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Recurrence function of the generalized Thue–Morse word

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Abstract: This paper is devoted to the study of the recurrence function of q-ary Thue–Morse word $\mathbf{t}_q$. After giving the structure of singular factors, we describe the maximal return time of factors of $\mathbf{t}_q$. We conclude by determining the recurrence function of generalized Thue–Morse word.

Keywords: Infinite words, Maximal return time, Singular factor, Recurrence function.
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1 Introduction

The recurrence function has been introduced in [5] by Morse and Hedlund where they studied it for the sturmian words. Let u be an infinite word. The recurrence function $R(n)$ of u is the smallest integer r such that any factor of u of length n occurs in any factor of u of length r.

Let us consider the alphabet $\mathcal{A}_q = \{0, 1, \ldots, q-1\}$. The Thue–Morse word over $\mathcal{A}$ is the infinite word $\mathbf{t}_{b,q}$ generated by the morphism $\mu_{b,q}$ defined on $\mathcal{A}_q$ by:

$$\mu_{b,q}(k) = k(k+1)\ldots(k+b-1)$$

where the letters are expressed modulo q.

The recurrence function of the binary Thue–Morse word has been studied in [3,4]. The ternary case has been studied in [7].

Return words play an important role in the determination of the recurrence function. A return word of a factor w in u is a word which begins with an occurrence of w and ends just before the next occurrence of w. This notion was introduced by Fabien Durand [6] to study the fixed points of primitive substitutions.

Furthermore, the singular factors have been introduced by Z-X. Wen and Z-Y. Wen [10]. In [8], the authors studied the singular factors of generalized Thue–Morse word. These factors constitute a useful tool in the generalization of the recurrence function of Thue–Morse word.

In this paper, we are interested in the study of the recurrence function of Thue–Morse word $\mathbf{t}_q(b=q)$ over an alphabet $\mathcal{A}_q$.

Our study is structured as follows. After giving some useful definitions in Section 2, we study some properties of return words and the maximal return time of $\mathbf{t}_q$ in Section 3. In Section 4, we determine the essential and non-essential singular factors of $\mathbf{t}_q$. Finally, we calculate the recurrence function of $\mathbf{t}_q$ in Section 5.

2 Definitions

An alphabet $\mathcal{A}$ is a set of finite symbols called letters. We designate by $\mathcal{A}^*$ the set of finite words on $\mathcal{A}$ and ε is the empty word. For all $u \in \mathcal{A}^*$, $|u|$ denotes the length of u.

An infinite word is a sequence of letters indexed by $\mathbb{N}$. We designate by $\mathcal{A}^\omega$ the set of infinite words on $\mathcal{A}$. A finite word w is a factor of u if there exist some words $u' \in \mathcal{A}^*$, $u'' \in \mathcal{A}^\omega$ such that $u = u'wu''$. If $u' = \varepsilon$, then w is a prefix of u. If $u'' = \varepsilon$, then w is a suffix of u.
An infinite word u is recurrent if any factor of u occurs infinitely many times in u. It is said to be uniformly recurrent if for any integer n there exists an integer n_0 such that any factor of u of length n occurs in any factor of u of length n_0.

Let w be a factor of $u = u_0u_1\ldots$ such that $|w| = l$. An integer j is an occurrence of w in u if $w = u_ju_{j+1}\ldots u_{j+l-1}$.

Let j and k be two successive occurrences of w in u, with $j < k$. The word $u_ju_{j+1}\ldots u_{k-1}$ is a return word of w in u. The set of return words of w in u is denoted $\mathrm{Ret}(w)$.

$$\mathrm{Ret}(w) = \{u_ju_{j+1}\ldots u_{k-1} |\ j,k \text{ two successive occurrences of } w \text{ in } u\}.$$

On $\mathcal{A}_q$ we define the application E_q by:

$$E_q(i) = \begin{cases} i+1, & \text{if } i \in \{0, 1, \ldots, q-2\} \\ 0, & \text{if } i = q-1. \end{cases}$$

A map φ defined on a monoid $\mathcal{A}^*$ is a morphism if for all $u, v \in \mathcal{A}^*$ one has: $\varphi(uv) = \varphi(u)\varphi(v)$.

A morphism is non-erasing if $\varphi(a) \neq \varepsilon$ for all $a \in \mathcal{A}$. It is prolongable on x_0, $x_0 \in \mathcal{A}$ if there exists $w \in \mathcal{A}^+$ such that $\varphi(x_0) = x_0 w$.

Definition 2.1. *A substitution is a non-erasing endomorphism for the concatenation of free monoid $\mathcal{A}^*$.*

Definition 2.2. [9] *Let ij be a factor of $\mathbf{t}_q$. Then ij is a forbidden factor in the image of a letter if $j \neq i+1$.*

Definition 2.3. [4] *The recurrence function of an infinite word u is the function $R_u : \mathbb{N} \to \mathbb{N} \cup \{+\infty\}$ defined by:*

$$R_u(n) = \inf(\{N \in \mathbb{N} : \forall v \in F_N(u), F_n(v) = F_n(u)\} \cup \{+\infty\}).$$

In other words, $R_u(n)$, if it exists, is the smallest integer r such that any factor of u of length r contains any factor of u of length n.

We write $R(n)$ instead of $R_u(n)$ if there is no confusion.

Let w be a factor of u. We call maximal return time of w in u the return word of w in u of maximal length. We denote it by $l(w)$.

Definition 2.4. [4] *Let u be a uniformly recurrent word and $w \in F(u)$. Then*

- $l(w) = \max\{|v| : v \in \mathrm{Ret}(w)\}$.
- *For all $n \in \mathbb{N}$, $L(n)$ denotes the maximal return time of the factors of length n of u given by:*

$$L(n) = \max\{l(w) : w \in F_n(u)\}.$$

Proposition 2.5. [4] *For any uniformly recurrent infinite word u and for any $n \in \mathbb{N}$, one has:*

$$R(n) = L(n) + n - 1.$$

3 Some properties of $L(n)$

The return words are used in the study of maximal return time of a word. So, we start by giving the properties of return words of $\mathbf{t}_q$. Then, we study its maximal return time.

3.1 Properties of return words of $\mathbf{t}_q$

Definition 3.1. [2] *Let w be a factor of a fixed point u of a substitution φ. The word $v_0v_1v_2\dots v_n \in F_{n+1}(u)$ is said to be an ancestor of w if:*

i) w is a factor of $\varphi(v_0v_1v_2\dots v_n)$,

ii) w is neither a factor of $\varphi(v_1v_2\dots v_n)$, nor $\varphi(v_0v_1v_2\dots v_{n-1})$.

Proposition 3.2. [8] *Let w be a factor of $\mathbf{t}_q$ and v an ancestor of w. Then, $\mu_q(v) = \delta_1 w \delta_2$ with $|\delta_1|, |\delta_2| \leq q-1$.*

Remark 3.3. *If w is bispecial and $|w| \geq 2q$, then w admits a unique ancestor.*

Proposition 3.4. [8] *Any nonbispecial factor of $\mathbf{t}_q$ admits a unique ancestor.*

Corollary 3.5. *If v is a factor of $\mathbf{t}_q$ of length at least 2, then v is the unique ancestor of $\mu_q(v)$.*

Proof. Let v be an ancestor of $\mu_q(v)$ of length at least 2. Then $|\mu_q(v)| = q|v|$. As $|v| \geq 2$, then $|\mu_q(v)| \geq 2q$. By Remark 3.3, $\mu_q(v)$ decomposes uniquely as a concatenation. So, v is the unique ancestor of $\mu_q(v)$. □

Lemma 3.6. *Let u be a factor of $\mathbf{t}_q$. Then, there exist factors v, δ_1 and δ_2 of $\mathbf{t}_q$ such that $u = \delta_1\mu_q(v)\delta_2$ with $|\delta_1|, |\delta_2| \leq q-1$.*

Proposition 3.7. [2] *Let u be an infinite word and w be a factor of u. If w has a unique right extension $a \in \mathcal{A}$, then $\mathrm{Ret}(wa) = \mathrm{Ret}(w)$. If w has a unique left extension $b \in \mathcal{A}$, then $\mathrm{Ret}(bw) = b\mathrm{Ret}(w)b^{-1} = \{bvb^{-1} | v \in \mathrm{Ret}(w)\}$, where bvb^{-1} is the word obtained by extending v on left by b and by erasing b at right of v.*

