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On a conjecture involving twin practical numbers

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Abstract: In this paper, we state a conjecture that for any positive integer p, there are infinitely many practical numbers q such that both q^p and $q^p + 2$ are practical numbers. We prove the conjecture when p is a prime number between 7 and 100, refining a result involving twin practical numbers of Wang and Sun. We also prove several results related to this conjecture.
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1 Introduction

A positive integer m is called a *practical number* if each $n = 1, \ldots, m$ can be written as the sum of some distinct divisors of m. Clearly all practical numbers are even except 1, and all even perfect numbers and powers of 2 are practical. Let $P(x)$ denote the number of practical numbers less than x. In 1984, Hausman and Shapiro [1] first gave an upper bound for $P(x)$. They showed that

$$P(x) \ll \frac{x}{(\log x)^{\beta}} \tag{1}$$

for any $\beta < \frac{1}{2}(1 - \frac{1}{\log 2})^2 \approx 0.0979$. A lower bound was first given by Margenstern [2] in 1991 such that

$$P(x) \gg \frac{x}{\exp\left(\frac{1}{2\log 2}(\log\log x)^2 + 3\log\log x\right)}. \tag{2}$$

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Tenenbaum [7, 8] improved the bounds (1) and (2) to

$$\frac{x}{\log x}(\log\log x)^{-\frac{5}{3}-\varepsilon} \ll P(x) \ll \frac{x}{\log x}\log\log x\log\log\log x. \tag{3}$$

Using a sieve method, Saias [6] improved Tenenbaum's result to

$$P(x) \asymp \frac{x}{\log x}. \tag{4}$$

Margenstern conjectured that the asymptotic formula

$$P(x) \sim \lambda_1 \frac{x}{\log x} \tag{5}$$

holds with some λ_1, and in 2020 Weingartner [10] proved this conjecture with $\lambda_1 \approx 1.33607$.

Like the famous twin prime conjecture, Margenstern [2] established that there are infinitely many twin practical numbers m and $m+2$, and another detailed proof was given by Melfi [3] in 1996. Let $P_2(x)$ denote the number of practical number pairs $(m, m+2)$ less than x. Margenstern's proof leads to a lower bound $P_2(x) \gg \log\log x$, and Melfi's proof leads to a better lower bound $P_2(x) \gg \log x$. In 2002, Melfi [4] improved the lower bound to

$$P_2(x) \gg \frac{x}{\exp\left(k(\log x)^{\frac{1}{2}}\right)} \tag{6}$$

with $k > 2 + \log\frac{3}{2}$. Margenstern also conjectured that the asymptotic formula

$$P_2(x) \sim \lambda_2 \frac{x}{(\log x)^2} \tag{7}$$

holds for some λ_2. The best unconditional result now is due to Pomerance and Weingartner [5], where they showed

$$\lambda_2 \frac{x}{(\log x)^{9.5367}} \ll P_2(x) \ll \lambda_2 \frac{x}{(\log x)^2}. \tag{8}$$

In 2022, Wang and Sun [9] generalized Margenstern's result and showed that there are infinitely many practical numbers q such that q^4 and q^4+2 are also practical numbers. They showed their result by modifying Melfi's cyclotomic method.

In this paper, we further state a conjecture and show some new results.

Conjecture 1.1. *For any positive integer C, there are infinitely many practical numbers q such that q^C and q^C+2 are also practical numbers.*

Clearly the results of Margenstern [2] and Melfi [3,4] imply the case $C=1$ of Conjecture 1.1, and Wang and Sun [9] solved the cases $C=2$ and $C=4$. Wang and Sun [9] solved the cases $C=2$ and $C=4$ by proving that the sequence $2^{35\times 3^k+1}+2$ are practical for every integer $k \geqslant 0$. In this paper, we find four new sequences that generate infinitely many practical numbers.

Theorem 1.1. $2^{135\times 7^k+1}+2$, $2^{165\times 3^k+1}+2$, $2^{175\times 3^k+1}+2$ *and* $2^{189\times 5^k+1}+2$ *are practical for every integer $k \geqslant 0$.*

Using Theorem 1.1 we can easily deduce the following three results.

