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# Enumeration of cyclic components of the power digraph $\Gamma(n, 13)$

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**Abstract:** For each positive integer $n$, we assign a digraph $\Gamma(n, 13)$ whose set of vertices is $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ and attach exactly one directed edge from a vertex $a$ to the vertex $b$ if and only if $a^{13} \equiv b \pmod{n}$. In this work, the number of cycles and the number of components in the digraph $\Gamma(n, 13)$ are presented for $n = p^k$ with the help of Carmichael's lambda function. It is proved that for $k \geq 1$ and for the prime $p = 3, 5, 7$ and $11$, the respective numbers of components in the digraph $\Gamma(p^k, 13)$ are $(4k-1)$, $(4k+1)$, $(18k-11)$ and $(6k-1)$. The structure of tree associated with power digraph $\Gamma(p^k, 13)$ for $p = 3, 5, 7$ and $11$ are also studied. We have also studied the structure of power digraph $\Gamma(p, 13)$, where $p = 2^{2^m} + 1, m \geq 0$ is a Fermat prime.
**Keywords:** Carmichael $\lambda$-function, Components, Cycles, Digraph, Fermat prime, Fixed point, Power digraph.
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# 1 Introduction

For each pair of integers $n(\geq 1)$ and $k(\geq 1)$, a power digraph modulo $n$ denoted by $\Gamma(n,k)$ is a digraph with vertex set $\mathbb{Z}_n = \{0,1,2,\ldots,n-1\}$ and the ordered pair $(a,b)$ is a directed edge of $\Gamma(n,k)$ from $a$ to $b$ if and only if $a^k \equiv b \pmod{n}$, where $a,b \in \mathbb{Z}_n$ (for details, see [17]). The idea has developed a powerful connection relating number theory, group theory and graph theory. The concept of power digraph was introduced in the year 1967 by Bryant [2] in which for each element $x$ of a finite group, a point $P(x)$ was assigned and joined with $P(x^2)$ by a directed line trailing in a graph called the graph of the group. Thereafter, Blanton *et al.* [1], Somer and Křížek [7, 15], Szalay [18] and Rogers [11] have investigated the properties of a variety of power digraphs corresponding to the quadratic congruence $a^2 \equiv b \pmod{n}$. In 1996, Lucheta *et al.* [8] studied the power digraph using the congruence $a^k \equiv b \pmod{p}$, where $k \geq 2$ is a positive integer and $p$ is a prime. Skowronek-Kaziów [14] and Rahmati [10] defined the power digraphs using the congruences $a^3 \equiv b \pmod{n}$ and $a^5 \equiv b \pmod{n}$ respectively and established some results. Mateen and Mahmood [9] discussed the power digraph with the help of the congruence $a^7 \equiv b \pmod{n}$ and gave explicit formula for fixed points and the condition for which the digraph has exactly 7 components. Sawkmie and Singh [12, 13]) studied the tree structure and uniqueness of the factorization of power digraph modulo $n$. Goswami *et al.* [4] investigated the formula for the number of fixed points as well as symmetry of the power digraph for the congruence $a^{11} \equiv b \pmod{n}$. It is important to mention that the problem of enumeration of cyclic vertices, cycles and components of power digraph $\Gamma(n,k)$ still remains open. In [19], Thakur *et al.* formulated an explicit formula to enumerate the cyclic vertices and components for the power digraph $\Gamma(n,11)$.

In this paper, we try to enumerate the number of components with respect to the congruence $a^{13} \equiv b \pmod{n}$ when $n = 3^k, 5^k, 7^k, 11^k$ and a Fermat prime. We organize the paper as follows:

In Section 2, we include some definitions from graph theory and number theory. This section also contains some properties of Carmichael lambda function. Finally, in Section 3, we have developed an explicit formula to enumerate the components of the digraphs $\Gamma(p^k,13)$ where $p < 13$ is an odd prime, $k$ is any positive integer. The same has also been done for $\Gamma(n,13)$, where $n$ is a Fermat prime. The notations used all throughout the paper are usual. For example, the greatest common divisor of two integers $m$ and $n$ is denoted by $\gcd(m,n)$, the order of $a$ modulo $n$ is denoted by $\mathrm{ord}_n(a)$, the greatest integer $\leq a$, i.e., the floor function, is denoted by $\lfloor a \rfloor$ and the smallest integer $\geq a$, i.e., the ceiling function, is denoted by $\lceil a \rceil$.

# 2 Preliminaries

For smooth reading of the paper we incorporate here few relevant terms and results from graph theory and number theory which are necessary for development of its main contents. As usual for a positive integer $n$, $\mathbb{Z}_n = \{0,1,2,\ldots,n-1\}$ will denote the complete set of residues modulo $n$. Here we consider a directed graph $\Gamma(n,13)$ whose vertex set is $\mathbb{Z}_n$ and in which any two vertices $a,b \in \mathbb{Z}_n$ are connected by exactly one directed edge from $a$ to $b$ if and only if

$$a^{13} \equiv b \pmod{n} \tag{1}$$

For example, the digraph $\Gamma(23, 13)$ is as shown below:

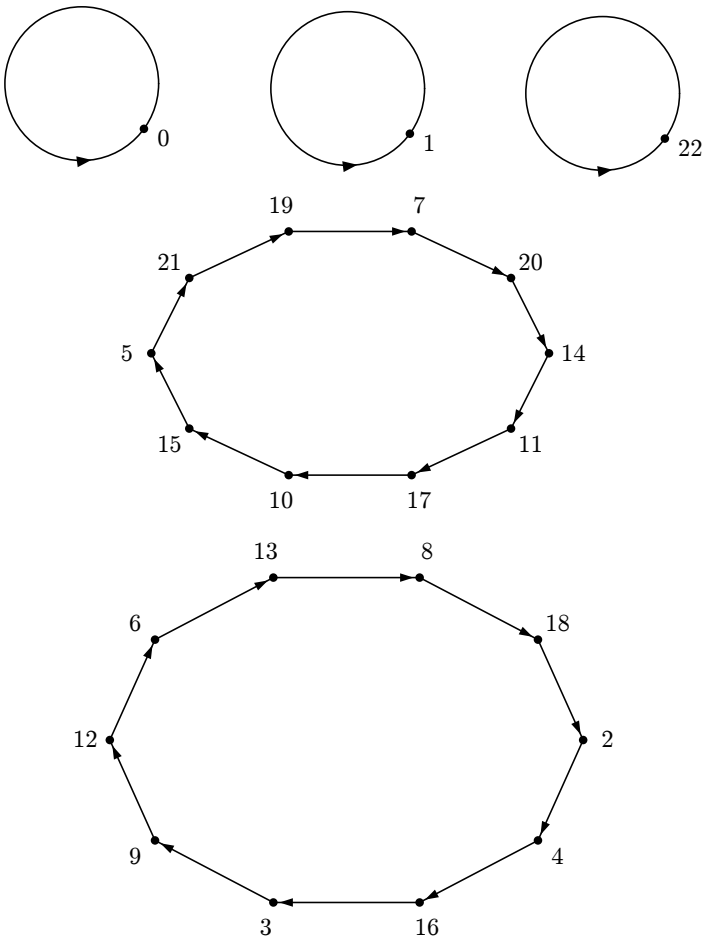



Figure 1. The digraph $\Gamma(23, 13)$

The distinct vertices $a_1, a_2, a_3, \ldots, a_t$ in $\mathbb{Z}_n$ will form a cycle of length $t$ if

$$\begin{aligned} a_1^{13} &\equiv a_2 \pmod{n} \\ a_2^{13} &\equiv a_3 \pmod{n} \\ a_3^{13} &\equiv a_4 \pmod{n} \\ &\vdots \\ a_t^{13} &\equiv a_1 \pmod{n} \end{aligned}$$

We call a cycle of length $t$ as a $t$- cycle and a cycle of length $1$ as a fixed point. A fixed point is isolated if it is not connected to any other vertex in $\Gamma(n, 13)$.

The indegree of a vertex $a \in \mathbb{Z}_n$, denoted by $\mathrm{indeg}(a)$ is the number of directed edges coming into the vertex $a$ and the outdegree of a vertex $a$, denoted by $\mathrm{outdeg}(a)$ is the number of directed edges leaving the vertex $a$. Since the residue of a number modulo $n$ is unique, so $\mathrm{outdeg}(a) = 1$ and $\mathrm{indeg}(a) \geq 0$ for each vertex $a \in \mathbb{Z}_n$. Also, for an isolated fixed point $a \in \mathbb{Z}_n$, $\mathrm{outdeg}(a) = \mathrm{indeg}(a) = 1$.

A component of a digraph is a subdigraph which is a maximal connected subgraph of the associated nondirected graph (for details, see [5]). As the outdegree of each vertex of the digraph $\Gamma(n, 13)$ is equal to 1, so the number of components of $\Gamma(n, 13)$ is equal to the number of all cycles. The cycles may or may not be isolated. Moreover, by using the properties of the congruence relation (1) and the definition of cycle, it can be established that the digraph $\Gamma(k^{13^t} - k, 13)$ has a $t$-cycle containing the vertex $k$, for an arbitrary integer $t \geq 1$.

For $n > 1$, let $\Gamma_1(n, 13)$ and $\Gamma_2(n, 13)$ be two subdigraphs of the digraph $\Gamma(n, 13)$, where $\Gamma_1(n, 13)$ is the subdigraph induced on the set of vertices $a \in \mathbb{Z}_n$ with $\gcd(a, n) = 1$ and $\Gamma_2(n, 13)$ is the subdigraph induced on the set of vertices $a \in \mathbb{Z}_n$ with $\gcd(a, n) \neq 1$. Clearly, the vertex set of $\Gamma_1(n, 13)$ is the unit group $\mathbb{Z}_n^*$ with order $\phi(n)$, where $\phi$ denotes Euler's totient function. Also, vertices 1 and $n - 1$ are in $\Gamma_1(n, 13)$ and the vertex 0 is always a vertex of $\Gamma_2(n, 13)$. It can be easily observed that $\Gamma_1(n, 13) \cup \Gamma_2(n, 13) = \Gamma(n, 13)$ and $\Gamma_1(n, 13) \cap \Gamma_2(n, 13) = \varnothing$.

A tree in which one vertex has been designated as the root is a rooted tree. The edges of a rooted tree can be assigned a natural orientation, either away from or towards the root and in either case the structure becomes a directed rooted tree. When a directed rooted tree has an orientation away from the root, it is called an out-tree and in case an orientation is towards the root, it is called an in-tree.

It is also appropriate to reproduce the following results on the Carmichael $\lambda$-function for their intimate connection with the content of the paper.

**Definition 2.1.** [3] *The Carmichael lambda function of a positive integer $n$, denoted by $\lambda(n)$ is defined as the smallest positive integer $m$ such that $a^m \equiv 1 \pmod{n}$ for every integer $a$ relatively prime to $n$.*

**Lemma 2.1.** [3] *Let $n$ be a positive integer, and let $\phi$ denote Euler's totient function. Then*

$$\begin{aligned}
\lambda(1) &= 1 = \phi(1),\\
\lambda(2) &= 1 = \phi(1),\\
\lambda(4) &= 2 = \phi(4),\\
\lambda(2^k) &= 2^{k-2} = \frac{1}{2}\phi(2^k) \quad \textit{for } k \geq 3,\\
\lambda(p^k) &= (p-1)p^{k-1} = \phi(p^k) \quad \textit{for any odd prime } p \textit{ and } k \geq 1,\\
\lambda(\Pi_{i=1}^r p_i^{k_i}) &= \operatorname{lcm}[\lambda(p_1^{k_1}), \lambda(p_2^{k_2}), \ldots, \lambda(p_r^{k_r})],
\end{aligned}$$

*where $p_1, p_2, \ldots, p_r$ are distinct primes for $k_i \geq 1, i = 1, 2, \ldots, r$.*

It follows from Lemma 2.1, that $\lambda(n) \mid \phi(n), \forall n$ and that $\lambda(n) = \phi(n)$ if and only if $n \in \{1, 2, 4, p^k, 2p^k\}$ where $p$ is an odd prime and $k \geq 1$. The following theorem generalizes the well-known Euler's Theorem which says that $a^{\phi(n)} \equiv 1 \pmod{n}$ if and only if $\gcd(a, n) = 1$. It shows that $\lambda(n)$ is the least possible universal order modulo $n$ [16].