Remark 3.8. *Let w be a factor of $\mathbf{t}_q$.*

- *If w has a unique right extension $a \in \mathcal{A}_q$, then $l(wa) = l(w)$.*
- *If w has a unique left extension $b \in \mathcal{A}_q$, then $l(bw) = l(w)$.*

Proposition 3.9. [2] *Let u be a fixed point of a primitive substitution φ and w a factor of u. If the unique ancestor of $\varphi(w)$ is the factor w, then*

$$\mathrm{Ret}(\varphi(w)) = \varphi(\mathrm{Ret}(w)).$$

3.2 Maximal return time

Recall that a bispecial factor of $\mathbf{t}_q$ of minimal length is a bispecial factor of $\mathbf{t}_q$ of length at most $2q-1$. The set bispecial q-prolongable factor of minimal length is given by

$$BSQ(\mathbf{t}_q) = \{w_k^{(i)} : 1 \leq k \leq q+1, i \in \mathcal{A}_q\},$$

where $w_k^{(i)} = i(i+1)\dots(i+k-2)$ and the set of bispecial biprolongable factors of minimal length is given by

$$BSB(\mathbf{t}_q) = \{v_k^{(i)} : 2 \leq k \leq q, i \in \mathcal{A}_q\},$$

where $v_k^{(i)} = \mu_q(i)i(i+1)\dots(i+k-2)$.

3.2.1 Algorithm for calculating the maximal return time

Definition 3.10. *Let w be a bispecial factor of $\mathbf{t}_q$ of minimal length. The ancestors tree of w is a graph which begins with an ancestor of w called the root and where the end of every path is also an ancestor of w.*

For a better understanding of ancestor tree see [**?**]. We simply adapt the description of their tree to ours. The following algorithm allows us to compute the maximal return time of a bispecial factor of $\mathbf{t}_q$ of minimal length. In this algorithm, $d^+(v)$ denotes the number of extensions of the factor v in $\mathbf{t}_q$.

Description of the algorithm

To find the maximal return time of w one proceeds as follows:

(i) List the set of ancestors of w.

(ii) Find two ancestors v_1 and v_2 of w such that w is a prefix of $\mu_q(v_1)$ and a suffix of $\mu_q(v_2)$.

(iii) Let W be the longest factor of $\mathbf{t}_q$ which admits v_1 as a prefix and v_2 as a suffix and $d^+(v_1)$ be the number of extensions of v_1 in $\mathbf{t}_q$. W is determined as follows:

- If w is a bispecial q-prolongable factor, we choose v_1 as a root of ancestors tree of w. Then v_1 will be associated to $d^+(v_1)$ nodes. Among these $d^+(v_1)$ nodes, only those which are not ancestors of w will be extended. We repeat this process recursively with these nodes until reaching only ancestors of w and the algorithm stops. The last ancestor found at the end of the longest path is v_2, and W is the longest path from v_1 to v_2.
- If w is a bispecial biprolongable factor, then $w = v_k^{(i)}, k \in [\![2, q]\!]$ and the set of ancestors of $v_k^{(i)}$ is given by

$$\mathrm{Anc}(v_k^{(i)}) = \{jj : j \in \mathcal{A}_q \backslash \{i+1, i+2, \ldots, i+k-2\}\}.$$

  Thus, $v_1 = ii$ and $v_2 = (i+k-1)(i+k-1)$ because $v_k^{(i)}$ is both a prefix of $\mu_q(ii)$ and a suffix of $\mu_q((i+k-1)(i+k-1))$. Then, we construct the longest factor admitting i as a prefix and $(i+k-2)$ as a suffix and that does not contain any occurrence of a letter $j \in \{i\} \cup [\![i+k-1, q]\!]$ denoted $u_k^{(i)}$. Finally, $W = i\mu_q^{q-1}((u_k^{(i)})^2)(i+k-1)$.

(iv) Apply μ_q to word obtained.

(v) Erase the suffix of $\mu_q(W)$ that is a prefix of w and calculate the length of new word obtained.

3.2.2 Maximal return time of bispecial factor of $\mathbf{t}_q$

Lemma 3.11. *If v is the unique ancestor of w of length greater than or equal to 2, then $l(w) = ql(v)$.*

Proof. Let v be the unique ancestor of w of length greater than or equal to 2. We distinguish two cases.

Case 1: w is bispecial. Then $w = \mu_q(v)$ and $\mathrm{Ret}(w) = \mu_q(\mathrm{Ret}(v))$ by Proposition 3.9. As a result:

$$\begin{aligned} l(w) &= \max\{|v|, v \in \mathrm{Ret}(w)\} \\ &= \max\{|v|, v \in \mu_q(\mathrm{Ret}(v))\} \\ &= \max\{|\mu_q(v_1)|, v_1 \in \mathrm{Ret}(v)\} \\ &= q\max\{|v_1|, v_1 \in \mathrm{Ret}(v)\} \\ &= ql(v). \end{aligned}$$

Case 2: w is nonbispecial. As v is the unique ancestor of w of length at least 2, then there exist factors δ_1, δ_2 of $\mathbf{t}_q$ such that $\mu_q(v) = \delta_1 w \delta_2$. Therefore $l(\delta_1 w \delta_2) = l(\mu_q(v))$. By proceeding similarly as in Case 1, we show that $l(\mu_q(v)) = ql(v)$. Furthermore, by Remark 3.8, $l(\delta_1 w \delta_2) = l(w)$. Hence, we have $l(w) = ql(v)$. □

The following proposition gives us the maximal return time of bispecial q-prolongable factors of minimal length of $\mathbf{t}_q$.

Proposition 3.12. *Let $k \in [\![2, q+1]\!]$ and $i \in \mathcal{A}_q$. The maximal return time of $w_k^{(i)}$ is given by:*

$$l(w_k^{(i)}) = \begin{cases} 2q-1, & \text{if } k = 2 \\ 3q-2, & \text{if } k = 3 \\ (2q-1)(k-1), & \text{if } k \in [\![4, q+1]\!]. \end{cases}$$

Proof. As $E_q(F_n(\mathbf{t}_q)) = F_n(\mathbf{t}_q)$, then we take $i = 0$ because the others are obtained by successive application of E_q. So, $w_k^{(0)} = 01\ldots(k-2)$. To determine the maximal return time of $w_k^{(0)}$, we distinguish three cases.

Case 1: For $k = 2, w_2^{(0)} = 0$. Observe that the letter 0 occurs in $\mu_q(i)$ for all $i \in \mathcal{A}_q$. Thus, the return words of 0 come from $\mu_q(ij)$ for all $i, j \in \mathcal{A}_q$. Therefore, the return word of 0 of maximal length is $\mu_q(01)0^{-1}$ because 0 is a prefix of $\mu_q(0)$ and a suffix of $\mu_q(1)$. It follows that $l(0) = |\mu_q(01)0^{-1}| = 2q-1$.

Case 2: For $k = 3$, $w_3^{(0)} = 01$. The set of ancestors of 01 is given by

$$\mathrm{Anc}(w_3^{(0)}) = \{i : i \in \mathcal{A}_q \backslash \{1\}\} \cup \{11\}.$$

As $w_3^{(0)}$ is a prefix of $\mu_q(0)$ and a suffix of $\mu_q(2)$, then the maximal return time of 01 comes from $\mu_q(012)$ because 012 is the factor of $\mathbf{t}_q$ of maximal length whose image under μ_q contains 01 both as a prefix and as a suffix. Therefore,

$$l(01) = |\mu_q(012)(01)^{-1}| = 3q-2.$$

Case 3: $k \in [\![4, q+1]\!]$. One has $w_k^{(0)} = 01\ldots(k-2)$. Observe that the set of ancestors of $w_k^{(0)}$ is given by

$$\mathrm{Anc}(w_k^{(0)}) = \{i : i \in \mathcal{A}_q \backslash \{1, \ldots, k-2\}\} \cup \{ii : 1 \leq i \leq k-2\}.$$

