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An extension of Dedekind's arithmetic function – II

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Abstract: We obtain refinements and new results related to an extension of Dedekind's arithmetic function, introduced in [1].
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1 Introduction

Let S_k denote the number of k consecutive numbers each less than n, and relatively prime to n. In 1869, Schemmel proved that for each integer $a \geq 1$ (see [2, 15]):

$$S_k(p^a) = \begin{cases} 0, & \text{if } p \leq k \\ p^{a-1}(p-k), & \text{if } p > k \end{cases}$$

and S_k is a multiplicative, with $S_k(1) = 1$. This form of Schemmel's function has the following other representation that we will use below:

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$$S_k(n) = n \cdot \prod_{p|n} \left(1 - \frac{k}{p}\right), \tag{1}$$

where in the product p runs through all prime divisors of $n \geq 2$ and every prime divisor of n is at least $k \geq 1$. When k is a prime divisor of n, then obviously $S_k(n) = 0$.

There are many known generalizations of $S_k(n)$, see the monograph [11]. For example, K. Nageswara Rao [5] in 1961 considered the generalization

$$S_k^{(s)}(n) = n^s \cdot \prod_{p|n} \left(1 - \frac{k}{p^s}\right). \tag{2}$$

Let $F = \{f_1(x), \dots f_k(x)\}$ $(k \geq 1)$ be a set of polynomials with integer coefficients and let A be the set of all ordered k-tuples of integers $(a_1, \dots, a_k)$ such that $0 \leq a_1, \dots, a_k \leq n$. Define $\varphi_F(n)$ as the number of elements in A such that $\gcd(f_1(a_1), \dots, f_k(a_k), n) = 1$. This generalization is due to H. Stevens (see [16]). For an asymptotic result on Steven's function, see [14]. There are many connections of these functions to other arithmetic functions, or applications in various fields of mathematics (see, e.g., [11]). For example, D. N. Lehmer [4] used $S_2(n)$ (i.e., $S_k(n)$ for $k = 2$) in the enumeration problem of certain magic squares..

In 2000, M. Deaconescu [3] proved the inequality

$$\varphi(n)(\varphi(n) - 1) \geq (n-1)S_2(n) \tag{3}$$

and made the conjecture that for $n \geq 2, S_2(n)$ divides $\varphi(n) - 1$ if and only if n is prime. This generalization of Lehmer's conjecture (see [14]) seems as difficult as the original Lehmer's problem is.

Recently, K. Atanassov [1] introduced an analogue of (1), and called it *the generalized Dedekind function*:

$$D_k(n) = n \cdot \prod_{p|n} \left(1 + \frac{k}{p}\right);\ D_k(1) = 1 \tag{4}$$

(in [1] is used the notation $\psi^{(k)}(n)$ for general k, and $\psi^*(n)$ for $k = 2$; and also $\varphi^*(n)$ for $S_2(n)$, but we will use the classical notations in the literature, i.e., for example, φ^* and ψ^* are usually denoted to the Cohen unitary functions). We will use the notation $D_2(n)$ for $k = 2$ in (4).

The aim of this paper is to offer new proofs and refinements of some results from [1], as well as certain new results of other type.

2 Main results

Theorem 1 of [1] can be generalized as follows:

Theorem 1. *Let n, k, a, b be positive integers such that $k^2 \geq ab$, and $2k \geq a + b$. Then one has*

$$D_k^2(n) \geq D_a(n) \cdot D_b(n) \text{ for } n \geq 1. \tag{5}$$

Proof. Let $n = p_1^{a_1} \cdots p_r^{a_r}$ be the prime factorization of n. Dividing both sides of (5) by n^2 and denoting $\frac{1}{p_i} = x_i$ $(i = 1, \ldots, r)$, it will be sufficient to prove that $(1+kx_i)^2 \geq (1+ax_i)(1+bx_i)$, or $k^2x_i + 2k \geq abx_i + a + b$. Since $k^2 \geq ab$, and $2k \geq a + b$, the result follows.