**Theorem 2.1.** (Carmichael's Theorem, see [3, 6]). *Let $a, n \in N$. Then*

$$a^{\lambda(n)} \equiv 1 \pmod{n} \quad \textit{if and only if } \gcd(a, n) = 1.$$

Moreover, there exists an integer $g$ such that $\operatorname{ord}_n(g) = \lambda(n)$, where $\operatorname{ord}_n(g)$ denotes the multiplicative order of $g$ modulo $n$ (by Theorem II of [3], pp. 233).

Assume now that $\lambda(n)$ has the following prime power factorization:

$$\lambda(n) = \Pi_{j=1}^{r} q_j^{l_j},$$

where $q_1 < q_2 < \cdots < q_r$ are primes and $l_j > 0$. It is evident that from the definition of $\lambda$ that $q_1 = 2$, if $n > 2$.

We recall that the number of $t$-cycles in the digraph $\Gamma(n,k)$ is denoted by $A_t(\Gamma(n,k))$ and in this connection we reproduce the following theorem:

**Theorem 2.2.** [16] *Let $n = \Pi_{i=1}^{r} p_i^{\alpha_i}$, where $p_i, i = 1, 2, \ldots, r$ are distinct primes. Then*

$$A_t\big(\Gamma(n,k)\big) = \frac{1}{t}\Big[\prod_{i=1}^{r}\big(\delta_i \gcd\big(\lambda(p_i^{\alpha_i}), k^t - 1\big) + 1\big) - \sum_{d|t, d\neq t} dA_d\big(\Gamma(n,k)\big)\Big],$$

*where $\delta_i = 2$ if $2 \mid k^t - 1$ and $8 \mid p_i^{\alpha_i}$, and $\delta_i = 1$, otherwise.*

We include the following lemma also for future reference:

**Lemma 2.2.** [19] *Suppose $x^n \equiv y \pmod{\alpha p^\beta}$. Then $x^{np} \equiv y^p \pmod{\alpha p^{\beta+1}}$.*

# 3 Main result: Enumeration of cycles and components

Our first result in order is based on Carmichael's $\lambda$-function which guarantees the existence of a $t$-cycle in the digraph $\Gamma(n, 13)$.

**Theorem 3.1.** *(i) Let $n > 2$. Then there exists a cycle of length $t$ in the digraph $\Gamma(n,13)$ if and only if $t = \mathrm{ord}_d(13)$, for some even positive divisor $d$ of $\lambda(n)$.*

*(ii) If there exists a $t$- cycle in $\Gamma(n,13)$, then there exists a $t$- cycle in $\Gamma_1(n,13)$.*

*Proof.* (i) Suppose there exists a cycle of length $t$ in the digraph $\Gamma(n,13)$. Let $a$ be a vertex of a $t-$cycle in the digraph $\Gamma(n,13)$, then $t$ is the least positive integer such that $a^{13^t} \equiv a \pmod{n}$.
Case I: If $\gcd(a,n) = 1$, then $a \in \Gamma_1(n,13)$. Since, the set of vertices of $\Gamma_1(n,13)$ is equal to $\mathbb{Z}_n^*$ so $a^{-1}$ exists and we have

$$\begin{aligned}
& a^{13^t - 1} \equiv 1 \pmod{n} \\
\Rightarrow\ & d_0 \mid 13^t - 1, \text{where } d_0 = \mathrm{ord}_n(a) \\
\Rightarrow\ & 13^t \equiv 1 \pmod{d_0}, \text{where } t \text{ is the least such positive integer.} \\
\Rightarrow\ & \mathrm{ord}_{d_0}(13) = t
\end{aligned}$$

Now, by Carmichael's Theorem $a^{\lambda(n)} \equiv 1 \pmod{n}$. Since $d_0 = \mathrm{ord}_n(a)$, $d_0 \mid \lambda(n)$. If $d_0$ is even, let $d = d_0$. Suppose $d_0$ is odd. Since $13^t \equiv 1 \pmod{d_0}$ and $13^t \equiv 1 \pmod{2}$, we have $13^t \equiv 1 \pmod{2d_0}$. If $d_0$ is odd, let $d = 2d_0$. Clearly, $\mathrm{ord}_d(13) = t$ otherwise it contradict that $\mathrm{ord}_{d_0}(13) = t$. Since $d_0$ is odd, $d_0 \mid \lambda(n)$ and $\lambda(n)$ is even for $n > 2$, $d \mid \lambda(n)$. Thus, if $\gcd(a,n) = 1$ and $a$ is a vertex of a $t$-cycle in the digraph $\Gamma(n,13)$, then $t = \mathrm{ord}_d(13)$, where $d$ is an even positive divisor of $\lambda(n)$.

Case II: If $gcd(a, n) \neq 1$, then $a \in \Gamma_2(n, 13)$.
Let $n_1 = \gcd(a, n)$. Since $a$ is a vertex of some $t$-cycle in $\Gamma(n, 13)$, so $t$ is the least positive integer such that $a^{13^t} - a \equiv 0 \pmod{n}$, i.e., $a(a^{13^t-1}-1) \equiv 0 \pmod{n}$. Clearly $\gcd(a, a^{13^t-1}-1) = 1$. Consequently, if a prime $p$ divides both $a$ and $n$, then the full power of $p$ occurring in $n$ divides $a$. Thus, with $n_1 = \gcd(a, n)$ and $n_2 = n/n_1$, we have $\gcd(n_1, n_2) = 1$ and $\gcd(a, n_2) = 1$. Let $n_2 = \frac{n}{n_1}$. Then $t$ is the least positive integer such that

$$a \equiv 0 \pmod{n_1} \quad \text{and} \quad a^{13^t-1} - 1 \equiv 0 \pmod{n_2} \tag{2}$$

where $\gcd(n_1, n_2) = 1$. If possible, let $s < t$ be a positive integer such that $a \equiv 0 \pmod{n_1}$ and $a^{13^s-1} - 1 \equiv 0 \pmod{n_2}$. Then by Chinese Remainder Theorem, we get $a^{13^s} \equiv a \pmod{n}$, which is a contradiction. Therefore, from (2)

$$\begin{aligned} & a^{13^t-1} \equiv 1 \pmod{n_2} \\ \Rightarrow\ & d_0 \mid 13^t - 1, \text{where } d_0 = \mathrm{ord}_{n_2}(a) \\ \Rightarrow\ & 13^t \equiv 1 \pmod{d_0}, \text{where } t \text{ is the least such positive integer.} \\ \Rightarrow\ & \mathrm{ord}_{d_0}(13) = t \end{aligned}$$

If $d_0$ is even, let $d = d_0$. Suppose $d_0$ is odd. Since $13^t \equiv 1 \pmod{d_0}$ and $13^t \equiv 1 \pmod{2}$, we have $13^t \equiv 1 \pmod{2d_0}$. If $d_0$ is odd, let $d = 2d_0$. Since $a^{\lambda(n_2)} \equiv 1 \pmod{n_2}, d_0 \mid \lambda(n_2)$. Since $\lambda(n_2) \mid \lambda(n)$ and $\lambda(n)$ is even for $n > 2, d \mid \lambda(n)$. Then $t = \mathrm{ord}_d(13)$, where $d$ is an even positive divisor of $\lambda(n)$.

Hence in both cases, if $a$ is a vertex of a $t$-cycle in the digraph $\Gamma(n, 13)$, then $t = \mathrm{ord}_d(13)$, where $d$ is an even positive divisor of $\lambda(n)$.

Conversely, let $t = \mathrm{ord}_d(13)$, for some even positive divisor $d$ of $\lambda(n)$. By Theorem II of [3], there exists an integer $g$ such that $\mathrm{ord}_n(g) = \lambda(n)$. Let $a = g^{\frac{\lambda(n)}{d}}$. Since $\mathrm{ord}_n(g) = \lambda(n)$ and $d \mid \lambda(n)$, the standard formula for the order of a power gives

$$\mathrm{ord}_n(a) = \frac{\lambda(n)}{\gcd(\lambda(n), \lambda(n)/d)} = d.$$

Again, since $t = \mathrm{ord}_d(13)$, so $t$ is the least positive integer such that $13^t \equiv 1 \pmod{d} \Rightarrow d \mid 13^t - 1$ .

Thus, $t$ is the least positive integer such that $a^{13^t-1} \equiv 1 \pmod{n}$, which gives $a^{13^t} \equiv a \pmod{n}$. Hence, $a$ is a vertex of some $t$-cycle in the digraph $\Gamma(n, 13)$.

(ii) The proof follows directly from (i). □

**Lemma 3.1.** *For $r \geq 1$, $\mathrm{ord}_{2\cdot 3^r}(13) = 3^{r-1}$.*

*Proof.* It is easy to check that $\mathrm{ord}_6(13) = 1$. So, we assume $r \geq 2$. We need to show that $13^{3^{r-1}} \equiv 1 \pmod{2 \cdot 3^r}$ and $13^d \not\equiv 1 \pmod{2 \cdot 3^r}$ for every positive proper divisor $d$ of $3^{r-1}$. Since every proper divisor of $3^{r-1}$ divides $3^{r-2}$, it suffices to show that $13^d \not\equiv 1 \pmod{2 \cdot 3^r}$ for $d = 3^{r-2}$. Equivalently, we claim that $13^{3^{r-2}} \equiv 1 + 4 \cdot 3^{r-1} \pmod{2 \cdot 3^r}$ and we use induction on $r \geq 2$ to confirm this.

A calculation verifies that $13^{3^1} \equiv 1 \pmod{2 \cdot 3^2}$, and $13^{3^0} \equiv 1 + 4 \cdot 3^1 \pmod{2 \cdot 3^2}$. This establishes the base case, i.e., for $r = 2$.

