are called simultaneously diagonalizable if there exists an invertible matrix $T \in M_n(\mathbb{R})$ such that

$$A = T^{-1} \operatorname{diag}(\lambda_1, \ldots, \lambda_n) T, \qquad B = T^{-1} \operatorname{diag}(\lambda_1', \ldots, \lambda_n') T.$$

It is known that:

i) if $A, B \in M_n(\mathbb{R})$ are simultaneously diagonalizable, then $A \cdot B = B \cdot A$.

ii) The converse of i) is valid, provided that one of the matrices A and B has no multiple eigenvalues (see [6]).

2 Results

We remark that $M_{p_1}^2 = M_{p_2}$ and, using the recurrence relation for biperiodic Padovan matrices, it follows immediately that $M_{p_3} = bM_{p_1} + I_3$, $M_{p_4} = aM_{p_1}^2 + M_{p_1}$, $M_{p_5} = M_{p_1}^2 + b^2 M_{p_1} + bI_3$, $M_{p_6} = a^2 M_{p_1}^2 + (a+b)\, M_{p_1} + I_3$. Next, we show that every M_{p_n} is a polynomial in M_{p_1}, namely:

Proposition 2.1. *For every integer $n \geq 0$, there exist real numbers α_n, β_n and γ_n such that*

$$M_{p_n} = \alpha_n \cdot M_{p_1}^2 + \beta_n \cdot M_{p_1} + \gamma_n \cdot I_3.$$

Proof. We prove by mathematical induction over $n \in \mathbb{N}$ the statement

$$P(n) : (\exists)\, \alpha_n, \beta_n, \gamma_n \in \mathbb{R} \text{ such that } M_{p_n} = \alpha_n \cdot M_{p_1}^2 + \beta_n \cdot M_{p_1} + \gamma_n \cdot I_3.$$

For $n = 0, 1, 2$, the statement follows from $M_{p_0} = I_3$ and $M_{p_2} = M_{p_1}^2$, with

$$(\alpha_0, \beta_0, \gamma_0) = (0, 0, 1), \quad (\alpha_1, \beta_1, \gamma_1) = (0, 1, 0), \quad (\alpha_2, \beta_2, \gamma_2) = (1, 0, 0).$$

We assume that, for some $n \geq 2$ and every integer $0 \leq k \leq n$, there exist real numbers α_k, β_k and γ_k such that

$$M_{p_k} = \alpha_k \cdot M_{p_1}^2 + \beta_k \cdot M_{p_1} + \gamma_k \cdot I_3.$$

We prove that there exist real numbers $\alpha_{n+1}, \beta_{n+1}$ and γ_{n+1} such that

$$M_{p_{n+1}} = \alpha_{n+1} \cdot M_{p_1}^2 + \beta_{n+1} \cdot M_{p_1} + \gamma_{n+1} \cdot I_3.$$

If n is an even number using the inductive hypothesis and the recurrence relation of bi-periodic Padovan matrices, we have:

$$M_{p_{n+1}} = bM_{p_{n-1}} + M_{p_{n-2}} =$$

$$b\left(\alpha_{n-1} \cdot M_{p_1}^2 + \beta_{n-1} \cdot M_{p_1} + \gamma_{n-1} \cdot I_3\right) + \alpha_{n-2} \cdot M_{p_1}^2 + \beta_{n-2} \cdot M_{p_1} + \gamma_{n-2} \cdot I_3 =$$

$$\alpha_{n+1} \cdot M_{p_1}^2 + \beta_{n+1} \cdot M_{p_1} + \gamma_{n+1} \cdot I_3,$$

where $\alpha_{n+1} = b\alpha_{n-1} + \alpha_{n-2}$, $\beta_{n+1} = b\beta_{n-1} + \beta_{n-2}$, $\gamma_{n+1} = b\gamma_{n-1} + \gamma_{n-2}$.
If n is an odd number, applying the inductive hypothesis and the recurrence relation of the bi-periodic Padovan matrices, we obtain:

$$M_{p_{n+1}} = aM_{p_{n-1}} + M_{p_{n-2}} =$$

$$a\left(\alpha_{n-1} \cdot M_{p_1}^2 + \beta_{n-1} \cdot M_{p_1} + \gamma_{n-1} \cdot I_3\right) + \alpha_{n-2} \cdot M_{p_1}^2 + \beta_{n-2} \cdot M_{p_1} + \gamma_{n-2} \cdot I_3 =$$