In particular, selecting $a = 2k - s$, $b = s$ $(s < 2k)$, the conditions are satisfied, as $k^2 \geq s(2k - s)$ becomes $(k - s)^2 \geq 0$. □

Let for the natural number n with prime divisors $p_1, \cdots, p_r$

$$\omega(n) = r,$$
$$\gamma(n) = p_1 \cdots p_r.$$

Theorem 2. *Let $n \geq 2$ have distinct prime divisors $p_1 < \cdots < p_r$ and let k be an integer such that $2 \leq k \leq p_1$. Then*

$$\frac{S_k(n) \cdot D_k(n)}{n^2} \leq 1 - \frac{k^{2\omega(n)}}{\gamma^2(n)} \leq 1 - \frac{k^2}{n^2} \cdot \omega^2(n) \text{ for } k \geq 2. \tag{6}$$

Proof. Apply the classical inequality

$$(1 - z_1) \cdots (1 - z_r) \leq 1 - z_1 \cdots z_r, \ z_i \in (0, 1] \tag{7}$$

$(i = 1, 2, \ldots, r; r \geq 1)$ for $z_i = (k/p_i)^2$. Then the first inequality of (6) follows, by remarking that the left side is $S_k(n)D_k(n)/n^2$ by relations (1) and (4). For the second inequality of (6), written in the form

$$\frac{k^{2\omega(n)}}{\gamma^2(n)} \geq \frac{k^2}{n^2}\omega^2(n),$$

remark that as $\frac{n}{\gamma(n)} \geq 1$, it will be sufficient to show that $k^{2\omega(n)} \geq k^2\omega^2(n)$ or $k^{\omega(n)} \geq k \cdot \omega(n)$. This is true by $k^{\omega(n)-1} \geq 2^{\omega(n)-1} \geq \omega(n)$.

Relation (6) improves Theorem 2 of [1]. □

Theorem 3. *With the notations of Theorem 2, one has*

$$S_k(n) + D_k(n) \geq 2n + \frac{2(\omega(n) - 1) \cdot k^{\omega(n)} \cdot n}{\gamma(n)} \geq 2n + 2(\omega(n) - 1) \cdot k^{\omega(n)} \text{ for } k < p_1 < \cdots < p_r. \tag{8}$$

Proof. We will apply the inequality (see [6]):

$$\prod_{i=1}^{r}(x_i + 1) + \prod_{i=1}^{r}(x_i - 1) \geq 2\prod_{i=1}^{r} x_i + 2(r - 1); \ r \geq 1, x_i \geq 1 (i = 1, \ldots, r). \tag{9}$$

By letting $x_i = \frac{1}{z_i}$, with $z_i \in (0, 1]$ in (9), we get

$$(1 + z_1) \cdots (1 + z_r) + (1 - z_1) \cdots (1 - z_r) \geq 2 + 2(r - 1)z_1 \cdots z_r. \tag{10}$$

Letting $z_i = \frac{k}{p_i}$, and using (1) and (4), we get

$$S_k(n) + D_k(n) \geq 2n + \frac{2(\omega(n) - 1) \cdot k^{\omega(n)} \cdot n}{\gamma(n)}.$$

The last inequality of (8) follows from $\frac{n}{\gamma(n)} \geq 1$ and $k^{\omega(n)} \geq k$. Relation (8) improves the inequality of Theorem 3 from [1]. □

Theorem 4. *Let $n \geq 1$ and $k \geq 2$ are integers, then*

$$D_k(n) \leq n\left(1+\frac{k}{2}\right)^{\omega(n)} \leq \frac{n \cdot k^{d(n)}}{2}. \tag{11}$$

Proof. As $1+\frac{k}{p} \leq 1+\frac{k}{2}$ for each prime $p \geq 2$, we can write $\frac{D_k(n)}{n} = (1+\frac{k}{p_1})\cdots(1+\frac{k}{p_r}) \leq (1+\frac{k}{2})^{\omega(n)}$, the first inequality of (11) follows. For the second inequality, write $k^{d(n)} = k \cdot k^{d(n)-1}$, and use the relations $k \geq 1+k/2$ and $d(n)-1 \geq \omega(n)$, which follows by $d(n) \geq 2^{\omega(n)} \geq \omega(n)+1$.