As inductive step let us suppose that for $r \geq 2$, we have $13^{3^{r-1}} \equiv 1 \pmod{2 \cdot 3^r}$, and $13^{3^{r-2}} \equiv 1 + 4 \cdot 3^{r-1} \pmod{2 \cdot 3^r}$. We need to show that $13^{3^r} \equiv 1 \pmod{2 \cdot 3^{r+1}}$, and $13^{3^{r-1}} \equiv 1 + 4 \cdot 3^r \pmod{2 \cdot 3^{r+1}}$. By Lemma 2.2, we have $13^{3^r} \equiv 1^3 \equiv 1 \pmod{2 \cdot 3^{r+1}}$, and $13^{3^{r-1}} \equiv (1 + 4 \cdot 3^{r-1})^3 \pmod{2 \cdot 3^{r+1}}$. Now, the binomial expansion yields

$$13^{3^{r-1}} \equiv (1 + 4 \cdot 3^{r-1})^3 \equiv 1 + 4 \cdot 3^r \pmod{2 \cdot 3^{r+1}}$$

This proves the lemma. □

**Theorem 3.2.** *For $k \geq 1, \Gamma(3^k, 13)$ has $4k - 1$ cycles.*

*Proof.* There is a $t$-cycle in $\Gamma(3^k, 13)$ if and only if there exists an even positive divisor $d$ of $\lambda(3^k) = 2 \cdot 3^{k-1}$ such that $\mathrm{ord}_d(13) = t$. Clearly, $\mathrm{ord}_2(13) = 1$ and by Lemma 3.1, $\mathrm{ord}_{2 \cdot 3^i}(13) = 3^{i-1}$ for $1 \leq i \leq k - 1$. Therefore, $A_t(\Gamma(3^k, 13)) \geq 1$ if $t = 1$ or $3^{i-1}, 1 \leq i \leq k - 1$ and $A_t(\Gamma(3^k, 13)) = 0$, otherwise. By Theorem 2.2, we have

$$A_t(\Gamma(\widehat{n}, \widehat{k})) = \frac{1}{t}\Big[\prod_{i=1}^{\widehat{r}} \big(\delta_i \gcd(\lambda(p_i^{\alpha_i}), \widehat{k}^t - 1) + 1\big) - \sum_{d|t, d \neq t} dA_d\big(\Gamma(\widehat{n}, \widehat{k})\big)\Big]$$

Since $\widehat{n} = \prod_{i=1}^{\widehat{r}} p_1^{\alpha_i} = 3^k$, we have $\widehat{r} = 1, p_1 = 3$, and $\alpha_1 = k$. Also, $\widehat{k} = 13$ and $\delta_1 = 1$. Thus

$$A_t\big(\Gamma(3^k, 13)\big) = \frac{1}{t}\Big[\gcd\big(\lambda(3^k), 13^t - 1\big) + 1 - \sum_{d|t, d \neq t} dA_d\big(\Gamma(3^k, 13)\big)\Big].$$

For $k = 1$, 2 is the only even divisor of $\lambda(3^k)$ and $\mathrm{ord}_2(13) = 1$. So, the graph $\Gamma(3^k, 13)$ has only 1-cycles and the number of 1-cycles is equal to

$$\begin{aligned} A_1\big(\Gamma(3^k, 13)\big) &= \frac{1}{1}\Big[\gcd\big(\lambda(3^k), 13^1 - 1\big) + 1 - \sum_{d|1, d \neq 1} dA_d\big(\Gamma(3^k, 13)\big)\Big] \\ &= \Big[\gcd\big(2, 12\big) + 1 - 0\Big] = 2 + 1 - 0 = 3 = 4 \cdot 1 - 1 \end{aligned}$$

For $k = 2$, the only even divisor of $\lambda(3^k)$ are 2 and $2 \cdot 3$. As $\mathrm{ord}_2(13) = 1 = \mathrm{ord}_{2 \cdot 3}(13)$, therefore the graph has only 1-cycles and the number of 1-cycles is equal to

$$\begin{aligned} A_1\big(\Gamma(3^k, 13)\big) &= \frac{1}{1}\Big[\gcd\big(\lambda(3^k), 13^1 - 1\big) + 1 - \sum_{d|1, d \neq 1} dA_d\big(\Gamma(3^k, 13)\big)\Big] \\ &= \Big[\gcd\big(2 \cdot 3, 12\big) + 1 - 0\Big] = 6 + 1 - 0 = 7 = 4 \cdot 2 - 1 \end{aligned}$$

For $k \geq 3$, the graph has 1-cycles and cycles of length $3^{i-1}, 2 \leq i \leq k - 1$, i.e., $3^r, 1 \leq r \leq k - 2$. The number of 1-cycles is equal to

$$\begin{aligned} A_1\big(\Gamma(3^k, 13)\big) &= \frac{1}{1}\Big[\gcd\big(\lambda(3^k), 13^1 - 1\big) + 1 - \sum_{d|1, d \neq 1} dA_d\big(\Gamma(3^k, 13)\big)\Big] \\ &= \Big[\gcd\big(2 \cdot 3^{k-1}, 12\big) + 1 - 0\Big] = 2 \cdot 3 + 1 - 0 = 7 \end{aligned}$$

For $1 \leq r \leq k-2$, the number of cycles of length $3^r$ is given by

$$A_{3^r}\big(\Gamma(3^k,13)\big) = \frac{1}{3^r}\Big[\gcd\big(\lambda(3^k), 13^{3^r}-1\big) + 1 - \sum_{d|3^r, d\neq 3^r} dA_d\big(\Gamma(3^k,13)\big)\Big]$$
$$= \frac{1}{3^r}\Big[\gcd\big(\lambda(3^k), 13^{3^r}-1\big) + 1 - A_1\big(\Gamma(3^k,13)\big) - \sum_{i=1}^{r-1} 3^i A_{3^i}\big(\Gamma(3^k,13)\big)\Big]$$

We use induction method on $r$ to show that $A_{3^r}\big(\Gamma(3^k,13)\big) = 4$ for $1 \leq r \leq k-2$. For $r = 1$

$$A_3\big(\Gamma(3^k,13)\big) = \frac{1}{3}\Big[\gcd\big(\lambda(3^k), 13^3-1\big) + 1 - A_1\big(\Gamma(3^k,13)\big)\Big]$$
$$= \frac{1}{3}\Big[2\cdot 3^2 + 1 - 7\Big] = 4$$

Suppose $r \geq 1$ and $A_{3^i}\big(\Gamma(3^k,13)\big) = 4$ for all $0 \leq i < r$. We need to show $A_{3^r}\big(\Gamma(3^k,13)\big) = 4$. We have

$$A_{3^r}\big(\Gamma(3^k,13)\big) = \frac{1}{3^r}\Big[\gcd\big(2\cdot 3^{k-1}, 13^{3^r}-1\big) + 1 - 1\cdot A_1\big(\Gamma(3^k,13)\big) - \sum_{i=1}^{r-1} 3^i \cdot A_{3^i}\big(\Gamma(3^k,13)\big)\Big]$$

By Lemma 3.1, $\mathrm{ord}_{2\cdot 3^{r+1}}(13) = 3^r$ and $\mathrm{ord}_{2\cdot 3^{r+2}}(13) = 3^{r+1}$. Thus, $13^{3^r} \equiv 1 \pmod{2\cdot 3^{r+1}}$ and $13^{3^{r+1}} \equiv 1 \pmod{2\cdot 3^{r+2}}$, but $13^{3^r} \not\equiv 1 \pmod{2\cdot 3^{r+2}}$. Hence, $2\cdot 3^{r+1} \mid (13^{3^r}-1)$, but $2\cdot 3^{r+2} \nmid (13^{3^r}-1)$. Therefore, $\gcd\big(2\cdot 3^{k-1}, 13^{3^r}-1\big) = 2\cdot 3^{r+1}$. We have

$$A_{3^r}\big(\Gamma(3^k,13)\big) = \frac{1}{3^r}\Big[2\cdot 3^{r+1} + 1 - 1\cdot 7 - \sum_{i=1}^{r-1} 3^i\cdot 4\Big]$$
$$= \frac{1}{3^r}\Big[2\cdot 3^{r+1} + 1 - 3 - 4 - \sum_{i=1}^{r-1} 3^i\cdot 4\Big].$$
$$= \frac{1}{3^r}\Big[2\cdot 3^{r+1} + 1 - 3 - \sum_{i=0}^{r-1} 3^i\cdot 4\Big]$$
$$= \frac{1}{3^r}\Big[2\cdot 3^{r+1} + 1 - 3 - \frac{3^r-1}{3-1}\cdot 4\Big] = 4.$$

Thus, $A_1\big(\Gamma(3^k,13)\big) = 7$ and $A_{3^r}\big(\Gamma(3^k,13)\big) = 4, 1 \leq r \leq k-2$. So, by using the addition principle of counting, the total number of cycles in $\Gamma(3^k,13) = 7 + \underbrace{4+4+\cdots+4}_{(k-2)\text{ terms}} = 4k-1$. Hence, there are $(4k-1)$ number of cycles in the digraph $\Gamma(3^k,13), k \geq 1$. □

**Remark.** The graph $\Gamma(3^k,13)$ has seven 1-cycles and four cycles of length $3^r, 1 \leq r \leq k-2$ except for $k = 1$ and 2.

**Theorem 3.3.** *Suppose $k \geq 2$. The digraph $\Gamma_2(3^k,13)$ is a directed rooted in-tree with root 0. The in-degree of the root 0 is $3^{k-\lceil k/13\rceil}$. Let $k_1 = \lceil(\lceil k/13\rceil)/13\rceil$ and $k_2 = \lfloor(k-1)/13\rfloor$. Then the number of branch points (non-leaf vertices adjacent to the root 0) in $\Gamma_2(3^k,13)$ is given below:*

- *If $k \leq 13$, then $\Gamma_2(3^k,13)$ has no branch points.*

- *If $k > 13$, then the number of branch points in $\Gamma_2(3^k, 13)$ is*

$$2 \cdot 3^{k-13k_2-1} \cdot \big(3^{13(k_2-k_1+1)} - 1\big)/(3^{13} - 1).$$

*Furthermore, if $k > 13$, the number of leaves adjacent to the root $0$ is*

$$3^{k-\lceil k/13 \rceil} - 1 - 2 \cdot 3^{k-13k_2-1} \cdot \big(3^{13(k_2-k_1+1)} - 1\big)/(3^{13} - 1).$$

*Proof.* Since every nonzero vertex $v$ in $\Gamma_2(3^k, 13)$ is divisible by 3, there exist integers $i$ and $t$ such that $v = 3^i t, i \geq 1, 1 \leq t \leq 3^{k-i}$ and $\gcd(3, t) = 1$. Let $\ell = \lceil \log_{13} k \rceil$. Then $13^\ell \geq k$. Thus $v^{13^\ell} \equiv 3^{i13^\ell} t^{13^\ell} \equiv 0 \pmod{3^k}$. Hence, $\Gamma_2(3^k, 13)$ is a directed rooted in-tree with root $0$.

Suppose $3^i t$ is a vertex of $\Gamma_2(3^k, 13)$ such that $\lceil k/13 \rceil \leq i \leq k-1$ and $t$ is an integer satisfying $1 \leq t < 3^{k-i}$ and $\gcd(3, t) = 1$. Since $i < k$, $3^i t$ is not the root $0$. Also, since $i \geq \lceil k/13 \rceil$, we have $i \geq k/13$. Thus, $(3^i t)^{13} \equiv 3^{13i} t^{13} \equiv 0 \pmod{3^k}$. Hence, $3^i t$ is adjacent to the root $0$. On the other hand, if $i < \lceil k/13 \rceil$, then $13i \leq 13(\lceil k/13 \rceil - 1) < k$. Thus, $3^i t$ is not adjacent to the root $0$. Hence, for fixed $\lceil k/13 \rceil \leq i \leq k-1$, the number of distinct vertices of the form $3^i t$ adjacent to the root $0$ is $\phi(3^{k-i}) = 2 \cdot 3^{k-i-1}$. Therefore, the number of vertices adjacent to the root $0$ is

$$\sum_{i=\lceil k/13 \rceil}^{k-1} \phi(3^{k-i}) = \sum_{i=\lceil k/13 \rceil}^{k-1} 2 \cdot 3^{k-i-1} = 3^{k-\lceil k/13 \rceil} - 1.$$

Again, $0$ is a fixed point of $\Gamma_2(3^k, 13)$. Hence, the indegree of $0$ is $(3^{k-\lceil k/13 \rceil} - 1) + 1$ i.e., $3^{k-\lceil k/13 \rceil}$.

