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On generalized continued fractions converging to an irrational number

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Abstract: In [5], we give a sufficient condition for generalized continued fractions to converge to an irrational number. Specifically, we determine how many non-integers must exist in $c_1, c_2, \ldots, c_i$ in order for the continued fraction $[0; c_1, c_2, \ldots]$ to be irrational, where c_i are rational numbers greater than $M > 1$ (denoted by $c_i \in \mathbb{Q}_{\geq M}$) and the denominator of c_i less than or equal to positive integer N. In [6], we extended this result from the case $M > 1$ to $M > 0$. In this paper, we give a further extension by removing the restrictions on the denominators of the partial quotients. Let M be a real number greater than 0. We consider continued fractions $[0; c_1, c_2, \ldots]$, where c_i are rational numbers greater than or equal to M for any positive integer i. We give a sufficient condition for it to converge to an irrational number and some interesting examples.
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1 Introduction

Let M be a positive real number. We consider the following continued fraction:

$$\alpha = [0; c_1, c_2, c_3, \ldots] := 0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \cdots}}}, \tag{1}$$

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where $c_i \in \mathbb{Q}_{\geq M}$ for any positive integer i. This continued fraction converges to a real number by the Seidel–Stern theorem (see [3] or [4]) . It is well known that if $c_i \in \mathbb{Z}_{>0}$ for any positive integer i, the continued fraction (1) converges to an irrational number (see [1]). We denote the denominator (respectively, the numerator) of c_i by $\operatorname{den} c_i$ (respectively, $\operatorname{num} c_i$). Then we have the following equation:

$$\alpha = \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \cdots}}} = \cfrac{\operatorname{den} c_1}{\operatorname{num} c_1 + \cfrac{(\operatorname{den} c_1)(\operatorname{den} c_2)}{\operatorname{num} c_2 + \cfrac{(\operatorname{den} c_2)(\operatorname{den} c_3)}{\operatorname{num} c_3 + \cdots}}}. \tag{2}$$

For the right hand side of this, the following result is well known (see [4] or [2]):

Theorem 1.1. *Let n_0 be a positive integer. If $\operatorname{num} c_i$ is greater than or equal to $(\operatorname{den} c_{i-1})(\operatorname{den} c_i)$ for any integer $i \geq n_0 \geq 2$, then the continued fraction* (2) *converges to an irrational number.*

This theorem implies that if $c_1 \geq 1$ and $c_i \geq \operatorname{den} c_{i-1} \geq 1$ for $i \geq 2$, then α is irrational. This criterion is sufficient but not necessary. In [5], the following extension was obtained:

Theorem 1.2 ([5, Theorem 1.2])**.** *Let S_n be the set of all c_i such that $1 \leq i \leq n$ and c_i is not an integer, and let $|S_n|$ denote the cardinality of S_n. Assume that $c_i \geq M > 1$ and $\operatorname{den} c_i \leq N$ for all $i \geq 1$. Then*

$$\limsup_{n\to\infty} \frac{|S_n|}{n} < \log_N M \Longrightarrow [0; c_1, c_2, \ldots] \in \mathbb{R} - \mathbb{Q}.$$

Furthermore, we extended this result from the case $M > 1$ to $M > 0$:

Theorem 1.3 ([6, Theorem 3])**.** *Let M be a positive real number greater than 0. We assume that $\operatorname{den} c_i \leq N$ for all $i \geq 1$ and put $\varphi_M := (M + \sqrt{M^2+4})/2$. Then*

$$\limsup_{n\to\infty} \frac{|S_n|}{n} < \log_N \varphi_M \Longrightarrow [0; c_1, c_2, \ldots] \in \mathbb{R} - \mathbb{Q}.$$

2 Main results

These results provide less restrictive sufficient conditions for a generalized continued fraction to be irrational, compared to Theorem 1.1. We further extend Theorem 1.3 by removing the restriction that the denominator of c_i is bounded by N. Specifically, we obtain a sufficient condition for α to be irrational by considering the limit superior of the sequence $(\prod_{i=1}^{n} \operatorname{den} c_i)^{1/n}$ as $n \to \infty$.

The main contribution of this note is a sufficient irrationality criterion that replaces pointwise restrictions on the denominators of rational partial quotients with a condition on their asymptotic geometric mean. Our main result is the following theorem:

Theorem 2.1. *With the above notations, put $D_n = \prod_{i=1}^{n} \operatorname{den} c_i$. Then*

$$\limsup_{n\to\infty} D_n^{\frac{1}{n}} < \varphi_M \Longrightarrow \alpha \in \mathbb{R} - \mathbb{Q},$$

or equivalently

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\log \operatorname{den} c_i < \log\varphi_M \Longrightarrow \alpha\in\mathbb{R}-\mathbb{Q}.$$

Since $\varphi_M > 1$ for every $M > 0$, Theorem 2.1 can establish the irrationality of α even when $\limsup_{i\to\infty}\operatorname{den} c_i = \infty$, or when there are infinitely many i such that $c_i < 1$. This extends the applicability of Theorems 1.1 and 1.3. We give examples illustrating these conditions at the end of the paper. To prove Theorem 2.1, we need the following lemma.