Relation (11) improves the inequality of Theorem 8 from [1]. □

Theorem 5. *For integers $n \geq 2$ and $k \geq 1$ one has*

$$D_k(n) \geq n + \frac{nk^{\omega(n)}}{\gamma(n)}. \tag{12}$$

Proof. In the classical inequality $(1+y_1)\cdots(1+y_r) \geq 1+y_1\cdots y_r \quad (y_i > 0)$, let $y_i = \frac{k}{p_i} \quad (i = 1,\ldots,r)$. Then we get

$$\frac{D_k(n)}{n} = \left(1+\frac{k}{p_1}\right)\cdots\left(1+\frac{k}{p_r}\right) \geq 1+\frac{k^{\omega(n)}}{\gamma(n)},$$

and the result follows.

Relation (12) improves the inequalities of Theorem 9 from [1], where in place of $k^{\omega(n)}$ there is $k^{\omega(n)-1}$. □

Theorem 6. *One has*

$$\frac{D_2(n)}{\sigma(n)} \geq 1+\frac{1}{\sigma(n)\cdot s(n)},\ for\ n>1, \tag{13}$$

where $\sigma(n)$ is the sum of divisors of n and $s(n) = \prod_{p|n}(p-1)$, where p runs through all distinct prime divisors of n.

Proof. Let $a \geq 1$ be an integer. First, let $n = 2^a$. Then $s(n) = 1$ and

$$\frac{D_2(n)}{\sigma(n)} = \frac{2^{a+1}}{2^{a+1}-1} = 1+\frac{1}{2^{a+1}-1} = 1+\frac{1}{\sigma(n)\cdot s(n)}.$$

Second, let p^a be a prime power. We will show that

$$\frac{D_2(p^a)}{\sigma(p^a)} \geq \frac{p^{a+1}}{p^{a+1}-1}. \tag{14}$$

As $D_2(p^a) = p^a \cdot (1+\frac{2}{p})$ and $\sigma(p^a) = \frac{p^{a+1}-1}{p-1}$, we easily obtain that

$$\frac{D_2(p^a)}{\sigma(p^a)} = \frac{p^{a+1}+p^{a-1}\cdot(p-2)}{p^{a+1}-1} \geq \frac{p^{a+1}}{p^{a+1}-1} = 1+\frac{1}{p^{a+1}-1},$$

as $p \geq 2$.

Third, as

$$\prod_{p^a||n}(p^{a+1}-1)=\sigma(n)\cdot s(n),$$

for arbitrary n, we obtain

$$\frac{D_2(n)}{\sigma(n)}\geq\prod_{p^a||n}\left(1+\frac{1}{p^{a+1}-1}\right)\geq 1+\frac{1}{\prod\limits_{p^a||n}(p^{a+1}-1)}=1+\frac{1}{\sigma(n)s(n)}$$

and the inequalities of (13) follow.

The equality in (13) holds if and only if $n=2^a$ for some integer $a\geq 1$. □

Inequality (13) improves the inequalities of Theorem 10 from [1].

Theorem 7. *For $n\geq 2$, one has*

$$S_2(n)\leq\frac{\varphi(n)(\varphi(n)-1)}{n-1}\leq\varphi(n)-1\leq n-d(n). \tag{15}$$

Proof. The first relation of (15) follows by Deaconescu's inequality (3). The second relation follows by the classical inequality $\varphi(n)\leq n-1$, while for the last inequality, written in the form $\varphi(n)+d(n)\leq n+1$, see [7] or [8].

Relation (15) improves the inequality $s_2(n)+d(n)\leq n$ of Theorem 4 from [1]. □

Another refinement of the inequality $s_2(n)\leq n-d(n)$ is provided by the next Theorem.