Suppose vertex $3^i t$ is adjacent to the root $0$ where $\lceil k/13 \rceil \leq i \leq k-1, 1 \leq t < 3^{k-i}$ and $\gcd(3, t) = 1$. If $13 \mid i$, then $i = 13j$ for some integer $j$ such that $\lceil k/13 \rceil \leq 13j \leq k-1$. Then $k_1 \leq j \leq k_2$. Since $\gcd(13, \phi(3^{k-13j})) = \gcd(13, 2 \cdot 3^{k-13j-1}) = 1$, the mapping $x \mapsto x^{13} \pmod{3^{k-13j}}$ is an automorphism of $U(3^{k-13j})$. So, there exists an integer $s$ in $U(3^{k-13j})$ such that $s^{13} \equiv t \pmod{3^{k-13j}}$. Thus, $(3^j s)^{13} \equiv 3^i t \pmod{3^k}$. Hence, $3^i t$ is a branch point of $\Gamma_2(3^k, 13)$. On the other hand, if $13 \nmid i$, then there is no vertex $3^j s$ such that $(3^j s)^{13} \equiv 3^i t \pmod{3^k}$. Thus, $3^i t$ is a leaf.

If $k \leq 13$, then every vertex of the form $3^{13j} t$, where $j$ is a positive integer, is the root $0$. Thus, $\Gamma_2(3^k, 13)$ has no branch points. On the other hand, if $k > 13$, we observe that the list of vertices of the form $3^{13j} t$, where $k_1 \leq j \leq k_2, 1 \leq t \leq 3^{k-13j}$ and $\gcd(3, t) = 1$, is a list of the distinct branch points of $\Gamma_2(3^k, 13)$. Hence, the number of branch points of $\Gamma_2(3^k, 13)$ is

$$\sum_{j=k_1}^{k_2} \phi(3^{k-13j}) = \sum_{j=k_1}^{k_2} 2 \cdot 3^{k-13j-1} = 2 \cdot 3^{k-13k_2-1} \cdot \big(3^{13(k_2-k_1+1)} - 1\big)/(3^{13} - 1).$$

Also, if $k > 13$, the number of leaves adjacent to the root $0$ is

$$3^{k-\lceil k/13 \rceil} - 1 - 2 \cdot 3^{k-13k_2-1} \cdot \big(3^{13(k_2-k_1+1)} - 1\big)/(3^{13} - 1). \qquad \square$$

**Lemma 3.2.** *Suppose $r \geq 1$. Then*

(a) $\operatorname{ord}_{2 \cdot 5^r}(13) = 4 \cdot 5^{r-1}$.

(b) $\operatorname{ord}_{4 \cdot 5^r}(13) = 4 \cdot 5^{r-1}$.

*Proof.* (a) It is easy to check that $\mathrm{ord}_{10}(13) = 4$. So, we assume $r \geq 2$. As $\gcd(13, 2 \cdot 5^r) = 1$, so by Euler's theorem we have $13^{4 \cdot 5^{r-1}} \equiv 1 \pmod{2 \cdot 5^r}$. We need to show that $13^d \not\equiv 1 \pmod{2 \cdot 5^r}$ for every positive proper divisor $d$ of $4 \cdot 5^{r-1}$. Since every proper divisor of $4 \cdot 5^{r-1}$ divides either $2 \cdot 5^{r-1}$ or $4 \cdot 5^{r-2}$, it suffices to show that $13^d \not\equiv 1 \pmod{2 \cdot 5^r}$ for $d \in \{2 \cdot 5^{r-1}, 4 \cdot 5^{r-2}\}$. Equivalently we claim that $13^{2 \cdot 5^{r-1}} \equiv -1 \pmod{2 \cdot 5^r}$ and $13^{4 \cdot 5^{r-2}} \equiv 1 + 2 \cdot 5^{r-1} \pmod{2 \cdot 5^r}$ and we use induction on $r \geq 2$ to confirm this.

A calculation verifies that $13^{2 \cdot 5^1} \equiv -1 \pmod{2 \cdot 5^2}$, and $13^{4 \cdot 5^0} \equiv 1 + 2 \cdot 5^1 \pmod{2 \cdot 5^2}$. This establishes the base case.

As inductive step let us suppose that for $r \geq 2$, we have $13^{2 \cdot 5^{r-1}} \equiv -1 \pmod{2 \cdot 5^r}$, and $13^{4 \cdot 5^{r-2}} \equiv 1 + 2 \cdot 5^{r-1} \pmod{2 \cdot 5^r}$. We need to show that $13^{2 \cdot 5^r} \equiv -1 \pmod{2 \cdot 5^{r+1}}$, and $13^{4 \cdot 5^{r-1}} \equiv 1 + 2 \cdot 5^r \pmod{2 \cdot 5^{r+1}}$. By Lemma 2.2, we have $13^{2 \cdot 5^r} \equiv (-1)^5 \pmod{2 \cdot 5^{r+1}}$ i.e., $13^{2 \cdot 5^r} \equiv -1 \pmod{2 \cdot 5^{r+1}}$, and $13^{4 \cdot 5^{r-1}} \equiv (1 + 2 \cdot 5^{r-1})^5 \pmod{2 \cdot 5^{r+1}}$. Now, the binomial expansion yields

$$13^{4 \cdot 5^{r-1}} \equiv (1 + 2 \cdot 5^{r-1})^5 \equiv 1 + 2 \cdot 5^r \pmod{2 \cdot 5^{r+1}}$$

(b) It is easy to check that $\mathrm{ord}_{20}(13) = 4$. So, we assume $r \geq 2$. We need to show that $13^{4 \cdot 5^{r-1}} \equiv 1 \pmod{4 \cdot 5^r}$ and $13^d \not\equiv 1 \pmod{4 \cdot 5^r}$ for every positive proper divisor $d$ of $4 \cdot 5^{r-1}$. Since every proper divisor of $4 \cdot 5^{r-1}$ divides either $2 \cdot 5^{r-1}$ or $4 \cdot 5^{r-2}$, it suffices to show that $13^d \not\equiv 1 \pmod{4 \cdot 5^r}$ for $d \in \{2 \cdot 5^{r-1}, 4 \cdot 5^{r-2}\}$ which is equivalent to show that $13^{2 \cdot 5^{r-1}} \equiv -1 + 2 \cdot 5^r \pmod{4 \cdot 5^r}$ and $13^{4 \cdot 5^{r-2}} \equiv 1 + 2^2 \cdot 3 \cdot 5^{r-1} \pmod{4 \cdot 5^r}$.

We use induction on $r \geq 2$ to confirm this. A calculation verifies that $13^{4 \cdot 5^1} \equiv 1 \pmod{4 \cdot 5^2}$, $13^{2 \cdot 5^1} \equiv -1 + 2 \cdot 5^2 \pmod{4 \cdot 5^2}$ and $13^{4 \cdot 5^0} \equiv 1 + 2 \cdot 5^1 \pmod{2 \cdot 5^2}$. Also $13^5 \not\equiv 1 \pmod{4 \cdot 5^2}$. This establishes the base case.

As inductive step, let us suppose that for $r \geq 2$, we have $13^{4 \cdot 5^{r-1}} \equiv 1 \pmod{4 \cdot 5^r}$, $13^{2 \cdot 5^{r-1}} \equiv -1 + 2 \cdot 5^r \pmod{4 \cdot 5^r}$, and $13^{4 \cdot 5^{r-2}} \equiv 1 + 2^2 \cdot 3 \cdot 5^{r-1} \pmod{4 \cdot 5^r}$. We need to show that $13^{4 \cdot 5^r} \equiv 1 \pmod{4 \cdot 5^{r+1}}$, $13^{2 \cdot 5^r} \equiv -1 + 2 \cdot 5^{r+1} \pmod{4 \cdot 5^{r+1}}$ and $13^{4 \cdot 5^{r-1}} \equiv 1 + 2^2 \cdot 3 \cdot 5^r \pmod{4 \cdot 5^{r+1}}$. By Lemma 2.2, we have $13^{4 \cdot 5^r} \equiv 1^5 \equiv 1 \pmod{4 \cdot 5^{r+1}}$, $13^{2 \cdot 5^r} \equiv (-1 + 2 \cdot 5^r)^5 \pmod{4 \cdot 5^{r+1}}$ and $13^{4 \cdot 5^{r-1}} \equiv (1 + 2^2 \cdot 3 \cdot 5^{r-1})^5 \pmod{4 \cdot 5^{r+1}}$. Now, the binomial expansion yields

$$13^{2 \cdot 5^r} \equiv (-1 + 2 \cdot 5^r)^5 \equiv -1 + 2 \cdot 5^{r+1} \pmod{4 \cdot 5^{r+1}}$$
$$13^{4 \cdot 5^{r-1}} \equiv (1 + 2^2 \cdot 3 \cdot 5^{r-1})^5 \equiv 1 + 2^2 \cdot 3 \cdot 5^r \pmod{4 \cdot 5^{r+1}}$$

This proves the lemma. □

**Theorem 3.4.** *For $k \geq 1, \Gamma(5^k, 13)$ has $4k + 1$ cycles.*

*Proof.* There is a $t$-cycle in $\Gamma(5^k, 13)$ if and only if there exists an even positive divisor $d$ of $\lambda(5^k) = 4 \cdot 5^{k-1}$ such that $\mathrm{ord}_d(13) = t$. Clearly, $\mathrm{ord}_2(13) = 1$ and $\mathrm{ord}_4(13) = 1$. By Lemma 3.2, $\mathrm{ord}_{2 \cdot 5^i}(13) = 4 \cdot 5^{i-1}$ and $\mathrm{ord}_{4 \cdot 5^i}(13) = 4 \cdot 5^{i-1}$ for $1 \leq i \leq k - 1$. Therefore, $A_t(\Gamma(5^k, 13)) \geq 1$

if $t = 1$ or $t = 4 \cdot 5^{i-1}, 1 \leq i \leq k-1$ i.e., $t = 4 \cdot 5^r, 0 \leq r \leq k-2$ and $A_t(\Gamma(5^k, 13)) = 0$, otherwise. By Theorem 2.2, we have

$$A_t\big(\Gamma(5^k,13)\big) = \frac{1}{t}\Big[\gcd\big(\lambda(5^k), 13^t - 1\big) + 1 - \sum_{d|t, d\neq t} dA_d\big(\Gamma(5^k,13)\big)\Big].$$

For $k = 1$, the graph has only 1-cycles and the number of 1-cycles is equal to

$$\begin{aligned} A_1\big(\Gamma(5^k,13)\big) &= \frac{1}{1}\Big[\gcd\big(\lambda(5^k), 13^1 - 1\big) + 1 - \sum_{d|1, d\neq 1} dA_d\big(\Gamma(5^k,13)\big)\Big] \\ &= \Big[\gcd\big(4 \cdot 5^{k-1}, 12\big) + 1 - 0\Big] = 4 + 1 - 0 = 5 \end{aligned}$$

For $k \geq 2$, the graph has 1-cycles and cycles of length $4 \cdot 5^r, 0 \leq r \leq k-2$. We have

$$\begin{aligned} A_1\big(\Gamma(5^k,13)\big) &= \frac{1}{1}\Big[\gcd\big(\lambda(5^k), 13^1 - 1\big) + 1 - \sum_{d|1, d\neq 1} dA_d\big(\Gamma(5^k,13)\big)\Big] \\ &= \Big[\gcd\big(4 \cdot 5^{k-1}, 12\big) + 1 - 0\Big] = 4 + 1 - 0 = 5 \end{aligned}$$