Theorem 1.2. *Let p denote a prime number, $7 \leqslant p < 100$. Then there are infinitely many practical numbers q such that q^p and $q^p + 2$ are also practical numbers.*

Corollary 1.1. *There are infinitely many practical numbers q such that:*

(1) *q^3 and $q^3 + 8$ are also practical numbers;*

(2) *q^5 and $q^5 + 32$ are also practical numbers.*

Corollary 1.2. *For any positive integer C, there exists an integer h such that there are infinitely many practical numbers q such that q^C and $q^C + 2^h$ are also practical numbers.*

2 Proof of Theorem 1.1

In order to prove our Theorems, we need the following structure theorem:

Lemma 2.1. [3, Lemma 1]*Let m be any practical number. Then mn is practical for every $n = 1, \ldots, \sigma(m) + 1$. In particular, mn is practical for every $1 \leqslant n \leqslant 2m$.*

Let $\Phi_m(x)$ denote the m-th cyclotomic polynomial, then we have

$$x^n - 1 = \prod_{d|n} \Phi_d(x)$$

for all $n \in \mathbb{N}^+$.

Now we start our proof of Theorem 1.1. Note that we may use m_k to represent different sequences in different subsections.

2.1 Sequence 1

We shall prove by induction that $2^{135\times 7^k+1} + 2$ is practical for every $k \geqslant 0$.

Write $m_k = 2^{135\times 7^k+1} + 2$ for $k = 0, 1, 2, \ldots$, then we only need to show that m_k is practical for every $k = 0, 1, 2, \ldots$. By a computer calculation we find that

$$\begin{aligned} m_0 &= 871122859317602466466238995025326621327 38 \\ &= 2 \times 3^4 \times 11 \times 19 \times 331 \times 811 \times 15121 \times 87211 \times 18837001 \times 385838642647891. \end{aligned}$$

By Lemma 2.1, we know that m_0 is practical. Let $x = 2^{7^k}$, then we have $m_k = 2(x^{135} + 1)$ and $m_{k+1} = 2(x^{945} + 1)$. Assume that $2(x^{135} + 1)$ is practical, then we only need to show that

$$2(x^{135} + 1)\Phi_{14}(x)\Phi_{42}(x)\Phi_{70}(x)\Phi_{126}(x)\Phi_{210}(x)\Phi_{378}(x)\Phi_{630}(x)\Phi_{1890}(x)$$

is practical.

Note that

$$\begin{aligned} \Phi_{14}(x) &= x^6 - x^5 + x^4 - x^3 + x^2 - x + 1, \\ \Phi_{42}(x) &= x^{12} + x^{11} - x^9 - x^8 + x^6 - x^4 - x^3 + x + 1, \\ \Phi_{70}(x) &= x^{24} + x^{23} - x^{19} - x^{18} - x^{17} - x^{16} + x^{14} + x^{13} + x^{12} + x^{11} \\ &\quad + x^{10} - x^8 - x^7 - x^6 - x^5 + x + 1, \\ \Phi_{126}(x) &= x^{36} + x^{33} - x^{27} - x^{24} + x^{18} - x^{12} - x^9 + x^3 + 1, \end{aligned}$$