Theorem 8. *For $n\geq 2$, one has*

$$S_2(n)\leq n\Big(1-\frac{2^{\omega(n)}}{\gamma(n)}\Big)\leq n-d(n). \tag{16}$$

Proof. Applying inequality (7) for $z_i=\frac{2}{p_i}$ $(i=1,\ldots,r)$, we get the first inequality of (16). For the second inequality, we have to prove that

$$n\cdot 2^{\omega(n)}\geq d(n)\cdot\gamma(n). \tag{17}$$

This can be rewritten as $p_1^{a_1}\cdots p_r^{a_r}\cdot 2^r\geq p_1\cdots p_r\cdot(a_1+1)\cdots(a_r+1)$, or $p_1^{a_1-1}\cdots p_r^{a_r-1}\cdot 2^r\geq(a_1+1)\cdots(a_r+1)$. It will be sufficient to show that $p^{a-1}\cdot 2\geq a+1$. For $a=1$ there is equality, and by induction upon a the relation follows. Indeed, $p^a\cdot 2=(p^{a-1}\cdot 2)\cdot p\geq(a+1)\cdot 2\geq a+2$, as $a>0$. □

A refinement of inequality $D_2(n)\geq n+d(n)\omega(n)$ of Theorem 5 from [1] is provided by the next Theorem.

Theorem 9. *For $n\geq 2$, one has*

$$D_2(n)\geq n+\frac{n}{\gamma(n)}\cdot\omega(n)\cdot 2^{\omega(n)}\geq n+d(n)\cdot\omega(n). \tag{18}$$

Proof. Applying the well-known inequality

$$(x_1+1)\cdots(x_r+1) \geq x_1\cdots x_r + r \text{ for } x_i \geq 1 \tag{19}$$

for $x_i = \frac{1}{y_i}$ $(0 < y_i \leq 1)$, we can write

$$(y_1+1)\cdots(y_r+1) \geq ry_1\cdots y_r + 1. \tag{20}$$

Let $y_i = \dfrac{2}{p_i}$ in (20). Then by (4) for $k=2$, we get $D_2(n) \geq n + \dfrac{n}{\gamma(n)} \cdot \omega(n) \cdot 2^{\omega(n)}$, so the first inequality of (18) follows. Now the second inequality of (18) follows by (17). □

Theorem 10. *One has the double inequality for every odd integer $n \geq 3$:*

$$P = \prod_{p\geq 3}\Big(1-\frac{4}{p^2}\Big) < \frac{S_2(n)D_2(n)}{n^2} < 1 \tag{21}$$

and

$$\limsup_{n\to\infty, n \text{ odd}} \frac{S_2(n)D_2(n)}{n^2} = 1; \tag{22}$$

$$\liminf_{n\to\infty, n \text{ odd}} \frac{S_2(n)D_2(n)}{n^2} = P \approx 0.378\ldots \tag{23}$$

Proof. One has

$$\frac{S_2(n)D_2(n)}{n^2} = \Big(1-\frac{4}{p_1^2}\Big)\cdots\Big(1-\frac{4}{p_r^2}\Big) < 1,$$

and the same is greater than $\prod\limits_{p\geq 3}\Big(1-\frac{4}{p^2}\Big) = P$.

As the infinite series $\sum\limits_{p\geq 3}\frac{1}{p^2}$ is convergent, the infinite product will be convergent, too. As for $n = p$ (prime), $\dfrac{S_2(p)D_2(p)}{p^2} \to 1$ as $p \to \infty$ and for $n = p_1\cdots p_k$ (where $p_k > \cdots > p_1 \geq 3$ are consecutive odd primes). For n equal to the product of the first m odd primes, the ratio tends to P as $m \to \infty$, proving (23).

The approximation $P \approx 0.378\ldots$ follows by computer calculations. □

Theorem 11. *For each integer $n \geq 2$, one has*

$$(S_2(n))^2 + 2(D_2(n))^2 \geq 3n^2 + \frac{2^{\omega(n)+1}\cdot n^2}{\gamma(n)} + \frac{12n^2}{\gamma(n)^2} \tag{24}$$

and

$$D_2(n)(D_2(n)+2S_2(n)) \geq 3n^2 + \frac{2^{\omega(n)+1}\cdot n^2}{\gamma(n)} - \frac{4^{\omega(n)}n^2}{\gamma(n)^2}. \tag{25}$$