For $0 \leq r \leq k-2$

$$\begin{aligned} A_{4\cdot5^r}\big(\Gamma(5^k,13)\big) &= \frac{1}{4\cdot5^r}\Big[\gcd\big(\lambda(5^k), 13^{4\cdot5^r} - 1\big) + 1 - \sum_{d|4\cdot5^r, d\neq 4\cdot5^r} dA_d\big(\Gamma(5^k,13)\big)\Big] \\ &= \frac{1}{4\cdot5^r}\Big[\gcd\big(\lambda(5^k), 13^{4\cdot5^r} - 1\big) + 1 - A_1\big(\Gamma(5^k,13)\big) - \sum_{i=0}^{r-1} 4\cdot5^i A_{4\cdot5^i}\big(\Gamma(5^k,13)\big)\Big] \\ &= \frac{1}{4\cdot5^r}\Big[\gcd\big(\lambda(5^k), 13^{4\cdot5^r} - 1\big) + 1 - 5 - \sum_{i=0}^{r-1} 4\cdot5^i A_{4\cdot5^i}\big(\Gamma(5^k,13)\big)\Big] \end{aligned}$$

We use induction method on $r$ to show that $A_{4\cdot5^r}\big(\Gamma(5^k,13)\big) = 4$ for $0 \leq r \leq k-2$. For $r = 0$

$$\begin{aligned} A_4\big(\Gamma(5^k,13)\big) &= \frac{1}{4}\Big[\gcd\big(\lambda(5^k), 13^4 - 1\big) + 1 - 5\Big] \\ &= \frac{1}{4}\Big[4 \cdot 5 + 1 - 5\Big] = 4 \end{aligned}$$

As inductive step let us suppose $r \geq 1$ and $A_{4\cdot5^i}\big(\Gamma(5^k,13)\big) = 4$ for all $0 \leq i < r$. We need to show $A_{4\cdot5^r}\big(\Gamma(5^k,13)\big) = 4$. We have

$$A_{4\cdot5^r}\big(\Gamma(5^k,13)\big) = \frac{1}{4\cdot5^r}\Big[\gcd\big(4 \cdot 5^{k-1}, 13^{4\cdot5^r} - 1\big) + 1 - 5 - \sum_{i=0}^{r-1} 4\cdot5^i \cdot A_{4\cdot5^i}\big(\Gamma(5^k,13)\big)\Big]$$

By Lemma 3.2, $\mathrm{ord}_{4\cdot5^{r+1}}(13) = 4\cdot5^r$ and $\mathrm{ord}_{4\cdot5^{r+2}}(13) = 4\cdot5^{r+1}$. Thus, $13^{4\cdot5^r} \equiv 1 \pmod{4\cdot5^{r+1}}$ and $13^{4\cdot5^{r+1}} \equiv 1 \pmod{4\cdot5^{r+2}}$, but $13^{4\cdot5^r} \not\equiv 1 \pmod{4\cdot5^{r+2}}$. Hence, $4\cdot5^{r+1} \mid (13^{4\cdot5^r} - 1)$, but $4\cdot5^{r+2} \nmid (13^{4\cdot5^r} - 1)$. Therefore, $\gcd\big(4\cdot5^{k-1}, 13^{4\cdot5^r} - 1\big) = 4\cdot5^{r+1}$. We have

$$\begin{aligned} A_{4\cdot5^r}\big(\Gamma(5^k,13)\big) &= \frac{1}{4\cdot5^r}\Big[4\cdot5^{r+1} + 1 - 5 - \sum_{i=0}^{r-1} 4\cdot5^i \cdot 4\Big] \\ &= \frac{1}{4\cdot5^r}\Big[4\cdot5^{r+1} + 1 - 5 - \frac{5^r - 1}{5 - 1}\cdot 4^2\Big] = 4. \end{aligned}$$

Thus, $A_1\big(\Gamma(5^k,13)\big) = 5$ and $A_{4\cdot 5^r}\big(\Gamma(5^k,13)\big) = 4, 0 \le r \le k-2$. So, using the addition principle of counting, the total number of cycles in $\Gamma(5^k,13) = 5 + \underbrace{4+4+\cdots+4}_{(k-1)\text{ terms}} = 4k+1$. Hence, there are $(4k+1)$ number of cycles in the digraph $\Gamma(5^k,13), k \ge 1$. □

**Theorem 3.5.** *Suppose $k \ge 2$. The digraph $\Gamma_2(5^k,13)$ is a directed rooted in-tree with root 0. The in-degree of the root 0 is $5^{k-\lceil k/13\rceil}$. Let $k_1 = \lceil(\lceil k/13\rceil)/13\rceil$ and $k_2 = \lfloor(k-1)/13\rfloor$. Then the number of branch points in $\Gamma_2(5^k,13)$ is given below:*

- *If $k \le 13$, then $\Gamma_2(5^k,13)$ has no branch points.*
- *If $k > 13$, then the number of branch points in $\Gamma_2(5^k,13)$ is*

$$4\cdot 5^{k-13k_2-1}\cdot\big(5^{13(k_2-k_1+1)}-1\big)/(5^{13}-1).$$

*Furthermore, if $k > 13$, the number of leaves adjacent to the root 0 is*

$$5^{k-\lceil k/13\rceil} - 1 - 4\cdot 5^{k-13k_2-1}\cdot\big(5^{13(k_2-k_1+1)}-1\big)/(5^{13}-1).$$

*Proof.* The proof is similar to the proof of Theorem 3.3. □

**Lemma 3.3.** *For $r \ge 1$,*

(a) $\mathrm{ord}_{2\cdot 7^r}(13) = 2\cdot 7^{r-1}$.

(b) $\mathrm{ord}_{6\cdot 7^r}(13) = 2\cdot 7^{r-1}$.

*Proof.* (a) It is easy to check that $\mathrm{ord}_{2\cdot 7}(13) = 2$. So, we assume $r \ge 2$. We need to show that $13^{2\cdot 7^{r-1}} \equiv 1 \pmod{2\cdot 7^r}$ and $13^d \not\equiv 1 \pmod{2\cdot 7^r}$ for every positive proper divisor $d$ of $2\cdot 7^{r-1}$. Since every proper divisor of $2\cdot 7^{r-1}$ divides either $7^{r-1}$ or $2\cdot 7^{r-2}$, it suffices to show that $13^d \not\equiv 1 \pmod{2\cdot 7^r}$ for $d \in \{7^{r-1}, 2\cdot 7^{r-2}\}$ which is equivalent to claim that $13^{7^{r-1}} \equiv -1 \pmod{2\cdot 7^r}$ and $13^{2\cdot 7^{r-2}} \equiv 1 + 2\cdot 5\cdot 7^{r-1} \pmod{2\cdot 7^r}$.

We use induction on $r \ge 2$ to show the result. A calculation verifies that $13^{2\cdot 7^1} \equiv 1 \pmod{2\cdot 7^2}$, $13^{7^1} \equiv -1 \pmod{2\cdot 7^2}$. Also, $13^{2\cdot 7^0} \equiv 1 + 2\cdot 5\cdot 7^1 \pmod{2\cdot 7^2}$. This establishes the base case, i.e., $r = 2$.

As inductive step let us suppose that for $r \ge 2$, we have $13^{2\cdot 7^{r-1}} \equiv 1 \pmod{2\cdot 7^r}$, $13^{7^{r-1}} \equiv -1 \pmod{2\cdot 7^r}$ and $13^{2\cdot 7^{r-2}} \equiv 1 + 2\cdot 5\cdot 7^{r-1} \pmod{2\cdot 7^r}$. We need to show that $13^{2\cdot 7^r} \equiv 1 \pmod{2\cdot 7^{r+1}}$, $13^{7^r} \equiv -1 \pmod{2\cdot 7^{r+1}}$ and $13^{2\cdot 7^{r-1}} \equiv 1 + 2\cdot 5\cdot 7^r \pmod{2\cdot 7^{r+1}}$. By Lemma 2.2, we have $13^{2\cdot 7^r} \equiv 1^7 \equiv 1 \pmod{2\cdot 7^{r+1}}$, $13^{7^{r+1}} \equiv (-1)^7 \pmod{2\cdot 7^{r+1}}$ and $13^{2\cdot 7^{r-1}} \equiv (1 + 2\cdot 5\cdot 7^{r-1})^7 \pmod{2\cdot 7^{r+1}}$. Now, the binomial expansion yields

$$13^{2\cdot 7^{r-1}} \equiv (1 + 2\cdot 5\cdot 7^{r-1})^7 \equiv 1 + 2\cdot 5\cdot 7^r \pmod{2\cdot 7^{r+1}}$$

(b) It is easy to check that $\mathrm{ord}_{6\cdot 7}(13) = 2$. So, we assume $r \ge 2$. We need to show that $13^{2\cdot 7^{r-1}} \equiv 1 \pmod{6\cdot 7^r}$ and $13^d \not\equiv 1 \pmod{6\cdot 7^r}$ for every positive proper divisor $d$ of $2\cdot 7^{r-1}$. Since every proper divisor of $2\cdot 7^{r-1}$ divides either $7^{r-1}$ or $2\cdot 7^{r-2}$, it suffices to

show that $13^d \not\equiv 1 \pmod{6 \cdot 7^r}$ for $d \in \{7^{r-1}, 2 \cdot 7^{r-2}\}$, which is equivalent to show that $13^{7^{r-1}} \not\equiv 1 \pmod{6 \cdot 7^r}$ and $13^{2 \cdot 7^{r-2}} \equiv 1 + 2^3 \cdot 3 \cdot 7^{r-1} \pmod{6 \cdot 7^r}$.

From part (a), $13^{7^{r-1}} \equiv -1 \pmod{2 \cdot 7^r}$. Also, $13^{7^{r-1}} \equiv 1 \pmod{3}$. Therefore, by the Chinese remainder theorem $13^{7^{r-1}} \not\equiv 1 \pmod{6 \cdot 7^r}$.

We use induction to show that $13^{2 \cdot 7^{r-1}} \equiv 1 \pmod{6 \cdot 7^r}$, and $13^{2 \cdot 7^{r-2}} \equiv 1 + 2^3 \cdot 3 \cdot 7^{r-1} \pmod{6 \cdot 7^r}$. A calculation verifies that $13^{2 \cdot 7^1} \equiv 1 \pmod{6 \cdot 7^2}$, and $13^{2 \cdot 7^0} \equiv 1 + 2^3 \cdot 3 \cdot 7 \pmod{6 \cdot 7^2}$. This establishes the base case.