$$\alpha_{n+1} \cdot M_{p_1}^2 + \beta_{n+1} \cdot M_{p_1} + \gamma_{n+1} \cdot I_3,$$

where $\alpha_{n+1} = a\alpha_{n-1} + \alpha_{n-2}$, $\beta_{n+1} = a\beta_{n-1} + \beta_{n-2}$, $\gamma_{n+1} = a\gamma_{n-1} + \gamma_{n-2}$. □

Corollary 2.2. *Let $(M_{p_n})_{n\geq 0}$ be the sequence of bi-periodic Padovan matrices defined in the introduction section. Then*

$$M_{p_n} \cdot M_{p_m} = M_{p_m} \cdot M_{p_n}, \ (\forall)\ n, m \in \mathbb{N}.$$

Proof. If $n = 0$ or $m = 0$, since $M_{p_0} = I_3$, the equality in the statement follows immediately. If $n, m \in \mathbb{N}^*$, applying Proposition 2.1, we immediately obtain

$$M_{p_n} \cdot M_{p_m} = M_{p_m} \cdot M_{p_n}. \qquad \square$$

We now study for which real numbers a, and b the Padovan bi-periodic matrices are diagonalizable. For some values of $a \geq 2$ we used *Magma Computer Algebra System* software and we saw that the matrix M_{p_1} is diagonalizable. We obtain the following result.

Proposition 2.3. *Let $(M_{p_n})_{n\geq 0}$ be the sequence of bi-periodic Padovan matrices with $a > \frac{3}{\sqrt[3]{4}}$ and b is an arbitrary nonzero real number. Then, we have:*

i) the matrices M_{p_n} are diagonalizable, for any n positive integer;

ii) if λ_{1n}, λ_{2n} and λ_{3n} are the eigenvalues of the matrix M_{p_n}, labelled consistently with the common diagonalization in the proof, for each n positive integer, then

$$\lambda_{in} = \begin{cases} a\lambda_{in-2} + \lambda_{in-3}, & \text{if } n \text{ is even}, \\ b\lambda_{in-2} + \lambda_{in-3}, & \text{if } n \text{ is odd}, \end{cases} \quad \text{for } n \geq 3,\ i = \overline{1,3}.$$

Proof. $M_{p_0} = I_3$ is diagonalizable and $\lambda_{10} = \lambda_{20} = \lambda_{30} = 1$.

$$M_{p_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & a & 0 \end{pmatrix}.$$

The characteristic equation is

$$\det\left(M_{p_1} - \lambda \cdot I_3\right) = 0,$$

which is equivalent to

$$\lambda^3 - a\lambda - 1 = 0.$$

The discriminant of this cubic equation is $\Delta = 4a^3 - 27$.

If $\Delta > 0$, that is $a > \frac{3}{\sqrt[3]{4}}$, the eigenvalues: λ_{11}, λ_{21}, λ_{31} of M_{p_1} are distinct real numbers, so the matrix M_{p_1} is diagonalizable.

Since $M_{p_2} = M_{p_1}^2$ it follows that the matrix M_{p_2} has the following eigenvalues $\lambda_{12} = \lambda_{11}^2$, $\lambda_{22} = \lambda_{21}^2$, $\lambda_{32} = \lambda_{31}^2$. Furthermore, we note that the matrix M_{p_1} cannot have two opposite eigenvalues. From all these, it follows that M_{p_2} has 3 real distinct eigenvalues, so the matrix M_{p_2} is diagonalizable.