As $w_k^{(0)}$ is a prefix of $\mu_q(0)$ and a suffix of $\mu_q(k-1)$, then the return word of $w_k^{(0)}$ of maximal length comes from $\mu_q(01\dots(k-2)12\dots(k-1))$ since $01\dots(k-2)12\dots(k-1)$ is the factor of $\mathbf{t}_q$ of maximal length whose image under μ_q admits $w_k^{(0)}$ as a prefix and as a suffix. Then the return word of $w_k^{(0)}$ coming from $\mu_q(01\dots(k-2)12\dots(k-1))$ is $\mu_q(01\dots(k-2)12\dots(k-1))(w_k^{(0)})^{-1}$. Thus, we have the following equalities:

$$\begin{aligned} l(w_k^{(0)}) &= |\mu_q(01\dots(k-2)12\dots(k-1))(w_k^{(0)})^{-1}| \\ &= 2(k-1)|\mu_q(0)| - (k-1) \\ &= (k-1)(2|\mu_q(0)| - 1) \\ &= (k-1)(2q-1). \end{aligned}$$

So, $l(w_k^{(0)}) = (2q-1)(k-1)$. □

Remark 3.13. *$l(0) = 2q-1$ implies that $L(1) = 2q-1$.*

In the following, we give the maximal return time of bispecial q-prolongable factors of $\mathbf{t}_q$. Let us put $v_k^{(i)} = \mu_q(i)i(i+1)\dots(i+k-2), 2 \leq k \leq q$. Recall the following corollary established in [9].

Corollary 3.14. *Let w be a separator factor of $\mathbf{t}_q$. If w separates ii and jj, then $j = i$ or $j = E_q^k(i)$, $k \in [\![1, q-1]\!]\backslash\{2\}$.*

(i) *If $|w| = q^{q-1} - 2$, then w separates two distinct squares. The separator factors of length $q^{q-1}-2$ are given by the set: $S = \{i^{-1}\mu_q^{q-1}(i)j^{-1} : i, j \in \mathcal{A}_q$ and $j = E_q(i)\}$.*

(ii) *If $|w| = 2(q^{q-1}-1)$, then w separates either two identical squares or two distinct squares.*

- *The separator factors of length $2(q^{q-1}-1)$ which separate two identical squares are given by the set: $S' = \{i^{-1}\mu_q^{q-1}(im)i^{-1} : i \in \mathcal{A}_q$ and $m = E_q^{q-1}(i)\}$.*
- *The separator factors of length $2(q^{q-1}-1)$ which separate two distinct squares are given by the set:*

$$\begin{aligned} S'' = (\bigcup_{k=2}^{q-2} \quad & \{ \quad i^{-1}\mu_q^{q-1}(im)j^{-1} : i \in \mathcal{A}_q, m = E_q^k(i) \text{ and } j = E_q(m)\}) \\ & \bigcup \quad \{i^{-1}\mu_q^{q-1}(ii)E_q(i)^{-1} : i \in \mathcal{A}_q\}. \end{aligned}$$

Remark 3.15. *If $w = i^{-1}\mu_q^{q-1}(im)j^{-1}$ is a separator factor of length $2(q^{q-1}-1)$, then im is a forbidden factor in the image of a letter.*

Proposition 3.16. *[9] The map $E_q^{(q-1)k}$ applied to a letter i yields the last letter of $\mu_q^k(i)$ for all $k \in [\![1, q-1]\!]$ and for all $i \in \mathcal{A}_q$.*

Remark 3.17. *For all $i \in \mathcal{A}_q$, the last letter of $\mu_q^{q-1}(i)$ is*

$$E_q^{(q-1)^2}(i) = i + (q-1)^2 \mod q = i+1 \mod q.$$

The following two Lemmas are established in [9].

Lemma 3.18. *For all $i \in \mathcal{A}_q$, the factor $\mu_q^{q-1}(i)$ does not contain a square of letters.*

Lemma 3.19. *Let $i, j \in \mathcal{A}_q$. If $j \neq i+1$, then the factor $\mu_q^{q-1}(ij)$ does not contain a square of letters.*

Lemma 3.20. *Let $i, j \in \mathcal{A}_q$. Then, $\mu_q^{q-1}(ij)$ contains a square if and only if $j = i+1$.*

Proof. Assume that $j = i+1$ and show that $\mu_q^{q-1}(ij)$ contains a square.

One has $\mu_q^{q-1}(ij) = \mu_q^{q-1}(i)\mu_q^{q-1}(j)$. By Remark 3.17, the last letter of $\mu_q^{q-1}(i)$ is $i+1 = j$. Therefore, $\mu_q^{q-1}(ij) = \mu_q^{q-1}(i)j^{-1}jjj^{-1}\mu_q^{q-1}(j)$. Then, $\mu_q^{q-1}(ij)$ contains a square of letters.

The converse follows immediately from the contraposition of Lemma 3.19. □

Corollary 3.21. *Let $i, j, k, l \in \mathcal{A}_q$.*

(i) *If $j = i+1$ and $k = j+1$, then $\mu_q^{q-1}(ijk)$ contains the squares jj and kk. Furthermore, the separator factor which separates jj and kk is of length $q^{q-1} - 2$.*

(ii) *If $j \neq i+1$ and $k = i-1$ and $l = j+1$, then $\mu_q^{q-1}(kijl)$ contains the squares ii and ll. Furthermore, the separator factor which separates ii and jj is of length $2(q^{q-1} - 1)$.*

Proof. (i) Assume that $j = i+1$ and $k = j+1$. One has:

$$\mu_q^{q-1}(ijk) = \mu_q^{q-1}(i)\mu_q^{q-1}(j)\mu_q^{q-1}(k) = \mu_q^{q-1}(i)j^{-1}jjj^{-1}\mu_q^{q-1}(j)k^{-1}kkk^{-1}\mu_q^{q-1}(k)$$

because by Remark 3.17, the last letter of $\mu_q^{q-1}(i)$ is $i+1 = j$ and the one of $\mu_q^{q-1}(j)$ is $j+1 = k$. Thus, the squares jj and kk occur in $\mu_q^{q-1}(ijk)$. Furthermore, $|j^{-1}\mu_q^{q-1}(j)k^{-1}| = q^{q-1} - 2$.

(ii) Assume that $j \neq i+1$ and $l = j+1$. As $\mu_q^{q-1}(ij)$ does not contain a square (because $j \neq i+1$), then $\mu_q^{q-1}(kijl)$ can be written as:

$$\mu_q^{q-1}(kijl) = \mu_q^{q-1}(k)\mu_q^{q-1}(ij)\mu_q^{q-1}(l) = \mu_q^{q-1}(k)i^{-1}iii^{-1}\mu_q^{q-1}(ij)l^{-1}lll^{-1}\mu_q^{q-1}(l)$$

because by Remark 3.17, the last letter of $\mu_q^{q-1}(k)$ is $k+1 = i$ and the one of $\mu_q^{q-1}(ij)$ is $j+1 = l$. Thus, $\mu_q^{q-1}(kijl)$ contains the squares ii and ll and the separator factor which separates them is of length $|i^{-1}\mu_q^{q-1}(ij)l^{-1}| = 2(q^{q-1} - 1)$. □

Proposition 3.22. *Let $k \in [\![4, q]\!]$ and*

$$W_k = 00s_{01}11s_{12}22\ldots(k-2)(k-2)s_{(k-2)1}11s_{12}22\ldots(k-2)(k-2)s_{(k-2)(k-1)}(k-1)(k-1),$$

where $(s_{ij})_{0\leq i\leq k-2, j=E_q(i)}$ are the separator factors of $\mathbf{t}_q$. Then W_k is a factor of $\mathbf{t}_q$.

Proof. Consider the factor $\theta_k = (q-1)012\ldots(k-2)01\ldots(k-2)(k-1)$. Let us show that θ_k is a factor of $\mathbf{t}_q$. Observe that $(k-2)0$ is a forbidden factor in the image of a letter. Otherwise, $0 = k-1$ and $k = 1$, which is absurd because $k \in [\![4, q]\!]$.