$$\begin{aligned}
\Phi_{210}(x) &= x^{48} - x^{47} + x^{46} + x^{43} - x^{42} + 2x^{41} - x^{40} + x^{39} + x^{36} - x^{35} \\
&\quad + x^{34} - x^{33} + x^{32} - x^{31} - x^{28} - x^{26} - x^{24} - x^{22} - x^{20} - x^{17} \\
&\quad + x^{16} - x^{15} + x^{14} - x^{13} + x^{12} + x^{9} - x^{8} \\
&\quad + 2x^{7} - x^{6} + x^{5} + x^{2} - x + 1, \\
\Phi_{378}(x) &= x^{108} + x^{99} - x^{81} - x^{72} + x^{54} - x^{36} - x^{27} + x^{9} + 1, \\
\Phi_{630}(x) &= x^{144} - x^{141} + x^{138} + x^{129} - x^{126} + 2x^{123} - x^{120} \\
&\quad + x^{117} + x^{108} - x^{105} + x^{102} - x^{99} + x^{96} - x^{93} - x^{84} \\
&\quad - x^{78} - x^{72} - x^{66} - x^{60} - x^{51} + x^{48} - x^{45} + x^{42} \\
&\quad - x^{39} + x^{36} + x^{27} - x^{24} + 2x^{21} - x^{18} + x^{15} + x^{6} - x^{3} + 1, \\
\Phi_{1890}(x) &= x^{432} - x^{423} + x^{414} + x^{387} - x^{378} + 2x^{369} - x^{360} + x^{351} \\
&\quad + x^{324} - x^{315} + x^{306} - x^{297} + x^{288} - x^{279} - x^{252} - x^{234} \\
&\quad - x^{216} - x^{198} - x^{180} - x^{153} + x^{144} - x^{135} + x^{126} - x^{117} \\
&\quad + x^{108} + x^{81} - x^{72} + 2x^{63} - x^{54} + x^{45} + x^{18} - x^{9} + 1.
\end{aligned}$$

Since $x \geqslant 2$, we have $2x^{n-1} \leqslant x^n \leqslant \frac{x^{n+1}}{2}$ for every $n \in \mathbb{N}$, and thus

$$\frac{1}{2}x^6 \leqslant \Phi_{14}(x) \leqslant x^6, \tag{9}$$

$$x^{12} \leqslant \Phi_{42}(x) \leqslant 2x^{12}, \tag{10}$$

$$x^{24} \leqslant \Phi_{70}(x) \leqslant 2x^{24}, \tag{11}$$

$$x^{36} \leqslant \Phi_{126}(x) \leqslant 2x^{36}, \tag{12}$$

$$\frac{1}{2}x^{48} \leqslant \Phi_{210}(x) \leqslant x^{48}, \tag{13}$$

$$x^{108} \leqslant \Phi_{378}(x) \leqslant 2x^{108}, \tag{14}$$

$$\frac{1}{2}x^{144} \leqslant \Phi_{630}(x) \leqslant x^{144}, \tag{15}$$

$$\frac{1}{2}x^{432} \leqslant \Phi_{1890}(x) \leqslant x^{432}. \tag{16}$$

By (9)–(13), we have

$$\Phi_{14}(x)\Phi_{42}(x)\Phi_{70}(x)\Phi_{126}(x)\Phi_{210}(x) \leqslant 8x^{126} \leqslant x^{129} \leqslant 2(x^{135}+1) \tag{17}$$

and

$$2(x^{135}+1)\Phi_{14}(x)\Phi_{42}(x)\Phi_{70}(x)\Phi_{126}(x)\Phi_{210}(x)$$

is practical by Lemma 2.1.

By (9)–(13), we have

$$2(x^{135}+1)\Phi_{14}(x)\Phi_{42}(x)\Phi_{70}(x)\Phi_{126}(x)\Phi_{210}(x) \geqslant \frac{1}{2}x^{261} \geqslant x^{260}. \tag{18}$$

By (14)–(15), we have

$$\Phi_{378}(x)\Phi_{630}(x) \leqslant 2x^{252} \leqslant x^{253}. \tag{19}$$

By (18)–(19), we have

$$\Phi_{378}(x)\Phi_{630}(x) \leqslant x^{253} \leqslant x^{260} \leqslant 2(x^{135}+1)\Phi_{14}(x)\Phi_{42}(x)\Phi_{70}(x)\Phi_{126}(x)\Phi_{210}(x) \tag{20}$$

and

$$2(x^{135}+1)\Phi_{14}(x)\Phi_{42}(x)\Phi_{70}(x)\Phi_{126}(x)\Phi_{210}(x)\Phi_{378}(x)\Phi_{630}(x)$$

is practical by Lemma 2.1.