Suppose that for $r \geq 2$, we have $13^{2 \cdot 7^{r-1}} \equiv 1 \pmod{6 \cdot 7^r}$, and $13^{2 \cdot 7^{r-2}} \equiv 1 + 2^3 \cdot 3 \cdot 7^{r-1} \pmod{6 \cdot 7^r}$. We need to show that $13^{2 \cdot 7^r} \equiv 1 \pmod{6 \cdot 7^{r+1}}$, and $13^{2 \cdot 7^{r-1}} \equiv 1 + 2^3 \cdot 3 \cdot 7^r \pmod{6 \cdot 7^{r+1}}$. By Lemma 2.2, we have $13^{2 \cdot 7^r} \equiv 1^7 \equiv 1 \pmod{6 \cdot 7^{r+1}}$, and $13^{2 \cdot 7^{r-1}} \equiv (1 + 2^3 \cdot 3 \cdot 7^{r-1})^7 \pmod{6 \cdot 7^{r+1}}$. A binomial expansion yields

$$13^{2 \cdot 7^{r-1}} \equiv (1 + 2^3 \cdot 3 \cdot 7^{r-1})^7 \equiv 1 + 2^3 \cdot 3 \cdot 7^r \pmod{6 \cdot 7^{r+1}}$$

This proves the lemma. □

**Theorem 3.6.** *For $k \geq 1, \Gamma(7^k, 13)$ has $18k - 11$ cycles.*

*Proof.* There is a $t$-cycle in $\Gamma(7^k, 13)$ if and only if there exists an even positive divisor $d$ of $\lambda(7^k) = 6 \cdot 7^{k-1}$ such that $\mathrm{ord}_d(13) = t$. Clearly, $\mathrm{ord}_2(13) = 1$ and $\mathrm{ord}_6(13) = 1$. By Lemma 3.3, $\mathrm{ord}_{2 \cdot 7^r}(13) = 2 \cdot 7^{r-1}$ and $\mathrm{ord}_{6 \cdot 7^r}(13) = 2 \cdot 7^{r-1}$ for $1 \leq r \leq k-1$. Therefore, $A_1(\Gamma(7^k, 13)) \geq 1$, $A_t(\Gamma(7^k, 13)) \geq 1$ if $t = 2 \cdot 7^r, 0 \leq r \leq k-2$ and $A_t(\Gamma(7^k, 13)) = 0$, otherwise. By Theorem 2.2, we have

$$A_t\big(\Gamma(7^k, 13)\big) = \frac{1}{t}\Big[\gcd\big(\lambda(7^k), 13^t - 1\big) + 1 - \sum_{d|t, d \neq t} d A_d\big(\Gamma(7^k, 13)\big)\Big].$$

Clearly, for $k = 1$, the graph $\Gamma(7^k, 13)$ has only 1-cycles and the number of 1-cycles is equal to

$$\begin{aligned} A_1\big(\Gamma(7^k, 13)\big) &= \frac{1}{1}\Big[\gcd\big(\lambda(7^k), 13^1 - 1\big) + 1 - \sum_{d|1, d \neq 1} d A_d\big(\Gamma(7^k, 13)\big)\Big] \\ &= \Big[\gcd\big(6 \cdot 7^{k-1}, 12\big) + 1 - 0\Big] = 6 + 1 = 7 \end{aligned}$$

For $k \geq 2$, the graph $\Gamma(7^k, 13)$ has 1-cycles and cycles of length $2 \cdot 7^r, 0 \leq r \leq k-2$. We have

$$\begin{aligned} A_1\big(\Gamma(7^k, 13)\big) &= \frac{1}{1}\Big[\gcd\big(\lambda(7^k), 13^1 - 1\big) + 1 - \sum_{d|1, d \neq 1} d A_d\big(\Gamma(7^k, 13)\big)\Big] \\ &= \Big[\gcd\big(6 \cdot 7^{k-1}, 12\big) + 1 - 0\Big] = 6 + 1 = 7 \end{aligned}$$

For $0 \leq r \leq k-2$

$$\begin{aligned} A_{2 \cdot 7^r}\big(\Gamma(7^k, 13)\big) &= \frac{1}{2 \cdot 7^r}\Big[\gcd\big(\lambda(7^k), 13^{2 \cdot 7^r} - 1\big) + 1 - \sum_{d|2 \cdot 7^r, d \neq 2 \cdot 7^r} d A_d\big(\Gamma(7^k, 13)\big)\Big] \\ &= \frac{1}{2 \cdot 7^r}\Big[\gcd\big(\lambda(7^k), 13^{2 \cdot 7^r} - 1\big) + 1 - A_1\big(\Gamma(7^k, 13)\big) - \sum_{i=0}^{r-1} 2 \cdot 7^i A_{2 \cdot 7^i}\big(\Gamma(7^k, 13)\big)\Big] \\ &= \frac{1}{2 \cdot 7^r}\Big[\gcd\big(\lambda(7^k), 13^{2 \cdot 7^r} - 1\big) + 1 - 7 - \sum_{i=0}^{r-1} 2 \cdot 7^i A_{2 \cdot 7^i}\big(\Gamma(7^k, 13)\big)\Big] \end{aligned}$$

We use induction on $r$ to show that $A_{2\cdot 7^r}\big(\Gamma(7^k,13)\big)=18$ for $0\le r\le k-2$. For $r=0$

$$\begin{aligned}A_2\big(\Gamma(7^k,13)\big)&=\frac{1}{2}\Big[\gcd\big(\lambda(7^k),13^2-1\big)+1-7\Big]\\&=\frac{1}{2}\Big[6\cdot 7+1-7\Big]=18\end{aligned}$$

Suppose $r\ge 1$ and $A_{2\cdot 7^i}\big(\Gamma(7^k,13)\big)=18$ for all $0\le i<r$. We need to show $A_{2\cdot 7^r}\big(\Gamma(7^k,13)\big)=18$. We have

$$A_{2\cdot 7^r}\big(\Gamma(7^k,13)\big)=\frac{1}{2\cdot 7^r}\Big[\gcd\big(2\cdot 7^{k-1},13^{2\cdot 7^r}-1\big)+1-7-\sum_{i=0}^{r-1}2\cdot 7^i\cdot A_{2\cdot 7^i}\big(\Gamma(7^k,13)\big)\Big]$$

By Lemma 3.3, $\operatorname{ord}_{6\cdot 7^{r+1}}(13)=2\cdot 7^r$ and $\operatorname{ord}_{6\cdot 7^{r+2}}(13)=2\cdot 7^{r+1}$. Thus, $13^{2\cdot 7^r}\equiv 1\pmod{6\cdot 7^{r+1}}$ and $13^{2\cdot 7^{r+1}}\equiv 1\pmod{6\cdot 7^{r+2}}$, but $13^{2\cdot 7^r}\not\equiv 1\pmod{6\cdot 7^{r+2}}$. Hence, $6\cdot 7^{r+1}\mid(13^{2\cdot 7^r}-1)$, but $6\cdot 7^{r+2}\nmid(13^{2\cdot 7^r}-1)$. Therefore, $\gcd\big(6\cdot 7^{k-1},13^{2\cdot 7^r}-1\big)=6\cdot 7^{r+1}$. We have

$$\begin{aligned}A_{2\cdot 7^r}\big(\Gamma(7^k,13)\big)&=\frac{1}{2\cdot 7^r}\Big[6\cdot 7^{r+1}+1-7-\sum_{i=0}^{r-1}2\cdot 7^i\cdot 18\Big]\\&=\frac{1}{2\cdot 7^r}\Big[6\cdot 7^{r+1}+1-7-2\cdot\frac{7^r-1}{7-1}\cdot 18\Big]=18.\end{aligned}$$

Thus, $A_1\big(\Gamma(7^k,13)\big)=7$ and $A_{2\cdot 7^r}\big(\Gamma(7^k,13)\big)=18, 0\le r\le k-2$. So, using the addition principle of counting, the total number of cycles in $\Gamma(7^k,13)$ is $7+\underbrace{18+18+\cdots+18}_{(k-1)\text{ terms}}=18k-11$. Hence, there are $(18k-11)$ number of cycles in the digraph $\Gamma(7^k,13), k\ge 1$. □

**Theorem 3.7.** *Suppose $k\ge 2$. The digraph $\Gamma_2(7^k,13)$ is a directed rooted in-tree with root 0. The in-degree of the root 0 is $7^{k-\lceil k/13\rceil}$. Let $k_1=\lceil(\lceil k/13\rceil)/13\rceil$ and $k_2=\lfloor(k-1)/13\rfloor$. Then the number of branch points in $\Gamma_2(7^k,13)$ is given below:*

- *If $k\le 13$, then $\Gamma_2(7^k,13)$ has no branch points.*
- *If $k>13$, then the number of branch points in $\Gamma_2(7^k,13)$ is*

$$6\cdot 7^{k-13k_2-1}\cdot\big(7^{13(k_2-k_1+1)}-1\big)/(7^{13}-1).$$

*Furthermore, if $k>13$, the number of leaves adjacent to the root 0 is*

$$7^{k-\lceil k/13\rceil}-1-6\cdot 7^{k-13k_2-1}\cdot\big(7^{13(k_2-k_1+1)}-1\big)/(7^{13}-1).$$

*Proof.* The proof is similar to the proof of Theorem 3.3. □

**Lemma 3.4.** *For $r\ge 1$,*

(a) $\operatorname{ord}_{2\cdot 11^r}(13)=10\cdot 11^{r-1}$.

(b) $\operatorname{ord}_{10\cdot 11^r}(13)=20\cdot 11^{r-1}$.

*Proof.* (a) Since, $\gcd(13, 2 \cdot 11^r) = 1$ so by Euler's theorem $13^{10 \cdot 11^{r-1}} \equiv 1 \pmod{2 \cdot 11^r}$. It is easy to check that $\mathrm{ord}_{2 \cdot 11}(13) = 10$. So, we assume $r \geq 2$. We need to show that $13^d \not\equiv 1 \pmod{2 \cdot 11^r}$ for every proper positive divisor $d$ of $10 \cdot 11^{r-1}$. Since every proper divisor of $10 \cdot 11^{r-1}$ divides either $2 \cdot 11^{r-1}$, $5 \cdot 11^{r-1}$ or $10 \cdot 11^{r-2}$, it suffices to show that $13^d \not\equiv 1 \pmod{2 \cdot 11^r}$ for $d \in \{2 \cdot 11^{r-1}, 5 \cdot 11^{r-1}, 10 \cdot 11^{r-2}\}$ which is equivalent to claim that $13^{2 \cdot 11^{r-1}} \not\equiv 1 \pmod{2 \cdot 11^r}$, $13^{5 \cdot 11^{r-1}} \equiv -1 \pmod{2 \cdot 11^r}$, and $13^{10 \cdot 11^{r-2}} \equiv 1 + 10 \cdot 11^{r-1} \pmod{2 \cdot 11^r}$.

By Fermat's little theorem, $4^{11} \equiv 4 \pmod{11}$. Therefore, $13^{2 \cdot 11^{r-1}} \equiv 4^{11^{r-1}} \equiv 4 \pmod{11}$. If $13^{2 \cdot 11^{r-1}} \equiv 1 \pmod{2 \cdot 11^r}$, then $2 \cdot 11^r \mid (13^{2 \cdot 11^{r-1}} - 1)$ and hence $11 \mid (13^{2 \cdot 11^{r-1}} - 1)$, which is a contradiction. Therefore, $13^{2 \cdot 11^{r-1}} \not\equiv 1 \pmod{2 \cdot 11^r}$.

Again, $13^{5 \cdot 11^{r-1}} \equiv -1 \pmod{11}$. If $13^{5 \cdot 11^{r-1}} \equiv 1 \pmod{2 \cdot 11^r}$, then $2 \cdot 11^r \mid (13^{5 \cdot 11^{r-1}} - 1)$ and hence $11 \mid (13^{5 \cdot 11^{r-1}} - 1)$, which is a contradiction. Therefore, $13^{5 \cdot 11^{r-1}} \not\equiv 1 \pmod{2 \cdot 11^r}$. But $13^{10 \cdot 11^{r-1}} \equiv 1 \pmod{2 \cdot 11^r}$ so $13^{5 \cdot 11^{r-1}} \equiv -1 \pmod{2 \cdot 11^r}$.