Moreover, $|(q-1)012\dots(k-2)| = |01\dots(k-2)(k-1)| = k$. If $k = q$, then

$$(q-1)012\dots(q-2)01\dots(k-2)(q-1) = \mu_q((q-1))\mu_q(0) = \mu_q((q-1)0).$$

As $(q-1)0$ is factor of $\mathbf{t}_q$, then $(q-1)012\dots(q-2)01\dots(k-2)(q-1)$ is also a factor of $\mathbf{t}_q$. If $k \in [\![4, q-1]\!]$, then $(q-1)012\dots(k-2)$ (respectively, $01\dots(k-2)(k-1)$) is a proper suffix of $\mu_q(k-1)$ (respectively, proper prefix of $\mu_q(0)$).

As a result, $(q-1)012\dots(k-2)01\dots(k-2)(k-1)$ is a factor of $\mu_q((k-1)0)$. Since $(k-1)0$ is a factor of $\mathbf{t}_q$, then $\theta_k = (q-1)012\dots(k-2)01\dots(k-2)(k-1)$ is also a factor of $\mathbf{t}_q$. Now, let us show that W_k is a factor of $\mathbf{t}_q$. By applying μ_q^{q-1} to $(q-1)012\dots(k-2)01\dots(k-2)(k-1)$, it follows that

$$\mu_q^{q-1}(\theta_k) \;=\; \mu_q^{q-1}(q-1)\mu_q^{q-1}(0)\mu_q^{q-1}(1)\dots\mu_q^{q-1}((k-2)0)\mu_q^{q-1}(1)\mu_q^{q-1}(2)\dots\mu_q^{q-1}(k-1).$$

$F_k = \mu_q^{q-1}(\theta_k)$. As θ_k is a factor of $\mathbf{t}_q$, then F_k is also a factor of $\mathbf{t}_q$. It follows from Remark 3.17 and Corollary 3.21 that F_k can be written as

$$\begin{aligned} F_k = {} & \mu_q^{q-1}(q-1)0^{-1}000^{-1}\mu_q^{q-1}(0)1^{-1}111^{-1}\mu_q^{q-1}(1)2^{-1}222^{-1}\mu_q^{q-1}(2)\dots\mu_q^{q-1}((k-2)0)1^{-1} \\ & 111^{-1}\mu_q^{q-1}(1)2^{-1}222^{-1}\mu_q^{q-1}(2)3^{-1}\dots\mu_q^{q-1}(k-1)(k-1)^{-1}\mu_q^{q-1}(k-1) \end{aligned}$$

because for all $i \in [\![0, k-2]\!]$, the last letter of $\mu_q^{q-1}(i)$ is $i+1$ and $0 \neq k-1$ ($(k-2)0$ is a forbidden factor in the image of a letter). Setting

$$s_{ij} = \begin{cases} (k-2)^{-1}\mu_q^{q-1}((k-2)0)1^{-1}, & \text{if } (i,j) = (k-2,1), \\ i^{-1}\mu_q^{q-1}(i)j^{-1}, & \text{if } 0 \leq i \leq k-2 \text{ and } j = E_q(i), \end{cases}$$

F_k can be written as

$$\begin{aligned} F_k \;&=\; \mu_q^{q-1}(q-1)0^{-1}00s_{01}11s_{12}22\dots(k-2)(k-2)s_{(k-2)1}11s_{12}22\dots(k-2)(k-2) \\ &\quad\; s_{(k-2)(k-1)}(k-1)(k-1) \\ &=\; \mu_q^{q-1}(q-1)0^{-1}W_k. \end{aligned}$$

Since F_k is a factor of $\mathbf{t}_q$ and W_k is a suffix of F_k, then W_k is a factor of $\mathbf{t}_q$. □

Let us recall the following useful Theorem.

Theorem 3.23. [1] *Let $q \geq 2$ and $b \geq 1$ be integers. The sequence $\mathbf{t}_{b,q}$ is overlap-free if and only if $q \geq b$.*

Remark 3.24. *It follows from the previous Theorem that the sequence $\mathbf{t}_q$ is overlap-free.*

Proposition 3.25. *For all $k \in [\![2, q]\!]$, the maximal return time of $v_k^{(i)}$ is given by:*

$$l(v_k^{(i)}) = \begin{cases} (2q^{q-1}+1)q - 1, & \text{if } k = 2 \\ q(4q^{q-1}+1) - 2, & \text{if } k = 3 \\ (2q^q - 1)k + q(-2q^{q-1}+1) + 1, & \text{if } k \in [\![4, q]\!]. \end{cases}$$

Proof. As $E_q(F_n(\mathbf{t}_q)) = F_n(\mathbf{t}_q)$, without loss of generality, we take $i = 0$. The cases of the other letters are obtained by successive application of E_q. So, to determine the maximal return time of $\mu_q(0)01\dots(k-2)$, we distinguish three cases.

Case 1: $k = 2, v_2^{(0)} = \mu_q(0)0$. Observe that the set of ancestors of $v_2^{(0)}$ is given by: $\mathrm{Anc}(v_2^{(0)}) = \{ii : i \in \mathcal{A}_q\}$. So, the return words of $v_2^{(0)}$ come from $\mu_q(iiw_{ii}ii), \mu_q(iiv_{ij}jj), \mu_q(iiw_{ij}jj)$. Therefore, the return word of $v_2^{(0)}$ of maximal length comes from $\mu_q(00w_{01}11)$ because $00w_{01}11$ is the factor of $\mathbf{t}_q$ whose image under μ_q admits $v_2^{(0)}$ as both a prefix and a suffix and does not contain any occurrence of 00 and 11. Then, $l(v_2^{(0)}) = |\mu_q(00w1)0^{-1}| = (2q^{q-1}+1)q - 1$.

Case 2: $k=3, v_3^{(0)}=\mu_q(0)01$. Observe that the set of ancestors of $\mu_q(0)01$ is given by $\mathrm{Anc}(v_3^{(0)})= \{ii : i \in \mathcal{A}_q\backslash\{1\}\}$. Observe that $v_3^{(0)}$ is both a prefix of $\mu_q(00)$ and a suffix of $\mu_q(22)$. Thus, the return word of $v_3^{(0)}$ of maximal length comes from $\mu_q(00v_{01}11w_{11}11v_{12}22)$. Therefore,

$$\begin{aligned} l(v_3^{(0)}) &= |\mu_q(00v_{01}11w_{11}11v_{12}2)(01)^{-1}| \\ &= (2(q^{q-1}-2)+2(q^{q-1}-1)+7)q-2 \\ &= (4q^{q-1}+1)q-2 \end{aligned}$$

Case 3: $k \in [\![4, q]\!]$. Observe that the set of ancestors of $v_k^{(0)}$ is given by

$$\begin{aligned} \mathrm{Anc}(v_k^{(0)}) &= \{ii : i \in \mathcal{A}_q\backslash\{1,2,\dots,(k-2)\}\} \\ &= \{00,(k-1)(k-1),kk,\dots,(q-1)(q-1)\}. \end{aligned}$$

Note that $v_k^{(0)}$ is a prefix of $\mu_q(00)$ and a suffix of $\mu_q((k-1)(k-1))$. Let us construct the longest factor which contains 00 as a prefix and $(k-1)(k-1)$ as a suffix and does not contain other ancestor of $v_k^{(0)}$.
By the proof of Proposition 3.22, $F_k = \mu_q^{q-1}(q-1)0^{-1}W_k$, which implies that $W_k = (\mu_q^{q-1}(q-1))^{-1}0F_k$. Thus, one has

$$\begin{aligned} W_k &= (\mu_q^{q-1}(q-1))^{-1}0F_k \\ &= 0\mu_q^{q-1}(0)\mu_q^{q-1}(1)\mu_q^{q-1}(2)\dots\mu_q^{q-1}((k-2)0)\mu_q^{q-1}(1)\mu_q^{q-1}(2)\dots\mu_q^{q-1}(k-2)(k-1) \\ &= 0\mu_q^{q-1}(012\dots(k-2)012\dots(k-2))(k-1) \\ &= 0\mu_q^{q-1}((w_k^{(0)})^2)(k-1). \end{aligned}$$

Observe that $w_k^{(0)} = 012\dots(k-2)$ is the longest factor of $\mathbf{t}_q$ which admits 0 as a prefix and $k-2$ as a suffix and does not contain others occurences of 0 and $k-2$. The sequence $\mathbf{t}_q$ does not contain a cube as factor otherwise it would not be overlap-free, which contradicts Theorem 3.23. Consequently, $(w_k^{(0)})^2$ is the longest factor of $\mathbf{t}_q$ which admits 0 as a prefix and $k-2$ as a suffix.
Therefore, $W_k = 0\mu_q^{q-1}((w_k^{(0)})^2)(k-1)$ is the longest factor which admits 00 as a prefix and $(k-2)(k-2)$ as a suffix.