By (9)–(15), we have

$$2(x^{135}+1)\Phi_{14}(x)\Phi_{42}(x)\Phi_{70}(x)\Phi_{126}(x)\Phi_{210}(x)\Phi_{378}(x)\Phi_{630}(x) \geqslant \frac{1}{4}x^{513} \geqslant x^{511}. \tag{21}$$

By (16) and (21), we have

$$\Phi_{1890}(x) \leqslant x^{432} \leqslant x^{511} \leqslant 2(x^{135}+1)\Phi_{14}(x)\Phi_{42}(x)\Phi_{70}(x)\Phi_{126}(x)\Phi_{210}(x)\Phi_{378}(x)\Phi_{630}(x) \tag{22}$$

and

$$2(x^{135}+1)\Phi_{14}(x)\Phi_{42}(x)\Phi_{70}(x)\Phi_{126}(x)\Phi_{210}(x)\Phi_{378}(x)\Phi_{630}(x)\Phi_{1890}(x)$$

is practical by Lemma 2.1.

Since m_0 is practical, we can show that m_k is practical for every $k = 0, 1, 2, \ldots$ by an inductive process. Now the proof of sequence $2^{135\times 7^k+1}+2$ is completed.

2.2 Sequence 2

We shall prove by induction that $2^{165\times 3^k+1}+2$ is practical for every $k \geqslant 0$.

Write $m_k = 2^{165\times 3^k+1}+2$ for $k = 0, 1, 2, \ldots$, then we only need to show that m_k is practical for every $k = 0, 1, 2, \ldots$. By a computer calculation we find that

$$\begin{aligned} m_0 &= 935361047891777867650358292938421132579796827504 66 \\ &= 2 \times 3^2 \times 11^2 \times 67 \times 331 \times 683 \times 2971 \times 20857 \times 48912491 \\ &\quad \times 415365721 \times 2252127523412251. \end{aligned}$$

By Lemma 2.1, we know that m_0 is practical. Let $x = 2^{3^k}$, then we have $m_k = 2(x^{165}+1)$ and $m_{k+1} = 2(x^{495}+1)$. Assume that $2(x^{165}+1)$ is practical, then we only need to show that

$$2(x^{165}+1)\Phi_{18}(x)\Phi_{90}(x)\Phi_{198}(x)\Phi_{990}(x)$$

is practical.

Note that

$$\begin{aligned} \Phi_{18}(x) &= x^6 - x^3 + 1, \\ \Phi_{90}(x) &= x^{24} + x^{21} - x^{15} - x^{12} - x^9 + x^3 + 1, \\ \Phi_{198}(x) &= x^{60} + x^{57} - x^{51} - x^{48} + x^{42} + x^{39} - x^{33} \\ &\quad - x^{30} - x^{27} + x^{21} + x^{18} - x^{12} - x^9 + x^3 + 1, \end{aligned}$$

$$\begin{aligned}\Phi_{990}(x) = {} & x^{240} - x^{237} + x^{234} + x^{225} - x^{222} + x^{219} \\ & + x^{207} - x^{204} + x^{201} - x^{195} + 2x^{192} - 2x^{189} \\ & + x^{186} - x^{180} + x^{177} - x^{174} - x^{162} + x^{159} \\ & - x^{156} + x^{150} - 2x^{147} + 2x^{144} - 2x^{141} + x^{138} \\ & - x^{132} + x^{129} - x^{126} + x^{123} - x^{120} + x^{117} \\ & - x^{114} + x^{111} - x^{108} + x^{102} - 2x^{99} + 2x^{96} \\ & - 2x^{93} + x^{90} - x^{84} + x^{81} - x^{78} - x^{66} + x^{63} \\ & - x^{60} + x^{54} - 2x^{51} + 2x^{48} - x^{45} + x^{39} - x^{36} \\ & + x^{33} + x^{21} - x^{18} + x^{15} + x^{6} - x^{3} + 1.\end{aligned}$$