Proof. In [9] the following two algebraic inequalities were proved: Let $x_i \geq 1$ $(i = 1, 2, \ldots, r)$,, then it holds that

$$\prod_{i=1}^{r}(x_i-1)^2+2\prod_{i=1}^{r}(x_i+1)^2\geq 3\prod_{i=1}^{r}x_i^2+2\prod_{i=1}^{r}x_i+3, \tag{26}$$

and

$$\prod_{i=1}^{r}(x_i+1)^2+2\prod_{i=1}^{r}(x_i^2-1)\geq 3\prod_{i=1}^{r}x_i^2+2\prod_{i=1}^{r}x_i-1. \tag{27}$$

Let $x_i=\frac{p_i}{2}\geq 1$. After simple computations, we get the inequalities (24), respectively (25).

Generalizations of (26) and (27) can be found in [12].

Thus, relations (26) and (27) can be generalized, too. □

Theorem 12. *For $n, k\geq 2$, one has*

$$D_2(n)<n\cdot\frac{\sigma(n)}{\varphi(n)}, \tag{28}$$

and

$$\frac{D_k(n)}{n}<\frac{\sigma(n)}{n}\cdot\Big(\frac{n}{\varphi(n)}\Big)^{k-1}, \tag{29}$$

where $\varphi(n)$ is Euler's totient function.

Proof. Inequality (28) can be rewritten as

$$\frac{D_2(n)}{n}<\frac{\sigma(n)}{n}\cdot\frac{n}{\varphi(n)}=A(n). \tag{30}$$

As each term of (30) is a multiplicative function, it will be sufficient to prove the inequality for prime powers $n=p^a$. One has $\frac{D_2(p^a)}{p^a}=\frac{p+2}{p}$, $A(p^a)=\Big(1+\frac{1}{p}+\cdots+\frac{1}{p^a}\Big)\frac{p}{p-1}$. For $a=1$, the inequality is true, as $\frac{p+2}{p}<\Big(1+\frac{1}{p}\Big)\cdot\frac{p}{p-1}=\frac{p+1}{p-1}$ becomes $(p+2)(p-1)<p(p+1)$, or $p^2+p-2<p^2+p$.

For $a>1$ remark that $A(p^a)>A(p^1)$, as a increases, $A(p^a)$ increases, too. Thus, one has $D_2(p^a)/p^a=D_2(p)/p<A(p^1)<A(p^a)$, so the inequality follows.

The proof of (29) is similar, but for $n=p$ we have to show that for $k=2$:

$$(p+2)(p-1)=p(p+1)-2<p(p+1),$$

while the inequality

$$(p+k)\cdot(p-1)^{k-1}<p^{k-1}\cdot(p+1). \tag{31}$$

follows by an induction upon $k\geq 2$, and the result follows.

It is known (see, e.g., [13]) that $\frac{\sigma(n)}{n}=\mathcal{O}(\log\log n)$ and $\frac{n}{\varphi(n)}=\mathcal{O}(\log\log n)$, so by (30) one has

$$\frac{D_2(n)}{n}=\mathcal{O}((\log\log n)^2). \tag{32}$$

□

The more precise result is contained in the next Theorem 13.

Theorem 13. *One has*

$$\limsup_{n\to\infty} \frac{D_2(n)}{n(\log\log n)^2} = Ce^{2\gamma}, \tag{33}$$

where γ is the Euler constant, and $C = \prod_p \left(1 - \frac{3}{p^2} + \frac{2}{p^3}\right) \approx 0.28675\ldots$.