As a result, the return word of $v_k^{(0)}$ of maximal length is $\mu_q(W_k)(v_k^{(0)})^{-1}$ since $v_k^{(0)}$ is a suffix of $\mu_q(k-1)$ and a prefix of $\mu_q(00)$ of length $k-1$. So we have:

$$\begin{aligned} l(v_k^{(0)}) &= \left|\mu_q(W_k)(v_k^{(0)})^{-1}\right| \\ &= \left|\mu_q(0\mu_q^{q-1}((w_k^{(0)})^2)(k-1))(v_k^{(0)})^{-1}\right| \\ &= [2(k-1)q^{q-1}+2]q-(k+q-1)], \text{ because } |(w_k^{(0)})^2| = 2(k-1) \\ &= (2q^q-1)k+q(-2q^{q-1}+1)+1. \end{aligned}$$

□

Lemma 3.26. *For all $i \in \mathcal{A}_q$ and for all $q \geq 2$ one has:*

$$l(ii) = (3q-2)q^{q-1}.$$

Proof. Observe that the ancestor of ii is $(i+1)i$. By Lemma 3.11, $l(ii) = q \times l((i+1)i)$. Similarly, the ancestor of $(i+1)i$ is $(i+2)i$. As a result, $l((i+1)i) = q \times l((i+2)i)$. Thus, $l(ii) = q^2 \times l((i+2)i)$. Successively, we show that $\forall k \in [\![1, q-1]\!]$,

$$l(ii) = q^k \times l((i+k)i). \tag{1}$$

In particular, for $k = q-1$, $l(ii) = q^{q-1}l((i+q-1)i)$. But $(i+q-1)i = E_q^{q-1}(i(i+1))$. Then,

$$l(ii) = q^{q-1}l(i(i+1)). \tag{2}$$

By Proposition 3.12, $l(i(i+1)) = l(w_2^{(i)}) = 3q-2$. From (1) we deduce that

$$l(ii) = (3q-2)q^{q-1}.$$

□

Remark 3.27. *Observe that $F_2(\mathbf{t}_q) = \{(i+k)i : i \in \mathcal{A}_q, 0 \leq k \leq q-1\}$.*

Proposition 3.28. *For all $i \in \mathcal{A}_q$ one has: $L(2) = l(ii) = (3q-2)q^{q-1}$.*

Proof. By Remark 3.27 one has:

$$F_2(u) = \{(i+k)i : i \in \mathcal{A}_q, 0 \leq k \leq q-1\}.$$

Thus, we have the following equalities:

$$\begin{aligned} L(2) &= \max\{l(w) : w \in F_2(u)\} \\ &= \max\{l((k+i)i) : 0 \leq k \leq q-1, i \in \mathcal{A}_q\} \\ &= \max\{l(k0) : 0 \leq k \leq q-1\} && \text{because} (k+i)i = E_q^i(k0) \\ &= \max\{\frac{1}{q^k}l(00) : 0 \leq k \leq q-1\} && \text{(by relation } (1)) \\ &= \max\{\frac{1}{q^k} : 0 \leq k \leq q-1\} \times l(00) \\ &= \frac{1}{q^0}l(00) \\ &= (3q-2)q^{q-1}. \end{aligned}$$

□

4 Essential and non-essential singular factors

Definition 4.1. *A factor w of an infinite word u is* singular *if $|w| = 1$ or there exist a factor v of u and letters x, x', y, y' such that $w = xvy$, $x \neq x'$, $y \neq y'$ and $x'vy, xvy'$ are factors of u.*

Let us recall that a singular factor w is said to be essential if $l(w) = L(|w|) > L(|w| - 1)$. Let i, j, z, and m be letters of $\mathcal{A}_q$ such that $j = E_q(i), m = E_q^{q-1}(i), z = E_q^{q-2}(i)$ and $j_k^{(i)} = E_q^{(k-1)}(i), \forall k \in [\![2, q]\!]$. We respectively define the sequence $(m_n)_{n\geq 1}$ and $(z_n)_{n\geq 1}$ by:

$$\begin{cases} m_1 = m \\ m_n = E_q^{q-1}(m_{n-1}) \end{cases} \quad \text{and} \quad \begin{cases} z_1 = z \\ z_n = E_q^{q-1}(z_{n-1}). \end{cases}$$

Recall that the following result established in [8].

Theorem 4.2. *Let $k \in [\![2, q]\!]$, $i \in \mathcal{A}_q$ and $n \in \mathbb{N}$.*

- *The unique singular factor s'_n generated by $\mu_q^n(w_k^{(i)})$ is given by,* $s'_n = m_{n+1}\mu_q^n(w_k^{(i)})j_k^{(i)}, \forall n \geq 1, k \in [\![3, q]\!]$.
- *For all $k \neq q$, the unique singular factor s''_n generated by $\mu_q^n(v_k^{(i)})$ is given by,* $s''_n = m_{n+1}\mu_q^n(v_k^{(i)})j_k^{(i)}, \forall n \geq 0, k \in [\![2, q-1]\!]$.

We have the following result.

Proposition 4.3. *Let $k \in [\![3, q]\!]$ and $i \in \mathcal{A}_q$. The singular factor generated by $w_k^{(i)}$ is not essential.*

Proof. We distinguish three cases:

Case 1: $k \in [\![3, q-2]\!]$. By Theorem 4.2, the singular factor generated by $w_k^{(i)}$ is $m_1\mu_q^0(w_k^{(i)})j_k^{(i)} = m_1 w_k^{(i)} j_k^{(i)}$. Thus, one has:

$$\begin{aligned} l(m_1 w_k^{(i)} j_k^{(i)}) &= l((i+q-1)i(i+1)\dots(i+k-2)(i+k-1)) \\ &= l(w_{k+2}^{(i+q-1)}) \\ &= l(w_{k+2}^{(i)}). \qquad \text{(because } w_{k+2}^{(i+q-1)} = E_q^{q-1}(w_{k+2}^{(i)}) \end{aligned}$$

Let us set $K = k + 2$. As $k \in [\![3, q-2]\!]$ then $K \in [\![5, q]\!]$. By Proposition 3.12, $l(w_K^{(i)}) = (2q-1)(K-1)$. Therefore,

$$l(m_1 w_k^{(i)} j_k^{(i)}) = l(w_{k+2}^{(i)}) = (2q-1)(k+1). \tag{3}$$

As the sequence $(l(m_1 w_k^{(i)} j_k^{(i)}))_{3\leq k\leq q-2}$ is strictly increasing, then $l(m_1 w_k^{(i)} j_k^{(i)}) \leq l(m_1 w_{q-2}^{(i)} j_{q-2}^{(i)})$. But $l(m_1 w_{q-2}^{(i)} j_{q-2}^{(i)}) = (2q-1)(q-1)$ by (2). Then, for all $k \in [\![3, q-2]\!]$, $l(m_1 w_k^{(i)} j_k^{(i)}) \leq (q-1)(2q-1)$. Let us check that $l(m_1 w_{q-2}^{(i)} j_{q-2}^{(i)}) < L(2)$.
For all $q \geq 3$, one has: $3q - 2 \geq 2q - 1$ and $q^{q-1} > q - 1$. Consequently,

$$(3q-2)q^{q-1} > (q-1)(2q-1),$$

which implies that $L(2) > l(m_1 w_{q-2}^{(i)} j_{q-2}^{(i)})$. Furthermore, for all $k \in [\![3, q-2]\!]$, $\left|m_1 w_k^{(i)} j_k^{(i)}\right| = k + 1 > 2$ but $L(2) > l(m_1 w_{q-2}^{(i)} j_{q-2}^{(i)})$. As a result, for all $k \in [\![3, q-2]\!]$, the singular factor generated by $w_k^{(i)}$ is not essential.