Lemma 2.1. [5, Lemma 4] *Let $p_n/q_n = [0; c_1, c_2, \ldots, c_n]$. Then we have that*

$$\lim_{n\to\infty}\frac{D_n}{q_n} = 0 \Longrightarrow \alpha\in\mathbb{R}-\mathbb{Q}.$$

Proof. We put $\alpha = p_{n+1}^{'}/q_{n+1}^{'} = [0; c_1, c_2, \ldots, c_n, \alpha_{n+1}]$, where $\alpha_{n+1} = [c_{n+1}; c_{n+2}, c_{n+3}, \ldots]$. By the classical continued fraction theory, rationals $p_{n+1}^{'}$ and $q_{n+1}^{'}$ satisfy the recurrence relations;

$$\begin{cases} p_{n+1}^{'} = \alpha_{n+1}p_n + p_{n-1}, \\ q_{n+1}^{'} = \alpha_{n+1}q_n + q_{n-1}, \end{cases}$$

especially

$$\begin{cases} p_{n+1} = c_{n+1}p_n + p_{n-1}, \\ q_{n+1} = c_{n+1}q_n + q_{n-1}, \end{cases}$$

with initial conditions $(p_{-1}, q_{-1}) = (1, 0),\ (p_0, q_0) = (0, 1)$. Then we obtain that

$$\alpha = \frac{p_{n+1}^{'}}{q_{n+1}^{'}} = \frac{\alpha_{n+1}p_n + p_{n-1}}{\alpha_{n+1}q_n + q_{n-1}}.$$

Hence we see that

$$\alpha - \frac{p_n}{q_n} = \frac{p_{n-1}q_n - p_nq_{n-1}}{q_n(\alpha_{n+1}q_n + q_{n-1})}.$$

Here by recurrence relations of p_n and q_n, we easily show that $p_{n-1}q_n - p_nq_{n-1} = (-1)^n$ by induction. Furthermore $c_i \geq M > 0$, then we see that $\alpha_{n+1} > M, q_n > 0$ for any positive integer n. Therefore we have inequalities:

$$0 < \left|\alpha - \frac{p_n}{q_n}\right| = \left|\frac{p_{n-1}q_n - p_nq_{n-1}}{q_n(\alpha_{n+1}q_n + q_{n-1})}\right| < \left|\frac{p_{n-1}q_n - p_nq_{n-1}}{q_n(Mq_n + q_{n-1})}\right| < \frac{1}{Mq_n^2}.$$

Since $0 < |\alpha q_n - p_n| < 1/Mq_n$, if p_n, q_n are positive integers, then α is irrational. Because, if $\alpha = a/b$ is rational number, we see that

$$0 < \left|\frac{a}{b}q_n - p_n\right| < \frac{1}{Mq_n}.$$

From these inequalities, since

$$0 < |aq_n - bp_n| < \frac{b}{Mq_n},$$

we see that $0 < |aq_n - bp_n| \to 0$ as $n \to \infty$. This contradicts $|aq_n - bp_n| \geq 1$.

In general, p_n, q_n are not positive integers by $c_n \in \mathbb{Q}_{\geq M}$, but they can be made positive integers by multiplying them by D_n. By recurrence relations of p_n, we have that

$$p_n D_n = c_n \cdot p_{n-1} D_n + p_{n-2} D_n = (\text{num } c_n) p_{n-1} D_{n-1} + (\text{den } c_n)(\text{den } c_{n-1}) p_{n-2} D_{n-2}.$$

Hence we easily see that $p_n D_n$ are positive integers for any $n \geq 1$ by induction. Similarly, we obtain that $q_n D_n$ are positive integers for any $n \geq 1$. Therefore we get inequalities:

$$0 < |\alpha q_n D_n - p_n D_n| < \frac{D_n}{M q_n}.$$

If the rightmost side approaches zero as n approaches infinity, a real number α is irrational. □