We use induction on $r \geq 2$ to confirm that $13^{10 \cdot 11^{r-2}} \equiv 1 + 10 \cdot 11^{r-1} \pmod{2 \cdot 11^r}$. A calculation verifies that $13^{10 \cdot 11^0} \equiv 1 + 10 \cdot 11 \pmod{2 \cdot 11^2}$. This establishes the base case, i.e., $r = 2$.

Let us suppose that for $r \geq 2$, we have $13^{10 \cdot 11^{r-2}} \equiv 1 + 10 \cdot 11^{r-1} \pmod{2 \cdot 11^r}$. We need to show that $13^{10 \cdot 11^{r-1}} \equiv 1 + 10 \cdot 11^r \pmod{2 \cdot 11^{r+1}}$. By Lemma 2.2, we have $13^{10 \cdot 11^{r-1}} \equiv (1 + 10 \cdot 11^{r-1})^{11} \pmod{2 \cdot 11^{r+1}}$. Now, the binomial expansion yields

$$13^{10 \cdot 11^{r-1}} \equiv (1 + 10 \cdot 11^{r-1})^{11} \equiv 1 + 10 \cdot 11^r \pmod{2 \cdot 11^{r+1}}$$

This proves the lemma.

(b) It is easy to check that $\mathrm{ord}_{10 \cdot 11}(13) = 20$. So, we assume $r \geq 2$. We need to show that $13^{20 \cdot 11^{r-1}} \equiv 1 \pmod{10 \cdot 11^r}$ and $13^d \not\equiv 1 \pmod{10 \cdot 11^r}$ for every proper positive divisor $d$ of $20 \cdot 11^{r-1}$. Since every proper divisor of $20 \cdot 11^{r-1}$ divides either $4 \cdot 11^{r-1}$, $10 \cdot 11^{r-1}$ or $20 \cdot 11^{r-2}$, it suffices to show that $13^d \not\equiv 1 \pmod{10 \cdot 11^r}$ for $d \in \{4 \cdot 11^{r-1}, 10 \cdot 11^{r-1}, 20 \cdot 11^{r-2}\}$ which is equivalent to claim that $13^{4 \cdot 11^{r-1}} \not\equiv 1 \pmod{10 \cdot 11^r}$, $13^{10 \cdot 11^{r-1}} \not\equiv 1 \pmod{10 \cdot 11^r}$, and $13^{20 \cdot 11^{r-2}} \equiv 1 + 20 \cdot 11^{r-1} \pmod{10 \cdot 11^r}$.

By Fermat's little theorem, $5^{11} \equiv 5 \pmod{11}$, so $13^{4 \cdot 11^{r-1}} \equiv 5^{11} \equiv 5 \pmod{11}$. If $13^{4 \cdot 11^{r-1}} \equiv 1 \pmod{10 \cdot 11^r}$, then $10 \cdot 11^r \mid (13^{4 \cdot 11^{r-1}} - 1)$ which implies $11 \mid (13^{4 \cdot 11^{r-1}} - 1)$, which is a contradiction.

Again, $13^{10 \cdot 11^{r-1}} \equiv -1 \pmod{5}$, so $13^{10 \cdot 11^{r-1}} \not\equiv 1 \pmod{10 \cdot 11^r}$.

We used induction on $r \geq 2$ to show that $13^{20 \cdot 11^{r-1}} \equiv 1 \pmod{10 \cdot 11^r}$ and $13^{20 \cdot 11^{r-2}} \equiv 1 + 20 \cdot 11^{r-1} \pmod{10 \cdot 11^r}$. A calculation verifies that $13^{20 \cdot 11} \equiv 1 \pmod{10 \cdot 11^2}$, and $13^{20 \cdot 11^0} \equiv 1 + 20 \cdot 11 \pmod{10 \cdot 11^2}$. This established the base case.

Suppose that for $r \geq 2$, we have $13^{20 \cdot 11^{r-1}} \equiv 1 \pmod{10 \cdot 11^r}$ and $13^{20 \cdot 11^{r-2}} \equiv 1 + 20 \cdot 11^{r-1} \pmod{10 \cdot 11^r}$. We need to show that $13^{20 \cdot 11^r} \equiv 1 \pmod{10 \cdot 11^{r+1}}$ and $13^{20 \cdot 11^{r-1}} \equiv 1 + 20 \cdot 11^r \pmod{10 \cdot 11^{r+1}}$. By Lemma 2.2, we have $13^{20 \cdot 11^r} \equiv 1 \pmod{10 \cdot 11^{r+1}}$ and $13^{20 \cdot 11^{r-1}} \equiv (1 + 20 \cdot 11^{r-1})^{11} \equiv 1 + 20 \cdot 11^r \pmod{10 \cdot 11^{r+1}}$.
This proves the lemma. □

**Theorem 3.8.** *For $k \geq 1, \Gamma(11^k, 13)$ has $6k-1$ cycles.*

*Proof.* There is a $t$-cycle in $\Gamma(11^k, 13)$ if and only if there exists an even positive divisor $d$ of $\lambda(11^k) = 10 \cdot 11^{k-1}$ such that $\mathrm{ord}_d(13) = t$. Clearly, $\mathrm{ord}_2(13) = 1$ and $\mathrm{ord}_{10}(13) = 4$. By Lemma 3.4, $\mathrm{ord}_{2\cdot 11^i}(13) = 10 \cdot 11^{i-1}$ and $\mathrm{ord}_{10\cdot 11^i}(13) = 20 \cdot 11^{i-1}$ for $1 \leq i \leq k-1$. Therefore, $A_1(\Gamma(11^k, 13)) \geq 1$, $A_4(\Gamma(11^k, 13)) \geq 1$ $A_t(\Gamma(11^k, 13)) \geq 1$ if $t = 10 \cdot 11^r, 20 \cdot 11^r, 0 \leq r \leq k-2$ and $A_t(\Gamma(11^k, 13)) = 0$, otherwise. By Theorem 2.2, we have

$$A_t\big(\Gamma(11^k, 13)\big) = \frac{1}{t}\Big[\gcd\big(\lambda(11^k), 13^t - 1\big) + 1 - \sum_{d|t, d\neq t} dA_d\big(\Gamma(11^k, 13)\big)\Big].$$

For $k = 1$, the graph has 1-cycles and cycles of length 4. We have

$$\begin{aligned} A_1\big(\Gamma(11^k, 13)\big) &= \frac{1}{1}\Big[\gcd\big(\lambda(11^k), 13^1 - 1\big) + 1 - \sum_{d|1, d\neq 1} dA_d\big(\Gamma(11^k, 13)\big)\Big] \\ &= \Big[\gcd\big(10 \cdot 11^{k-1}, 12\big) + 1 - 0\Big] = 2 + 1 - 0 = 3 \end{aligned}$$

and

$$\begin{aligned} A_4\big(\Gamma(11^k, 13)\big) &= \frac{1}{4}\Big[\gcd\big(\lambda(11^k), 13^4 - 1\big) + 1 - \sum_{d|4, d\neq 4} dA_d\big(\Gamma(11^k, 13)\big)\Big] \\ &= \frac{1}{4}\Big[\gcd\big(10 \cdot 11^{k-1}, 13^4 - 1\big) + 1 - 1 \cdot 3 - 2 \cdot 0\Big] = \frac{1}{4}(10 + 1 - 3) = 2 \end{aligned}$$

Therefore, the total number of cycles is equal to $3 + 2 = 5$.

For $k \geq 2$, we have

$$\begin{aligned} A_1\big(\Gamma(11^k, 13)\big) &= \frac{1}{1}\Big[\gcd\big(\lambda(11^k), 13^1 - 1\big) + 1 - \sum_{d|1, d\neq 1} dA_d\big(\Gamma(11^k, 13)\big)\Big] \\ &= \Big[\gcd\big(10 \cdot 11^{k-1}, 12\big) + 1 - 0\Big] = 2 + 1 - 0 = 3 \end{aligned}$$

and

$$\begin{aligned} A_4\big(\Gamma(11^k, 13)\big) &= \frac{1}{4}\Big[\gcd\big(\lambda(11^k), 13^4 - 1\big) + 1 - \sum_{d|4, d\neq 4} dA_d\big(\Gamma(11^k, 13)\big)\Big] \\ &= \frac{1}{4}\Big[\gcd\big(10 \cdot 11^{k-1}, 13^4 - 1\big) + 1 - 1 \cdot 3 - 2 \cdot 0\Big] = \frac{1}{4}(10 + 1 - 3) = 2 \end{aligned}$$

For $0 \leq r \leq k-2$,

$$\begin{aligned} A_{10\cdot 11^r}\big(\Gamma(11^k, 13)\big) &= \frac{1}{10 \cdot 11^r}\Big[\gcd\big(\lambda(11^k), 13^{10\cdot 11^r} - 1\big) + 1 - \sum_{d|10\cdot 11^r, d\neq 10\cdot 11^r} dA_d\big(\Gamma(11^k, 13)\big)\Big] \\ &= \frac{1}{10 \cdot 11^r}\Big[\gcd\big(\lambda(11^k), 13^{10\cdot 11^r} - 1\big) + 1 - A_1\big(\Gamma(11^k, 13)\big) \\ &\quad - \sum_{i=0}^{r-1} 10 \cdot 11^i A_{10\cdot 11^i}\big(\Gamma(11^k, 13)\big)\Big] \\ &= \frac{1}{10 \cdot 11^r}\Big[\gcd\big(\lambda(11^k), 13^{10\cdot 11^r} - 1\big) + 1 - 3 \\ &\quad - \sum_{i=0}^{r-1} 10 \cdot 11^i A_{10\cdot 11^i}\big(\Gamma(11^k, 13)\big)\Big] \end{aligned}$$

We use induction on $r$ to show that $A_{10\cdot 11^r}\big(\Gamma(11^k,13)\big)=2$ for $0\le r\le k-2$. For $r=0$

$$\begin{aligned}A_{10}\big(\Gamma(11^k,13)\big)&=\frac{1}{10}\Big[\gcd\big(\lambda(11^k),13^{10}-1\big)+1-3\Big]\\&=\frac{1}{10}\Big[2\cdot 11+1-3\Big]=2\end{aligned}$$

Suppose $r\ge 1$ and $A_{10\cdot 11^i}\big(\Gamma(11^k,13)\big)=2$ for all $0\le i<r$. We need to show that $A_{10\cdot 11^r}\big(\Gamma(11^k,13)\big)=2$. We have

$$A_{10\cdot 11^r}\big(\Gamma(11^k,13)\big)=\frac{1}{10\cdot 11^r}\Big[\gcd\big(10\cdot 11^{k-1},13^{10\cdot 11^r}-1\big)+1-3-\sum_{i=0}^{r-1}10\cdot 11^i\cdot A_{10\cdot 11^i}\big(\Gamma(11^k,13)\big)\Big]$$

By Lemma 3.4, $\operatorname{ord}_{2\cdot 11^{r+1}}(13)=10\cdot 11^r$ and $\operatorname{ord}_{2\cdot 11^{r+2}}(13)=10\cdot 11^{r+1}$. Thus, $13^{10\cdot 11^r}\equiv 1 \pmod{2\cdot 11^{r+1}}$ and $13^{10\cdot 11^{r+1}}\equiv 1 \pmod{2\cdot 11^{r+2}}$, but $13^{10\cdot 11^r}\not\equiv 1 \pmod{2\cdot 11^{r+2}}$. Hence, $2\cdot 11^{r+1}\mid(13^{10\cdot 11^r}-1)$, but $2\cdot 11^{r+2}\nmid(13^{10\cdot 11^r}-1)$. Also, $13^{10\cdot 11^r}\equiv -1 \pmod 5$. Thus, $5\nmid(13^{10\cdot 11^r}-1)$. Therefore, $\gcd\big(10\cdot 11^{k-1},13^{10\cdot 11^r}-1\big)=2\cdot 11^{r+1}$. We have

$$\begin{aligned}A_{10\cdot 11^r}\big(\Gamma(11^k,13)\big)&=\frac{1}{10\cdot 11^r}\Big[2\cdot 11^{r+1}+1-3-\sum_{i=0}^{r-1}10\cdot 11^i\cdot 2\Big]\\&=\frac{1}{10\cdot 11^r}\Big[2\cdot 11^{r+1}+1-3-10\cdot\frac{11^r-1}{11-1}\cdot 2\Big]=2.\end{aligned}$$