Case 2: $k = q-1$. Then $m_1 w_{q-1}^{(i)} j_{q-1}^{(i)} = (i+q-1)i(i+1)\dots(i+q-3)(i+q-2) = \mu_q(i+q-1)$. By Proposition 3.12, $l(m_1 w_{q-1}^{(i)} j_{q-1}^{(i)}) = l(\mu_q(i+q-1)) = q(2q-1)$.
Let us check that $l(m_1 w_{q-1}^{(i)} j_{q-1}^{(i)}) < L(2)$. For all $q \geq 3$, one has: $q^{q-1} > q$ and $3q-2 > 2q-1$. Therefore,

$$(3q-2)q^{q-1} > q(2q-1).$$

So, $L(2) > l(m_1 w_{q-1}^{(i)} j_{q-1}^{(i)})$. But $\left|m_1 w_{q-1}^{(i)} j_{q-1}^{(i)}\right| > 2$ for all $q \geq 3$. Then, the singular factor generated by $w_{q-1}^{(i)}$ is not essential.

Case 3: $k = q$. Then

$$m_1 w_q^{(i)} j_q^{(i)} = (i+q-1)i(i+1)\dots(i+q-2)(i+q-1) = \mu_q(i+q-1)(i+q-1).$$

Thus,

$$\begin{aligned} l(m_1 w_q^{(i)} j_q^{(i)}) &= l(\mu_q((i+q-1)(i+q-1))) \\ &= l(\mu_q(i)i) \text{ (because } \mu_q(i+q-1)(i+q-1) = E_q^{q-1}(\mu_q(i)i)) \\ &= (2q^{q-1}+1)q-1, \text{ (by Proposition 3.25.)} \end{aligned}$$

Let us check that $l(m_1 w_q^{(i)} j_q^{(i)}) < L(2)$.

$$L(2) - l(m_1 w_q^{(i)} j_q^{(i)}) = (3q-2)q^{q-1} - (2q^{q-1}+1) + 1 = q^{q-1}(q-2) - (q-1).$$

For all $q \geq 3$,

$$q^{q-1}(q-2) - (q-1) \geq q^{q-1} - (q-1) \geq q(q^{q-2}-1).$$

For all $q \geq 3$, $q(q^{q-1}-1) > 0$. Thus, $L(2) > l(m_1 w_q^{(i)} j_q^{(i)})$ and $\left|m_1 w_q^{(i)} j_q^{(i)}\right| > 2$. Therefore, the singular factor generated by $w_q^{(i)}$ is not essential. □

Theorem 4.4. [8] *The singular factors of* $\mathbf{t}_q$ *generated by* $\mu_q^n(i)$, $n \geq 1$ *are given by:*

$$S_n = \{m_n\mu_q^n(i)i, m_n\mu_q^n(i)j, z_n\mu_q^n(i)i, z_n\mu_q^n(i)j : i \in \mathcal{A}_q, j = E_q(i)\}, \forall n \geq 1.$$

Thus, the set of singular factors generated by $\mu_q(i)$ is given by:

$$S_1 = \{m_1\mu_q(i)i, m_1\mu_q(i)j, z_1\mu_q(i)i, z_1\mu_q(i)j : i \in \mathcal{A}_q, j = E_q(i)\}.$$

In the following, we give the maximal return time of singular factors generated by $\mu_q(i)$.

Proposition 4.5. *Let* $i \in \mathcal{A}_q$ *and* $j = E_q(i)$. *Then, the maximal return time of each singular factor generated by* $\mu_q(i)$ *is given by:*

(i) $l(m_1\mu_q(i)i) = (4q^{q-1}+1)q-2$;

(ii) $l(m_1\mu_q(i)j) = (3q-2)q^q$;

(iii) $l(z_1\mu_q(i)i) = (3q-2)q^q$;

(iv) $l(z_1\mu_q(i)j) = 3q(2q-1)$.

Proof. As $E_q(F_n(\mathbf{t}_q)) = F_n(\mathbf{t}_q)$, then we take $i = 0$. The cases for the other letters are treated similarly.

(i) One has: $l((q-1)\mu_q(0)0) = l(\mu_q(q-1)(q-1)0)$.
By Proposition 3.25,

$$\begin{aligned} l(\mu_q(q-1)(q-1)0) &= l(\mu_q(0)01), \text{ because } \mu_q(q-1)(q-1)0 = E_q^{q-1}(\mu_q(0)01) \\ &= l(v_3^{(0)}) \\ &= (4q^{q-1}+1)q-2. \end{aligned}$$

(ii) The factor $m_1\mu_q(0)1 = (q-1)\mu_q(0)1$ occurs in $\mu_q(001)$. By Lemma 3.11,

$$l((q-1)\mu_q(0)1) = q \times l(001).$$

Since 00 extends uniquely to the right by 1, then $l(001) = l(00)$. Therefore,

$$l((q-1)\mu_q(0)1) = q \times l(00).$$

By Corollary 3.26, $l(00) = (3q-2)q^{q-1}$. Hence $l((q-1)\mu_q(0)1) = (3q-2)q^q$.

(iii) Observe that the factor $z_1\mu_q(0)0 = (q-2)\mu_q(0)0$ occurs in $\mu_q((q-1)00)$.
By Lemma 3.11 one has:

$$l((q-2)\mu_q(0)0) = q \times l((q-1)00) = q \times l(00) = (3q-2)q^q.$$

(iv) Observe that the factor $z_1\mu_q(0)1 = (q-2)\mu_q(0)1$ occurs in $\mu_q((q-1)01)$.
By Lemma 3.11,

$$l((q-2)\mu_q(0)1) = q \times l((q-1)01).$$

But $(q-1)01 = E_q^{q-1}(012)$. Then,

$$l((q-2)\mu_q(0)1) = q \times l(012).$$

By Proposition 3.12

$$l(012) = l(w_4^{(0)}) = 3(2q-1).$$

Hence, $l((q-2)\mu_q(0)1) = 3q(2q-1)$. □

Proposition 4.6. *For all $i \in \mathcal{A}_q$, the set S'_n of essential singular factors generated by $\mu_q^n(i)$, $n \in \mathbb{N}^*$ is given by:*

$$S'_n = \{m_n\mu_q^n(i)j, z_n\mu_q^n(i)i : i \in \mathcal{A}_q, j = E_q(i)\}, \forall n \geq 1.$$

Proof. We distinguish two cases.

Case 1: $n = 1$. By Proposition 4.5, $l(m_1\mu_q(i)j) = l(z_1\mu_q(i)i)$.
It remains to compare $l(m_1\mu_q(i)j)$, $l(m_1\mu_q(i)i)$ and $l(z_1\mu_q(i)j)$. We take $i = 0$ because the letters play a symmetric role.

Compare $l((q-1)\mu_q(0)1)$ *and* $l((q-2)\mu_q(0)1)$.
For all $q \geq 3$, $3q \leq q^q$ and $2q-1 < 3q-2$.
Consequently,

$$3q(2q-1) < (3q-2)q^q.$$

Hence,

$$l((q-2)\mu_q(0)1) < l((q-1)\mu_q(0)1). \tag{4}$$

Compare $l((q-1)\mu_q(0)1)$ *and* $l((q-1)\mu_q(0)0)$.
By Proposition 4.5,

$$l((q-1)\mu_q(0)1) = (3q-2)q^q \text{ and } l((q-1)\mu_q(0)0) = (4q^{q-1}+1)q-2.$$