Proof of Theorem 2.1. We consider the generalized continued fraction:

$$\alpha = [0; c_1, c_2, c_3, \ldots] = \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \cdots}}},$$

where $c_i \in \mathbb{Q}_{\geq M}$. We obtain an inequality for q_n:

$$q_n = c_n q_{n-1} + q_{n-2} \geq M q_{n-1} + q_{n-2}.$$

This follows from the recurrence relation for q_n and the assumption for c_i to be greater than or equal to M. Putting $r_n = M r_{n-1} + r_{n-2}$, with initial conditions $r_{-1} = 0$ and $r_0 = 1$, we determine the asymptotic behavior of r_n using an argument similar to that in [6]. We see that

$$\begin{aligned} q_n \geq r_n &= \frac{1}{\sqrt{M^2+4}} \left(\left(\frac{M + \sqrt{M^2+4}}{2} \right)^{n+1} - \left(\frac{M - \sqrt{M^2+4}}{2} \right)^{n+1} \right) \\ &= \frac{1}{\sqrt{M^2+4}} \left(\varphi_M^{n+1} - \left(-\frac{1}{\varphi_M} \right)^{n+1} \right). \end{aligned}$$

Since $\varphi_M > 1$, we obtain that

$$\lim_{n\to\infty} \frac{\sqrt{M^2+4}}{\varphi_M^{n+1}} r_n = \lim_{n\to\infty} \left(1 - \left(-\frac{1}{\varphi_M^2} \right)^{n+1} \right) = 1.$$

Then we have that $r_n \sim \varphi_M^{n+1}/\sqrt{M^2+4} \ \ (n \to \infty)$, where $f(n) \sim g(n) \ \ (n \to \infty)$ implies $f(n)/g(n) \to 1 \ \ (n \to \infty)$. Hence we obtain that

$$\frac{1}{q_n} \leq \frac{1}{r_n} \sim \frac{\sqrt{M^2+4}}{\varphi_M^{n+1}} \quad (n \to \infty).$$

By Lemma 2.1, if

$$\lim_{n\to\infty} \frac{\sqrt{M^2+4}}{\varphi_M} \cdot \frac{D_n}{\varphi_M^n} = 0, \ \text{ i.e., } \ \lim_{n\to\infty} \frac{D_n}{\varphi_M^n} = 0,$$

then a real number $\alpha = [0; c_1, c_2, \ldots]$ is irrational. Since

$$\lim_{n\to\infty} \frac{D_n}{\varphi_M^n} = \lim_{n\to\infty} \left(\frac{D_n^{\frac{1}{n}}}{\varphi_M} \right)^n = 0,$$

if

$$\limsup_{n\to\infty} \frac{D_n^{\frac{1}{n}}}{\varphi_M} < 1 \quad \text{i.e.,} \quad \limsup_{n\to\infty} D_n^{\frac{1}{n}} < \varphi_M, \tag{3}$$

then $\lim_{n\to\infty}(D_n^{\frac{1}{n}}/\varphi_M)^n = 0$. Also, the condition (3) is equivalent to

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log \operatorname{den} c_i < \log \varphi_M.$$

Therefore, this completes the proof of Theorem 2.1. □

Since $D_n^{\frac{1}{n}} \geq 1$ and $(\sum_{i=1}^{n} \log \operatorname{den} c_i)/n \geq 0$, we can state the following corollary.

Corollary 2.1.

$$\limsup_{n\to\infty} D_n^{\frac{1}{n}} = 1 \ \text{ or } \ \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log \operatorname{den} c_i = 0 \Longrightarrow \alpha = [0; c_1, c_2, \ldots] \in \mathbb{R} - \mathbb{Q},$$

Proof. Clearly, φ_M is greater than 1 and $\log \varphi_M$ is greater than 0. Therefore, a real number α is irrational for every positive real number M by Theorem 2.1. □

3 Examples

We give four examples including cases in which the denominators of the partial quotients are unbounded.

Example 1. For any real number $0 \leq \varepsilon < 1$ and for some constants $a, b \geq 0$, we assume that $\prod_{i=1}^{n} \operatorname{den} c_i \leq n^{an^{\varepsilon}+b}$. Then we see that

$$\frac{1}{n} \sum_{i=1}^{n} \log \operatorname{den} c_i \leq \frac{an^{\varepsilon}+b}{n} \log n = a \cdot \frac{\log n}{n^{1-\varepsilon}} + b \cdot \frac{\log n}{n}.$$

Since $(\log n)/n^{1-\varepsilon}$ and $(\log n)/n$ approach 0 as n approaches infinity, we have that a real number α is irrational by Corollary 2.1.