Proof. The expression

$$\frac{D_2(n)}{n} = \prod_{p|n} \Big(1 + \frac{2}{p}\Big)$$

takes the greatest value when the primes p are the least, so when

$$n = n_x = \prod_{p\le x} p.$$

Then

$$\frac{D_2(n_x)}{n_x} = \prod_{p\le x} \Big(1 + \frac{2}{p}\Big).$$

Remarking that

$$\Big(1 + \frac{2}{p}\Big)\Big(1 - \frac{1}{p}\Big)^2 = \Big(1 + \frac{2}{p}\Big)\Big(1 - \frac{2}{p} + \frac{1}{p^2}\Big) = 1 - \frac{3}{p^2} + \frac{2}{p^3},$$

we get

$$1 + \frac{2}{p} = \Big(1 - \frac{1}{p}\Big)^{-2}\Big(1 - \frac{3}{p^2} + \frac{2}{p^3}\Big),$$

so

$$\prod_{p\le x} \Big(1 + \frac{2}{p}\Big) = \Big[\prod_{p\le x} \Big(1 - \frac{1}{p}\Big)^{-1}\Big]^2 \prod_{p\le x} \Big(1 - \frac{3}{p^2} + \frac{2}{p^3}\Big).$$

The product $\prod_{p\le x} \left(1 - \frac{3}{p^2} + \frac{2}{p^3}\right)$ converges absolutely to a constant

$$C = \prod_{p \text{ prime}} \Big(1 - \frac{3}{p^2} + \frac{2}{p^3}\Big) \approx 0.2867\ldots$$

(with a computer computation). By Mertens' theorem, one has

$$\prod_{p\le x} \Big(1 - \frac{1}{p}\Big)^{-1} \sim e^{\gamma} \log x$$

as $x \to \infty$ (see [7]). Thus from the above we get

$$\frac{D_2(n_x)}{n_x} \sim Ce^{2\gamma}(\log x)^2.$$

Now,

$$\log n = \log\Big(\prod_{p\le x} p\Big) = \sum_{p\le x} \log p = \Theta(x),$$

Chebyshev's function. By the prime number theorem, one has $\Theta(x) \sim x$, so $\log n \sim x$. This gives $\log\log n \sim \log x$, and (33) follows.

Numerically, $C \approx 0.28674\ldots$ and $e^{2\gamma} \approx 3.17221\ldots$, so $Ce^{2\gamma} \approx 0.9096\ldots$. □

A similar method can be used to show that

$$\limsup_{n\to\infty} \frac{D_k(n)}{n(\log\log n)^k} = C_k \cdot e^{k\gamma},$$

where

$$C_k = \prod_p \Big(1+\frac{k}{p}\Big)\Big(1-\frac{1}{p}\Big)^k.$$

Theorem 14. *For $n \geq 3$ odd, one has*

$$D_2(n) \leq n \cdot \Big(\gamma(n)\Big)^{\lambda}, \tag{34}$$

where

$$\lambda = \frac{\log 5}{\log 3} - 1. \tag{35}$$

Proof. Let $f(x) = \frac{x+2}{x^{1+\lambda}}, x > 0$. Then

$$f'(x) = \frac{x^\lambda \cdot [x-(1+\lambda)(x+2)]}{x^{2+2\lambda}} = -\frac{1}{x^{2+\lambda}} \cdot [\lambda x + 2(1+\lambda)] < 0$$

for $\lambda > 0, x > 0$. Thus the function $f(x)$ is strictly decreasing, implying $f(p) \leq f(3)$ for $p \geq 3$ a prime. Thus $\frac{p+2}{p^{1+\lambda}} \leq \frac{5}{3^{1+\lambda}} = 1$ by (35). Thus, we can write the inequality

$$1 + \frac{2}{p} \leq p^\lambda. \tag{36}$$

By taking all primes $p|n$, we get from (36)

$$\frac{D_2(n)}{n} \leq (\gamma(n))^\lambda,$$

and the result follows. □

Remark 1. *The constant λ is transcendental, as can be shown by a method used, e.g., in [10].*

Remark 2. *By considering the function $g(x) = \frac{x-2}{x^{1-\mu}}$, we can show similarly that if $n \geq 3$ is odd and n is not divisible by 3, then*

$$\frac{S_2(n)}{n} \geq \frac{1}{(\gamma(n))^\mu}, \tag{37}$$

where

$$\mu = \frac{\log(\frac{5}{3})}{\log 5} \tag{38}$$

Theorem 15. *One has*

$$\sum_{n\leq x} \frac{D_2(n)}{n} = Cx + \mathcal{O}(\log^2 x), \tag{39}$$

$$\sum_{n\leq x} D_2(n) = \frac{C}{2}x^2 + \mathcal{O}(x\log^2 x), \tag{40}$$

where $C = \prod_{p \text{ prime}} \Big(1+\frac{2}{p^2}\Big) \approx 2.1908\ldots$.