But, $M_{p_1} \cdot M_{p_2} = M_{p_2} \cdot M_{p_1}$ and according to 8.3.26, p.429 ([6]) we obtain that the matrices M_{p_1} and M_{p_2} are simultaneously diagonalizable, so there exists an invertible matrix $T \in \mathcal{M}_3(\mathbb{R})$ such that:

$$M_{p_1} = T^{-1} \cdot \begin{pmatrix} \lambda_{11} & 0 & 0 \\ 0 & \lambda_{21} & 0 \\ 0 & 0 & \lambda_{31} \end{pmatrix} \cdot T. \tag{2.1}$$

and

$$M_{p_2} = T^{-1} \cdot \begin{pmatrix} \lambda_{12} & 0 & 0 \\ 0 & \lambda_{22} & 0 \\ 0 & 0 & \lambda_{32} \end{pmatrix} \cdot T. \tag{2.2}$$

From the relations (2.1), (2.2) and the recurrence relation of the bi-periodic Padovan matrices, it follows that

$$M_{p_3} = T^{-1} \cdot \begin{pmatrix} b\lambda_{11} + \lambda_{10} & 0 & 0 \\ 0 & b\lambda_{21} + \lambda_{20} & 0 \\ 0 & 0 & b\lambda_{31} + \lambda_{30} \end{pmatrix} \cdot T. \tag{2.3}$$

and

$$M_{p_4} = T^{-1} \cdot \begin{pmatrix} a\lambda_{12} + \lambda_{11} & 0 & 0 \\ 0 & a\lambda_{22} + \lambda_{21} & 0 \\ 0 & 0 & a\lambda_{32} + \lambda_{31} \end{pmatrix} \cdot T. \tag{2.4}$$

From (2.3), it results that $\lambda_{i3} = b\lambda_{i1} + \lambda_{i0}$, for $i = \overline{1,3}$.
From (2.4), it results that $\lambda_{i4} = a\lambda_{i2} + \lambda_{i1}$, for $i = \overline{1,3}$.
We prove by mathematical induction over $n \geq 1$ the statement

$$P(n) : M_{p_n} = T^{-1} \cdot \begin{pmatrix} \lambda_{1n} & 0 & 0 \\ 0 & \lambda_{2n} & 0 \\ 0 & 0 & \lambda_{3n} \end{pmatrix} \cdot T$$

and

$$\lambda_{in} = \begin{cases} a\lambda_{in-2} + \lambda_{in-3}, & \text{if } n \text{ is even,} \\ b\lambda_{in-2} + \lambda_{in-3}, & \text{if } n \text{ is odd,} \end{cases} \quad \text{for } n \geq 3,\ i = \overline{1,3}.$$

The common diagonalization has already been established for $n = 0, 1, 2$. From (2.3), it results that $P(3)$ is true. From (2.4), it results that $P(4)$ is true.
We assume that $P(l)$ is true for each $0 \leq l \leq n$, with $n \geq 4$, and we show that $P(n+1)$ is true. We consider two cases.

Case 1: When n is odd. Applying (1.1) and the inductive hypothesis, it follows that

$$M_{p_{n+1}} = aM_{p_{n-1}} + M_{p_{n-2}}$$

$$= T^{-1} \cdot \begin{pmatrix} a\lambda_{1n-1} + \lambda_{1n-2} & 0 & 0 \\ 0 & a\lambda_{2n-1} + \lambda_{2n-2} & 0 \\ 0 & 0 & a\lambda_{3n-1} + \lambda_{3n-2} \end{pmatrix} \cdot T. \tag{2.5}$$

Case 2: When n is even. Applying (1.1) and the inductive hypothesis, it follows that

$$M_{p_{n+1}} = bM_{p_{n-1}} + M_{p_{n-2}}$$

$$= T^{-1} \cdot \begin{pmatrix} b\lambda_{1n-1} + \lambda_{1n-2} & 0 & 0 \\ 0 & b\lambda_{2n-1} + \lambda_{2n-2} & 0 \\ 0 & 0 & b\lambda_{3n-1} + \lambda_{3n-2} \end{pmatrix} \cdot T. \tag{2.6}$$

From (2.5) and (2.6), it results that $P(n+1)$ is true.

We obtain that $P(n)$ is true for each positive integer $n \geq 3$. From here it follows that the matrices M_{p_n} are diagonalizable, for any n positive integer and its eigenvalues are

$$\lambda_{in} = \begin{cases} a\lambda_{in-2} + \lambda_{in-3}, & \text{if } n \text{ is even,} \\ b\lambda_{in-2} + \lambda_{in-3}, & \text{if } n \text{ is odd,} \end{cases} \quad \text{for } n \geq 3,\ i = \overline{1,3}. \qquad \square$$

From Proposition 2.3, it follows that, when $a > 3/\sqrt[3]{4}$, all matrices M_{p_n} are diagonalizable for every nonzero real b; the parameter b enters the recurrence relations of their eigenvalues.