Example 2. We put $\operatorname{den} c_i = \sqrt{i}$ if i is square, $\operatorname{den} c_i = 1$ otherwise. Then we have that

$$D_n = \prod_{\substack{i:\text{square} \\ 1\leq i\leq n}} \sqrt{i} = \prod_{i=1}^{[\sqrt{n}]} i \leq [\sqrt{n}]^{[\sqrt{n}]} < \sqrt{n}^{\sqrt{n}},$$

where $[x]$ denotes the integer part of x. Hence, we see that

$$1 \leq D_n^{\frac{1}{n}} < \sqrt{n}^{\frac{1}{\sqrt{n}}} \to 1 \quad (n \to \infty).$$

Therefore, a real number α is irrational by Corollary 2.1.

Example 3. Let $\pi(x)$ be the prime-counting function. We define $\operatorname{den} c_i = [\log(\pi(i)+2)]$ if i is prime, $\operatorname{den} c_i = 1$ otherwise. Then we obtain that

$$1 \leq D_n = \prod_{\substack{i:\text{prime} \\ 1 \leq i \leq n}} [\log(\pi(i)+2)] \leq \prod_{\substack{i:\text{prime} \\ 1 \leq i \leq n}} \log(\pi(i)+2) = \prod_{k=3}^{\pi(n)+2} \log k.$$

Hence we obtain the inequalities

$$1 \leq D_n^{\frac{1}{n}} \leq \left(\prod_{k=3}^{\pi(n)+2} \log k \right)^{\frac{1}{n}} \leq (\log(\pi(n)+2))^{\frac{\pi(n)}{n}}.$$

Taking logarithms, these inequalities yield

$$0 \leq \frac{1}{n} \log D_n \leq \frac{1}{n} \sum_{k=3}^{\pi(n)+2} \log\log k \leq \frac{\pi(n)}{n} \log\log(\pi(n)+2).$$

By the prime number theorem, since $\pi(n) \sim n/\log n$, i.e., $\pi(n)/n \sim 1/\log n \ \ (n \to \infty)$, we have that the rightmost side asymptotically zero. Then we have that $D_n^{\frac{1}{n}}$ approaches 1. Therefore, a real number α is irrational by Corollary 2.1.

Example 4. We define $\operatorname{den} c_i = \pi(i)$ if i is prime, $\operatorname{den} c_i = 1$ otherwise. Then we obtain that the equalities

$$D_n = \prod_{\substack{i:\text{prime} \\ 1 \leq i \leq n}} \pi(i) = (\pi(n))!.$$

Hence we have that

$$\frac{1}{n} \sum_{i=1}^{n} \log \operatorname{den} c_i = \frac{1}{n} \log \pi(n)!.$$

By Stirling's approximation, the right hand side can be approximated by

$$\frac{\pi(n)\log\pi(n) - \pi(n) + \mathcal{O}(\log\pi(n))}{n} = \frac{\pi(n)\log\pi(n)}{n} - \frac{\pi(n)}{n} + \frac{\mathcal{O}(\log\pi(n))}{n}.$$

Using the prime number theorem, we see that the right hand side asymptotically approaches

$$\frac{\log\pi(n)}{\log n}.$$

Since $\log\pi(n) \sim \log n - \log\log n$ by the prime number theorem, we see that

$$\frac{\log\pi(n)}{\log n} \sim 1 - \frac{\log\log n}{\log n} \to 1.$$

Hence we have that

$$\frac{1}{n}\sum_{i=1}^{n}\log \operatorname{den} c_i = \frac{1}{n}\log \pi(n)! \to 1 \quad (n \to \infty).$$

Especially, since

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\log \operatorname{den} c_i = 1,$$

if $\log \varphi_M > 1$, a real number α is irrational by Theorem 2.1. This condition is equivalent to

$$M > e - \frac{1}{e} = 2.35040238\ldots.$$

Therefore, if $M > e - 1/e$ and $\operatorname{den} c_i$ defined by above, then a real number α is irrational.

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References

[1] Khinchin, A. Ya. (1949). *Continued Fractions* (2nd ed.). Gosudarstvennoe Izdatel'stvo Tekhniko-Teoreticheskoi Literatury. Moscow–Leningrad (in Russian); English translation, University of Chicago Press, Chicago-London, 1964.

[2] Legendre, A. M. (1794). *Éléments de Géométrie*. Firmin Didot, Paris.

[3] Lorentzen, L., & Waadeland, H. (1992). *Continued Fractions with Applications*. North-Holland.

[4] Perron, O. (1913). *Die Lehre von den Kettenbrüchen*. B.G. Teubner, Leipzig.

[5] Yoshida, H. (2025). A simple condition for generalized continued fractions to converge to an irrational number. *Research in Number Theory*, 11, Article ID 36.

[6] Yoshida, H. (2025). An extension of conditions for generalized continued fractions to converge to an irrational number. *The Ramanujan Journal*, 68, Article ID 73.