Similarly,

$$\begin{aligned}A_{20\cdot 11^r}\big(\Gamma(11^k,13)\big)=\frac{1}{20\cdot 11^r}\Big[&\gcd\big(10\cdot 11^{k-1},13^{20\cdot 11^r}-1\big)+1-3-4\cdot 2-\\&\sum_{i=0}^{r}10\cdot 11^i\cdot A_{10\cdot 11^i}\big(\Gamma(11^k,13)\big)-\sum_{i=0}^{r-1}20\cdot 11^i\cdot A_{20\cdot 11^i}\big(\Gamma(11^k,13)\big)\Big]=4.\end{aligned}$$

Thus, $A_1\big(\Gamma(11^k,13)\big)=3$, $A_4\big(\Gamma(11^k,13)\big)=2$, $A_{10\cdot 11^r}\big(\Gamma(11^k,13)\big)=2, 0\le r\le k-2$ and $A_{20\cdot 11^r}\big(\Gamma(11^k,13)\big)=4, 0\le r\le k-2$. So, using the addition principle of counting, the total number of cycles in $\Gamma(11^k,13)=3+2+\underbrace{2+2+\cdots+2}_{(k-1)\text{ terms}}+\underbrace{4+4+\cdots+4}_{(k-1)\text{ terms}}=6k-1$. Hence, there are $(6k-1)$ number of cycles in the digraph $\Gamma(11^k,13), k\ge 1$. □

**Theorem 3.9.** *Suppose $k\ge 2$. The digraph $\Gamma_2(11^k,13)$ is a directed rooted in-tree with root $0$. The in-degree of the root $0$ is $11^{k-\lceil k/13\rceil}$. Let $k_1=\lceil(\lceil k/13\rceil)/13\rceil$ and $k_2=\lfloor(k-1)/13\rfloor$. Then the number of branch points in $\Gamma_2(11^k,13)$ is given below:*

- *If $k\le 13$, then $\Gamma_2(11^k,13)$ has no branch points.*
- *If $k>13$, then the number of branch points in $\Gamma_2(11^k,13)$ is*

$$10\cdot 11^{k-13k_2-1}\cdot\big(11^{13(k_2-k_1+1)}-1\big)/(11^{13}-1).$$

*Furthermore, if $k>13$, the number of leaves adjacent to the root $0$ is*

$$11^{k-\lceil k/13\rceil}-1-10\cdot 11^{k-13k_2-1}\cdot\big(11^{13(k_2-k_1+1)}-1\big)/(11^{13}-1).$$

*Proof.* The proof is similar to the proof of Theorem 3.3. □

**Lemma 3.5.** *If $r \geq 3$, then $\mathrm{ord}_{2^r}(13) = 2^{r-2}$.*

*Proof.* It suffices to establish

$$13^{2^{r-2}} \equiv 1 \pmod{2^r}, \qquad 13^{2^{r-3}} \equiv 1 + 2^{r-1} \pmod{2^r}.$$

Indeed, every positive proper divisor of $2^{r-2}$ divides $2^{r-3}$, and the second congruence shows that $13^{2^{r-3}} \not\equiv 1 \pmod{2^r}$.

We used induction on $r \geq 3$. A calculation verifies that $13^2 \equiv 1 \pmod{2^3}$ and $13^1 \equiv 5 \equiv 1 + 2^{3-1} \pmod{2^3}$, which establishes the base case.

Suppose that for $r \geq 3$, we have $13^{2^{r-2}} \equiv 1 \pmod{2^r}$ and $13^{2^{r-3}} \equiv 1 + 2^{r-1} \pmod{2^r}$. We need to show that $13^{2^{r-1}} \equiv 1 \pmod{2^{r+1}}$ and $13^{2^{r-2}} \equiv 1 + 2^r \pmod{2^{r+1}}$. By Lemma 2.2, we have $13^{2^{r-1}} \equiv 1 \pmod{2^{r+1}}$ and $13^{2^{r-2}} \equiv (1 + 2^{r-1})^2 \equiv 1 + 2^r \pmod{2^{r+1}}$. □

**Theorem 3.10.** *For any Fermat prime $p = 2^{2^m} + 1, m \geq 0$, $\Gamma(p, 13)$ has $2^{m+1} + 1$ cycles.*

*Proof.* There is a $t$-cycle in $\Gamma(p, 13)$ if and only if there exists an even positive divisor $d$ of $\lambda(p) = 2^{2^m}$ such that $\mathrm{ord}_d(13) = t$. Clearly, $\mathrm{ord}_2(13) = 1 = \mathrm{ord}_4(13)$ and by Lemma 3.5, $\mathrm{ord}_{2^i}(13) = 2^{i-2}$, $3 \leq i \leq 2^m$. Therefore, $A_t(\Gamma(p, 13)) \geq 1$ if $t = 1$ or $2^{i-2}, 3 \leq i \leq 2^m$, i.e., $2^r, 1 \leq r \leq 2^m - 2$ and $A_t(\Gamma(p, 13)) = 0$, otherwise.

We now apply Theorem 2.2 to count the cycles, considering the cases $m = 0$, $m = 1$, and $m \geq 2$ separately.

For $m = 0$, the graph $\Gamma(p, 13)$ has only 1-cycles and the number of 1-cycles is equal to $A_1(\Gamma(p, 13)) = \gcd(\lambda(p), 13 - 1) + 1 - 0 = 2 + 1 = 3 = 2^{0+1} + 1$.

For $m = 1$, the graph $\Gamma(p, 13)$ has only 1-cycles and the number of 1-cycles is equal to $A_1(\Gamma(p, 13)) = \gcd(\lambda(p), 13 - 1) + 1 - 0 = 4 + 1 = 5 = 2^{1+1} + 1$.

For $m \geq 2$, the graph $\Gamma(p, 13)$ has 1-cycles and cycles of length $2^r, 1 \leq r \leq 2^m - 2$. We have

$$\begin{aligned} A_1\big(\Gamma(p, 13)\big) &= \frac{1}{1}\Big[\gcd\big(\lambda(p), 13^1 - 1\big) + 1 - \sum_{d|1, d\neq 1} dA_d\big(\Gamma(p, 13)\big)\Big] \\ &= \Big[\gcd\big(2^{2^m}, 12\big) + 1 - 0\Big] = 5. \end{aligned}$$

$$\begin{aligned} A_2\big(\Gamma(p, 13)\big) &= \frac{1}{2}\Big[\gcd\big(\lambda(p), 13^2 - 1\big) + 1 - \sum_{d|2, d\neq 2} dA_d\big(\Gamma(p, 13)\big)\Big] \\ &= 2. \end{aligned}$$

By the method of induction on $r$, we have

$$A_{2^r}(\Gamma(p, 13)) = 2, 1 \leq r \leq 2^m - 2$$

Therefore, for a Fermat prime $p$, $\Gamma(p, 13)$ has five 1-cycles and two cycles of length $2^r$, $1 \leq r \leq 2^m - 2$. Therefore, the number of cycles is equal to $\Gamma(p, 13) = 5 + \underbrace{2 + 2 + \cdots + 2}_{(2^m - 2) \text{ terms}} =$ $2^{m+1} + 1$. □

# 4 Conclusion

In this paper, we have tried to formulate an explicit formula to enumerate the cyclic components of the digraphs $\Gamma(p^k, 13)$ where $p < 13$ is an odd prime and $k$ is any positive number. We also studied the number of cyclic components of the digraph $\Gamma(p, 13)$ where $p$ is a Fermat prime.


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# References


[1] Blanton Jr, E. L., Hurd, S. P., & McCranie, J. S. (1992). On a digraph defined by squaring modulo $n$. *The Fibonacci Quarterly*, 30(4), 322–334 .

[2] Bryant, S. (1967). Groups, graphs and Fermat's last theorem. *The American Mathematical Monthly*, 74(2), 152–156.

[3] Carmichael, R. D. (1910). Note on a new number theory function. *Bulletin of the American Mathematical Society*, 16(5), 232–238.

[4] Goswami, P., Thakur, S. K., & Ray, G. C. (2023). The structure of power digraph connected with the congruence $a^{11} \equiv b \pmod{n}$. *Proyecciones (Antofagasta)*, 42(2), 457–477.

[5] Harary, F. (1969). *Graph Theory*. Addison-Wesley Publication Company, London.

[6] Křížek, M., Luca, F., & Somer, L. (2001). *17 Lectures on Fermat Numbers: From Number Theory to Geometry*. CMS Books in Mathematics, Vol. 9, Springer-Verlag, New York.

[7] Křížek, M., & Somer, L. (2004). Sophie Germain little suns. *Mathematica Slovaca*, 54(5), 433–442.

[8] Lucheta, C., Miller, E., & Reiter, C. (1996). Digraphs from powers modulo $p$. *The Fibonacci Quarterly*, 34, 226–239.

[9] Mateen, M. H., & Mahmood, M. K. (2019). Power digraphs associated with the congruence $x^n \equiv y \pmod{m}$. *Punjab University Journal of Mathematics*, 51(5), 93–102.

[10] Rahmati, M. (2017). Some digraphs attached with the congruence $x^5 \equiv y \pmod{n}$. *Journal of Mathematical Extension*, 11(1), 47–56.

[11] Rogers, T. D. (1996). The graph of the square mapping on the prime fields. *Discrete Mathematics*, 148, 317–324.

[12] Sawkmie, A., & Singh, M. M. (2015). On the tree structure of the power digraphs modulo $n$. *Czechoslovak Mathematical Journal*, 65(4), 923–945.

[13] Sawkmie, A., & Singh, M. M. (2018). On the uniqueness of the factorization of power digraphs modulo $n$. *Rendiconti del Seminario Matematico della Università di Padova*, 140, 185–219.

[14] Skowronek-Kaziów, J. (2009). Properties of digraphs connected with some congruence relations. *Czechoslovak Mathematical Journal*, 59, 39–49.

[15] Somer, L., & Křížek, M. (2004). On a connection of number theory with graph theory. *Czechoslovak Mathematical Journal*, 54, 465–485.

[16] Somer, L., & Křížek, M. (2009). On symmetric digraphs of the congruence $x^k \equiv y \pmod{n}$. *Discrete Mathematics*, 309, 1999–2009.

[17] Somer, L., & Křížek, M. (2011). The structure of digraphs associated with the congruence $x^k \equiv y \pmod{n}$. *Czechoslovak Mathematical Journal*, 61, 337–358.

[18] Szalay, L. (1992). Egy diszkrét iteráció a számelméletben. *Berzsenyi Dániel Tanárképző Főiskola Tudományos Közleményei*, 8, 71–91 (in Hungarian).

[19] Thakur, S. K., Goswami, P., & Ray, G. C. (2023). Enumeration of cyclic vertices and components over the congruence $a^{11} \equiv b \pmod{n}$. *Notes on Number Theory and Discrete Mathematics*, 29(3), 525–537.