Corollary 2.4. *Let $(M_{p_n})_{n\geq 0}$ be the sequence of bi-periodic Padovan matrices with $a > \frac{3}{\sqrt[3]{4}}$ and b is an arbitrary nonzero real number. Then, the matrices M_{p_n} and M_{p_m} are simultaneously diagonalizable, $(\forall)\ n, m \in \mathbb{N}$.*

Proof. Let λ_{1n}, λ_{2n} and λ_{3n} be the eigenvalues of the matrix M_{p_n} and let λ_{1m}, λ_{2m} and λ_{3m} be the eigenvalues of the matrix M_{p_m}.
According to Proposition 2.3, the matrices M_{p_n} and M_{p_m} are diagonalizable, $(\forall)\ n, m \in \mathbb{N}$ and

$$M_{p_n} = T^{-1} \cdot \begin{pmatrix} \lambda_{1n} & 0 & 0 \\ 0 & \lambda_{2n} & 0 \\ 0 & 0 & \lambda_{3n} \end{pmatrix} \cdot T \text{ and } M_{p_m} = T^{-1} \cdot \begin{pmatrix} \lambda_{1m} & 0 & 0 \\ 0 & \lambda_{2m} & 0 \\ 0 & 0 & \lambda_{3m} \end{pmatrix} \cdot T,$$

so, the matrices M_{p_n} and M_{p_m} are simultaneously diagonalizable, for any positive integers n and m. $\square$

Now, we examine whether the matrix M_{p_1} is diagonalizable or not in the cases $a = \frac{3}{\sqrt[3]{4}}$, respectively $a < \frac{3}{\sqrt[3]{4}}$. We obtain the following results.

Proposition 2.5. *If $a \leq \frac{3}{\sqrt[3]{4}}$, the matrix M_{p_1} is not diagonalizable over $\mathbb{R}$.*

Proof. Case 1: When $a = \frac{3}{\sqrt[3]{4}}$. The characteristic equation is

$$\det(M_{p_1} - \lambda \cdot I_3) = 0,$$

which is equivalent to

$$\lambda^3 - \frac{3}{\sqrt[3]{4}} \cdot \lambda - 1 = 0.$$

The discriminant of this cubic equation is $\Delta = 4\left(\frac{3}{\sqrt[3]{4}}\right)^3 - 27 = 0$.
The eigenvalues are $\lambda_1 = -\frac{1}{\sqrt[3]{2}}$, with the algebraic multiplicity $ma_{\lambda_1} = 2$ and $\lambda_2 = \sqrt[3]{4}$, with the algebraic multiplicity $ma_{\lambda_2} = 1$. The eigenspace associated with the eigenvalue λ_1 is

$$S(\lambda_1) = \left\{ \left(\alpha \cdot \sqrt[3]{4}, -\alpha \cdot \sqrt[3]{2}, \alpha \right) | \alpha \in \mathbb{R} \right\}.$$

It follows that the geometric multiplicity of λ_1 is $mg_{\lambda_1} = \dim_{\mathbb{R}} S(\lambda_1) = 1$. So, $ma_{\lambda_1} \neq mg_{\lambda_1}$ and the matrix M_{p_1} is not diagonalizable.

Case 2: When $a < \frac{3}{\sqrt[3]{4}}$.

The characteristic equation is

$$\lambda^3 - a\lambda - 1 = 0.$$

If $a < \frac{3}{\sqrt[3]{4}}$, the discriminant of this cubic equation is $\Delta = 4a^3 - 27 < 0$. So, M_{p_1} has the eigenvalues $\lambda_1 \in \mathbb{R}$ and $\lambda_2, \lambda_3 \in \mathbb{C}\setminus\mathbb{R}$. It follows that, for $a < \frac{3}{\sqrt[3]{4}}$, the matrix $M_{p_1} \in M_3(\mathbb{R})$ is not diagonalizable over $\mathbb{R}$. □

Together, Propositions 2.3 and 2.5 show that the whole family $(M_{p_n})_{n\geq 0}$ is simultaneously diagonalizable over $\mathbb{R}$ if and only if $a > 3/\sqrt[3]{4}$, independently of the nonzero real parameter b. The diagonalizability of individual matrices below or at this threshold is not classified here.