But for all $q \geq 3$, $q^q > q$. Then,

$$l((q-1)\mu_q(0)0) < 5q^q. \tag{$*$}$$

Furthermore, for all $q \geq 3$, $(3q-2) \geq 7$. Therefore,

$$(3q-2)q^q \geq 7q^q > 5q^q.$$

So, $(*)$ implies that

$$l((q-1)\mu_q(0)0) < l((q-1)\mu_q(0)1). \tag{5}$$

From (4) and (5) we deduce that $(q-1)\mu_q(0)1$ and $(q-2)\mu_q(0)0$ are essential singular factors. Thus,

$$S_1' = \{m_1\mu_q(i)j, z_1\mu_q(i)i : i \in \mathcal{A}_q, j = E_q(i)\}.$$

Case 2: $n \geq 2$. Observe that $m_{n-1}\mu_q^{n-1}(i)i$ is the ancestor of $m_n\mu_q^n(i)i$. By Lemma 3.11, $l(m_n\mu_q^n(i)i) = ql(m_{n-1}\mu_q^{n-1}(i)i)$. Similarly, the ancestor of $m_{n-1}\mu_q^{n-1}(i)i$ is $m_{n-2}\mu_q^{n-2}(i)i$. Therefore, $l(m_{n-1}\mu_q^{n-1}(i)i) = ql(m_{n-2}\mu_q^{n-2}(i)i)$. Hence

$$l(m_n\mu_q^n(i)i) = q^2 l(m_{n-2}\mu_q^{n-2}(i)i).$$

Thus, successively, we show that

$$l(m_n\mu_q^n(i)i) = q^{n-1} l(m_1\mu_q(i)i).$$

Similarly, we show that

$$l(m_n\mu_q^n(i)j) = q^{n-1} l(m_1\mu_q(i)j), \tag{6}$$

$$l(z_n\mu_q^n(i)i) = q^{n-1} l(z_1\mu_q(i)i), \tag{7}$$

$$l(z_n\mu_q^n(i)j) = q^{n-1} l(z_1\mu_q(i)j). \tag{8}$$

By Proposition 4.5, $l(m_1\mu_q(i)j) = l(z_1\mu_q(i)i)$. As a result,

$$l(m_n\mu_q^n(i)j) = l(z_n\mu_q^n(i)i).$$

Moreover, by Case 1,

$$l(m_1\mu_q(i)j) > l(m_1\mu_q(i)i) \text{ and } l(m_1\mu_q(i)j) > l(z_1\mu_q(i)j).$$

So, from (6), (7) and (8) we deduce the following inequalities:

$$l(m_n\mu_q^n(i)j) > l(m_n\mu_q^n(i)i) \text{ and } l(m_n\mu_q^n(i)j) > l(z_n\mu_q^n(i)j).$$

Hence,

$$S'_n = \{m_n\mu_q^n(i)j, z_n\mu_q^n(i)i : i \in \mathcal{A}_q, j = E_q(i)\}. \qquad \square$$

Remark 4.7. *The essential singular factors are of length $q^n + 2, \forall n \geq 1$. Indeed,*

$$\left|m_n\mu_q^n(i)j\right| = \left|z_n\mu_q^n(i)i\right| = q^n + 2.$$

Proposition 4.8. *Let $k \in [\![2, q-1]\!]$ and $i \in \mathcal{A}_q$. The singular factor generated by $v_k^{(i)}$ is not essential.*

Proof. We distinguish two cases.

Case 1: $k \in [\![2, q-2]\!]$. By Theorem 4.2, the singular factor generated by $v_k^{(i)}$ is $m_1\mu_q^0(v_k^{(i)})j_k^{(i)} = m_1v_k^{(i)}j_k^{(i)}$. Thus, one has:

$$\begin{aligned} m_1v_k^{(i)}j_k^{(i)} &= (i+q-1)\mu_q(i)i(i+1)\dots(i+k-2)(i+k-1) \\ &= \underbrace{(i+q-1)i(i+1)\dots(i+q-2)}_{\mu_q(i+q-1)}(i+q-1)i(i+1)\dots(i+k-2)(i+k-1) \\ &= E_q^{q-1}(\mu_q(i)i(i+1)\dots(i+k)) \\ &= E_q^{q-1}(v_{k+2}^{(i)}). \end{aligned}$$

As a result,

$$l(m_1v_k^{(i)}j_k^{(i)}) = l(E_q^{q-1}(v_{k+2}^{(i)})) = l(v_{k+2}^{(i)}) = l(v_K^{(i)}) \text{ with } K = k+2.$$

As $k \in [\![2, q-2]\!]$, then $K \in [\![4, q]\!]$. By Proposition 3.25,

$$l(v_K^{(i)}) = (2q^q - 1)K + q(-2q^{q-1} + 1) + 1,$$

which implies that for all $k \in [\![2, q-2]\!]$:

$$\begin{aligned} l(m_1v_k^{(i)}j_k^{(i)}) &= (2q^q-1)(k+2) + q(-2q^{q-1}+1) + 1 \\ &= (2q^q-1)k + (2q^q-1) + q \\ &= (2q^q-1)(k+1) + q. \end{aligned}$$

Since the sequence $(l(m_1v_k^{(i)}j_k^{(i)}))_{2\leq k\leq q-2}$ is strictly increasing, then for all $k \in [\![2, q-2]\!]$,

$$l(m_1v_k^{(i)}j_k^{(i)}) \leq l(m_1v_{q-2}^{(i)}j_{q-2}^{(i)}).$$

But

$$l(m_1v_{q-2}^{(i)}j_{q-2}^{(i)}) = (2q^q-1)(q-1) + q = 2q^q(q-1) + 1.$$

Then, $\forall k \in [\![2, q-2]\!]$,

$$l(m_1v_k^{(i)}j_k^{(i)}) \leq 2q^q(q-1) + 1.$$

Note that $|m_1v_k^{(i)}j_k^{(i)}| = q + k + 1, \forall k \in [\![2, q-2]\!]$. Then, $|m_1v_k^{(i)}j_k^{(i)}| > q+3$, for all $k \in [\![2, q-2]\!]$. To show that the singular factors $m_1v_k^{(i)}j_k^{(i)}, k \in [\![2, q-2]\!]$ are not essential, it suffices to show that $l(m_1v_k^{(i)}j_k^{(i)}) \leq L(n)$ with $n < q+3$.

So, let us take $n = q+2$ and compare $l(m_1 v_k^{(i)} j_k^{(i)})$ and $l(q+2)$, for all $k \in [\![2, q-2]\!]$. As

$$l(m_1 v_k^{(i)} j_k^{(i)}) \leq l(m_1 v_{q-2}^{(i)} j_{q-2}^{(i)}), \forall k \in [\![2, q-2]\!]$$

it suffices to compare $L(q+2)$ and $l(m_1 v_{q-2}^{(i)} j_{q-2}^{(i)})$. We start by determining $L(q+2)$. We have: $L(q+2) = \max\{l(w) : w \in S_1', |w| \leq q+2\}$. By Proposition 4.6, the set of essential singular factors of length $q+2$ is given by

$$S_1' = \{m_1 \mu_q(i) j, z_1 \mu_q(i) i : i \in \mathcal{A}_q, j = E_q(i)\}.$$

Therefore,

$$\begin{aligned} L(q+2) &= \max\{l(z_1 \mu_q(i) i), l(m_1 \mu_q(i) j)\} \\ &= \max\{l(z_1 \mu_q(i) i)\} \text{ because } l(z_1 \mu_q(i) i) = l(m_1 \mu_q(i) j) \\ &= (3q-2) q^q. \end{aligned}$$

Let us study the sign of the difference between $L(q+2)$ and $l(m_1 v_{q-2}^{(i)} j_{q-2}^{(i)})$.

$$L(q+2) - l(m_1 v_{q-2}^{(i)} j_{q-2}^{(i)}) = (3q-2)q^q - 2q^q(q-1) - 1 = q^{q+1} - 1.$$

As a result, $L(q+2) > l(m_1 v_{q-2}^{(i)} j_{q-2}^{(i)})$ for all $q \geq 4$. However, $|m_1 v_k^{(i)} j_k^{(i)}| = q+k+1$, which is strictly greater than $q+2$ for all $k \geq 2$. Then, for all $k \in [\![2, q-2]\!]$, the singular factor generated by $v_k^{(i)}$ is not essential.