Proof. Let

$$f(n) = \frac{D_2(n)}{n} = \prod_{p|n} \Big(1 + \frac{2}{p}\Big).$$

Then first we will determine by Dirichlet convolution the function $g(n)$ given by

$$f(n) = (1 * g)(n) = \sum_{d|n} g(d).$$

As $f(p^k) = 1 + \frac{2}{p}$ and $f(p) = g(1) + g(p)$, we get $g(p) = \frac{2}{p}$; $f(p^2) = 1 + g(p) + g(p^2)$ from where it follows that $g(p^2) = 0$. Thus $g(n)$ is only non-zero, when n is squarefree, i.e.,

$$g(d) = \frac{\mu^2(d) \cdot 2^{\omega(d)}}{d}$$

for all d.

One has, by standard methods,

$$\sum_{n\le x} f(n) = \sum_{n\le x} \sum_{d|n} g(d) = \sum_{d\le x} g(d) \Big[\frac{x}{d}\Big] = x \sum_{d\ge 1} \frac{g(d)}{d} - x \sum_{d>x} \frac{g(d)}{d} - \sum_{d\le x} g(d) \{\frac{x}{d}\},$$

where $\Big\{\frac{x}{d}\Big\}$ is the fractional part of $\frac{x}{d}$.

Now

$$\sum_{d\le x} g(d) \Big\{\frac{x}{d}\Big\} \ll \sum_{d\le x} (g(d)) \le \sum_{d\le x} \frac{2^{\omega(d)}}{d}.$$

A well-known result of number theory states that (see, e.g. [13])

$$A(x) = \sum_{d\le x} 2^{\omega(d)} \ll x \log x. \tag{41}$$

Now, by Abel's summation formula

$$\sum_{d\le x} \frac{2^{\omega(d)}}{d} = \frac{A(x)}{x} + \int_1^x \frac{A(t)}{t^2}\,\mathrm{d}t \ll \frac{x \log x}{x} + \int_1^x \frac{t \log t}{t^2}\,\mathrm{d}t \ll \log x + \frac{1}{2}\log^2 x \ll \log^2 x.$$

On the other hand,

$$x \sum_{d>x} \frac{g(d)}{d} \ll x \cdot \sum_{d>x} \frac{2^{\omega(d)}}{d^2},$$

and by Abel's summation formula again we get

$$\sum_{d>x} \frac{2^{\omega(d)}}{d^2} = -\frac{A(x)}{x^2} + 2\int_x^\infty \frac{A(t)}{t^3}\,\mathrm{d}t \ll \frac{x \lg x}{x^2} + \int_x^\infty \frac{\log t}{t^2}\,\mathrm{d}t.$$

As

$$\int_x^\infty \frac{\log t}{t^2}\,\mathrm{d}t = \frac{\log x}{x} + \frac{1}{x} \ll \frac{\log x}{x},$$

we get finally that

$$\sum_{n\le x} f(n) = Cx + \mathcal{O}(\log^2 x).$$

Now, by a simple Abel summation, we get

$$\sum_{n\le x} nf(n) = \sum_{n\le x} D_2(n) = \frac{C}{2}x^2 + \mathcal{O}(x\log^2 x),$$

so Theorem 15 is proved. □

Remark 3. *It can be proved similarly that*

$$\sum_{n\le x} \frac{D_k(n)}{n} = C_k \cdot x + \mathcal{O}((\log x)^k), \tag{42}$$

and

$$\sum_{n\le x} D_k(n) = \frac{C_k}{2} \cdot x^2 + \mathcal{O}(x(\log x)^k), \tag{43}$$

where C_k is some constant.

3 Conclusion

In the near future, other properties of the new Dedekind arithmetic function will be investigated and its relationship with other arithmetic functions will be discussed.

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