Proposition 2.6. *Let $(M_{p_n})_{n\geq 0}$ be the sequence of bi-periodic Padovan matrices. The following statements are true:*

i) The set $A = \left\{\alpha \cdot I_3 + \beta \cdot M_{p_1} + \gamma \cdot M_{p_1}^2 | \alpha, \beta, \gamma \in \mathbb{R}\right\}$ is a finitely generated algebra over the field $\mathbb{R}$;

ii) $M_{p_n} \in A$, for any $n \in \mathbb{N}$.

Proof. i) Applying Cayley–Hamilton theorem, it follows that $M_{p_1}^m \in A$, for any $m \in \mathbb{N}$, $m \geq 3$. We obtain quickly that A is a subalgebra of the algebra $M_3(\mathbb{R})$ over the field $\mathbb{R}$, namely A is the $\mathbb{R}$-algebra generated by the matrix M_{p_1}.
ii) It follows immediately by applying Proposition 2.1. □

In proving our next results we used the following remark.

Remark 2.7. ([6]).

i) $\mathrm{Tr}(T \cdot A \cdot T^{-1}) = \mathrm{Tr}(A)$, *for any $A \in M_n(\mathbb{C})$ and for any invertible matrix $T \in M_n(\mathbb{C})$.*

ii) Let n be a positive integer, let K be a commutative field and let $A \in M_n(K)$. If the matrix A has the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the matrix A^l has the eigenvalues $\lambda_1^l, \lambda_2^l, \ldots, \lambda_n^l$, where $l \in \mathbb{N}^*$.

For $a = 2$, we obtain the following result about matrices $M_{p_1}^n$, with n positive integers.

Proposition 2.8. *Let $(L_n)_{n\geq 0}$ be the sequence of Lucas numbers and let $(M_{p_n})_{n\geq 0}$ be the sequence of bi-periodic Padovan matrices with $a = 2$ and b is an arbitrary nonzero real number. Then* $\mathrm{Tr}\left(M_{p_1}^n\right) = L_n + (-1)^n$.

Proof. If, we denote $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$, according to Binet's formula for Lucas numbers, we have $L_n = \alpha^n + \beta^n$.

From (2.1), there exists an invertible matrix $T \in \mathcal{M}_3(\mathbb{R})$ such that:

$$M_{p_1} = T^{-1} \cdot \begin{pmatrix} -1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix} \cdot T.$$

From here, applying Remark 2.7 ii) and i), it follows that $\mathrm{Tr}\left(M_{p_1}^n\right) = L_n + (-1)^n$. □

If we denote by $B = Q \cdot M_{p_1}$, we quickly obtain that the eigenvalues of the matrix B are $\lambda_1 = \lambda_2 = \lambda_3 = 1$, but B is not diagonalizable.

Proposition 2.9. *i)* $B^n = \begin{pmatrix} 1 & na & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, *for any* $n \in \mathbb{N}^*$.

ii) If a *is a real number, the set of matrices* $G = \left\{ \begin{pmatrix} 1 & na & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} | n \in \mathbb{Z} \right\}$ *forms a cyclic group with respect to matrix multiplication.*

Proof. i) $B = Q \cdot M_{p_1} = \begin{pmatrix} 1 & a & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. It is quickly shown by induction over $n \in \mathbb{N}^*$ that $B^n = \begin{pmatrix} 1 & na & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

ii) The matrix B is invertible and $B^{-1} = \begin{pmatrix} 1 & -a & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. It is easily proven by induction over $n \in \mathbb{N}^*$ that $\left(B^{-1}\right)^n = \begin{pmatrix} 1 & -na & -n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Using these, it is easy to show that G is a cyclic subgroup of the group $\left(GL_3\left(\mathbb{R}\right), \cdot\right)$ of invertible matrices of order 3, with real number elements. Since the $(1,3)$ entry of B^n equals n, the powers B^n are distinct. Thus $G = \langle B \rangle \cong (\mathbb{Z}, +)$ is an infinite cyclic subgroup of the real Heisenberg group of upper triangular 3×3 matrices with all diagonal entries equal to 1. □

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