Case 2: $k = q-1$. Then, $v_{q-1}^{(i)} = \mu_q(i) i (i+1) \dots (i+q-3)$. By Theorem 4.2, the singular factor generated by $v_{q-1}^{(i)}$ is

$$\begin{aligned} m_1 v_{q-1}^{(i)} j_{q-1}^{(i)} &= (i+q-1)\mu_q(i) i (i+1) \dots (i+q-3)(i+q-2) \\ &= \underbrace{(i+q-1) i (i+1) \dots (i+q-2)}_{\mu_q(i+q-1)} \underbrace{(i+q-1) i (i+1) \dots (i+q-3)(i+q-2)}_{\mu_q(i+q-1)} \\ &= \mu_q((i+q-1))\mu_q((i+q-1)) \\ &= \mu_q((i+q-1)(i+q-1)). \end{aligned}$$

Observe that $(i+q-1)(i+q-1)$ is the unique ancestor of $\mu_q((i+q-1)(i+q-1))$. Thus, it follows that:

$$l(m_1 v_{q-1}^{(i)} j_{q-1}^{(i)}) = q l((i+q-1)(i+q-1)) = q l(E_q^{q-1}(ii)) = q l(ii) = (3q-2)q^q.$$

Then, $l(m_1 v_{q-1}^{(i)} j_{q-1}^{(i)}) = L(q+2)$ and $|m_1 v_{q-1}^{(i)} j_{q-1}^{(i)}| = 2q > q+2$ for $q \geq 3$. As a result, the singular factor generated by $v_{q-1}^{(i)}$ is not essential. □

Lemma 4.9. *Let $k \in [\![2, q]\!]$, $i \in \mathcal{A}_q$ and $n \in \mathbb{N}$. Then we have the following equalities:*

- $l(m_{n+1} \mu_q^n(w_k^{(i)}) j_k^{(i)}) = q^n l(m_1 w_k^{(i)} j_k^{(i)})$.
- $l(m_{n+1} \mu_q^n(v_k^{(i)}) j_k^{(i)}) = q^n l(m_1 v_k^{(i)} j_k^{(i)})$.

Proof. Observe that the ancestor of $m_{n+1}\mu_q^n(w_k^{(i)})j_k^{(i)}$ is $m_n\mu_q^{n-1}(w_k^{(i)})j_k^{(i)}$. By Lemma 3.11,

$$l(m_{n+1}\mu_q^n(w_k^{(i)})j_k^{(i)}) = q \times l(m_n\mu_q^{n-1}(w_k^{(i)})j_k^{(i)})$$

By applying Lemma 3.11 n times, we show that

$$l(m_{n+1}\mu_q^n(w_k^{(i)})j_k^{(i)}) = q^n \times l(m_1 w_k^{(i)} j_k^{(i)}).$$

Similarly, we show that

$$l(m_{n+1}\mu_q^n(v_k^{(i)})j_k^{(i)}) = q^n l(m_1 v_k^{(i)} j_k^{(i)}).$$ □

Proposition 4.10. *Let $i \in \mathcal{A}_q$ and $n \in \mathbb{N}$.*

- *For all $k \in [\![3, q]\!]$, the singular factor generated by $\mu_q^n(w_k^{(i)})$ is not essential.*
- *For all $k \in [\![2, q-1]\!]$, the singular factor generated by $\mu_q^n(v_k^{(i)})$ is not essential.*

Proof. We have: $\left|m_{n+1}\mu_q^n(w_k^{(i)})j_k^{(i)}\right| = q^n \left|w_k^{(i)}\right| + 2$.

But for all $k \in [\![3, q]\!]$, $\left|w_k^{(i)}\right| > 1$. Then,

$$q^n \left|w_k^{(i)}\right| + 2 > q^n + 2. \tag{9}$$

Moreover, by Proposition 4.3, $l(m_1 w_k^{(i)} j_k^{(i)}) < L(2)$. Therefore, $q^n l(m_1 w_k^{(i)} j_k^{(i)}) < q^n L(2)$, for all $k \in [\![3, q]\!]$. So,

$$l(m_{n+1}\mu_q^n(w_k^{(i)})j_k^{(i)}) < l(m_n\mu_q^n(i)j). \tag{10}$$

From (9) and (10), we deduce that for all $k \in [\![3, q]\!]$ and for all $n \in \mathbb{N}$, the singular factor generated by $\mu_q^n(w_k^{(i)})$ is not essential.
Analogously, we show that the singular factor generated by $\mu_q^n(v_k^{(i)})$ is not essential. □

Theorem 4.11. *The essential singular factors of $\mathbf{t}_q$ are:*

(i) *The letters of the alphabet $\mathcal{A}_q$;*

(ii) *the squares of letters in $\mathbf{t}_q$;*

(iii) *the essential singular factors generated by $\mu_q^n(i)$, $n \in \mathbb{N}^*$ is given by the set:*

$$S'_n = \{m_n\mu_q^n(i)j, z_n\mu_q^n(i)i : i \in \mathcal{A}_q, j = E_q(i)\}, \forall n \geq 1.$$

Proof. (i) $L(1) = 2q - 1 > 0$. Then $L(1) > L(0)$. Thus, the letters of the alphabet are essential singular factors.

(ii) By Proposition 3.28, $L(2) = l(ii)$ for all $i \in \mathcal{A}_q$. Furthermore, $L(2) > L(1)$. It follows that the squares of letters in $\mathbf{t}_q$ are essential singular factors.

(iii) The proof follows from Proposition 4.6. □

5 Recurrence function of $\mathbf{t}_q$

To calculate the recurrence function of $\mathbf{t}_q$, we consider only the essential singular factors since the non essential singular factors have no influence on the calculation.

Proposition 5.1. *Let $n, k, q \in \mathbb{N}$ such that $n \in [\![q^k + 2, q^{k+1} + 1]\!]$. Then the maximal return time of factors of length n of $\mathbf{t}_q$ is given by:*

$$L(n) = \begin{cases} (l(00) + 1) \times 2^k, 2^k + 2 \leq n \leq 2^{k+1} + 2, & \text{if } q = 2 \\ [(3q-2)q^{q-1}] \times q^k, q^k + 2 \leq n \leq q^{k+1} + 1, \forall k \geq 1, & \text{if } q \geq 3. \end{cases}$$

Proof. Let us proceed by induction on k. Without loss of generality, we take $i = 0$ since the others are obtained by using E_q.

- For $q = 2$ see [4].
- For $q \geq 3$, we proceed by induction on k. For $k = 1$, by Theorem 4.11, the essential singular factors of length $q + 2$ are given by the set

  $$S_1' = \{m_1\mu_q(0)1, z_1\mu_q(0)0\}.$$

  By Proposition 4.5, $l(m_1\mu_q(0)1) = l(z_1\mu_q(0)0) = (3q-2)q^q$. Thus, $L(q+2) = (3q-2)q^q$. Assume that the property holds up to order k and show that it holds for order $k+1$. Observe that the ancestor of $z_{k+1}\mu_q^{k+1}(0)0$ is $z_k\mu_q^k(0)0$. So,

  $$l(z_{k+1}\mu_q^{k+1}(0)0) = q \times l(z_k\mu_q^k(0)0).$$

  By induction hypothesis, $l(z_k\mu_q^k(0)0) = [(3q-2)q^{q-1}] \times q^k$. Consequently, $l(z_{k+1}\mu_q^{k+1}(0)0) = [(3q-2)q^{q-1}] \times q^{k+1}$. Moreover, by Remark 4.7, the essential singular factors are of length $q^k + 2, \forall k \geq 1$. Hence,

  $$L(n) = \left[(3q-2)q^{q-1}\right] \times q^k, q^k + 2 \leq n \leq q^{k+1} + 1, \forall k \geq 1. \qquad \square$$

Remark 5.2. *The property holds for $k = 0$.*

Theorem 5.3. *For all $q \geq 3$, the maximal return time $L(n)$ is given by:*

$$L(n) = \begin{cases} 1, & \text{if } n = 0 \\ 2q-1, & \text{if } n = 1 \\ (3q-2)q^{q-1}, & \text{if } n = 2 \\ [(3q-2)q^{q-1}] \times q^k, & \text{if } q^k + 2 \leq n \leq q^{k+1} + 1, \forall k \geq 1. \end{cases}$$

As $R(n) = L(n) + n - 1$, it follows that:

$$R(n) = \begin{cases} 0, & \text{if } n = 0 \\ 2q-1, & \text{if } n = 1 \\ (3q-2)q^{q-1} + 1, & \text{if } n = 2 \\ [(3q-2)q^{q-1}] \times q^k + n - 1, & \text{if } q^k + 2 \leq n \leq q^{k+1} + 1, \forall k \geq 1. \end{cases}.$$

Proof. The proof follows from Remark 3.13, Proposition 3.28 and Proposition 5.1. $\square$

6 Conclusion

We determined the recurrence function of generalized Thue–Morse word $\mathbf{t}_q$. For the study of this one, we developed techniques which allow us to find the maximal return time without listing the set of return words. However, the recurrence function of word $\mathbf{t}_{b,q}$ $(b \neq q)$ remains unknown.

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