Since $x \geqslant 2$, we have $2x^{n-1} \leqslant x^n \leqslant \frac{x^{n+1}}{2}$ for every $n \in \mathbb{N}$, and thus

$$\frac{1}{2}x^6 \leqslant \Phi_{18}(x) \leqslant x^6, \tag{23}$$

$$x^{24} \leqslant \Phi_{90}(x) \leqslant 2x^{24}, \tag{24}$$

$$x^{60} \leqslant \Phi_{198}(x) \leqslant 2x^{60}, \tag{25}$$

$$\frac{1}{2}x^{240} \leqslant \Phi_{990}(x) \leqslant x^{240}. \tag{26}$$

By (23)–(25), we have

$$\Phi_{18}(x)\Phi_{90}(x)\Phi_{198}(x) \leqslant 4x^{90} \leqslant x^{92} \leqslant 2(x^{165}+1) \tag{27}$$

and

$$2(x^{165}+1)\Phi_{18}(x)\Phi_{90}(x)\Phi_{198}(x)$$

is practical by Lemma 2.1.

By (23)–(25), we have

$$2(x^{165}+1)\Phi_{18}(x)\Phi_{90}(x)\Phi_{198}(x) \geqslant x^{255}. \tag{28}$$

By (26) and (28), we have

$$\Phi_{990}(x) \leqslant x^{240} \leqslant x^{255} \leqslant 2(x^{165}+1)\Phi_{18}(x)\Phi_{90}(x)\Phi_{198}(x) \tag{29}$$

and

$$2(x^{165}+1)\Phi_{18}(x)\Phi_{90}(x)\Phi_{198}(x)\Phi_{990}(x)$$

is practical by Lemma 2.1.

Since m_0 is practical, we can show that m_k is practical for every $k = 0, 1, 2, \ldots$ by an inductive process. Now the proof of sequence $2^{165\times 3^k+1} + 2$ is completed.

2.3 Sequence 3

We shall prove by induction that $2^{175\times 3^k+1} + 2$ is practical for every $k \geqslant 0$.

Write $m_k = 2^{175\times 3^k+1} + 2$ for $k = 0, 1, 2, \ldots$, then we only need to show that m_k is practical for every $k = 0, 1, 2, \ldots$. By a computer calculation we find that

$$\begin{aligned} m_0 &= 957809713041180536473966891968943239761711951364751 38 \\ &= 2 \times 3 \times 11 \times 43 \times 251 \times 281 \times 1051 \times 4051 \times 86171 \times 110251 \\ &\quad \times 347833278451 \times 34010032331525251. \end{aligned}$$

By Lemma 2.1, we know that m_0 is practical. Let $x = 2^{3^k}$, then we have $m_k = 2(x^{175}+1)$ and $m_{k+1} = 2(x^{525}+1)$. Assume that $2(x^{175}+1)$ is practical, then we only need to show that

$$2(x^{175}+1)\Phi_6(x)\Phi_{30}(x)\Phi_{42}(x)\Phi_{150}(x)\Phi_{210}(x)\Phi_{1050}(x)$$

is practical.

Note that

$$\begin{aligned} \Phi_6(x) &= x^2 - x + 1, \\ \Phi_{30}(x) &= x^8 + x^7 - x^5 - x^4 - x^3 + x + 1, \\ \Phi_{42}(x) &= x^{12} + x^{11} - x^9 - x^8 + x^6 - x^4 - x^3 + x + 1, \\ \Phi_{150}(x) &= x^{40} + x^{35} - x^{25} - x^{20} - x^{15} + x^5 + 1, \\ \Phi_{210}(x) &= x^{48} - x^{47} + x^{46} + x^{43} - x^{42} + 2x^{41} - x^{40} \\ &\quad + x^{39} + x^{36} - x^{35} + x^{34} - x^{33} + x^{32} - x^{31} \\ &\quad - x^{28} - x^{26} - x^{24} - x^{22} - x^{20} - x^{17} + x^{16} \\ &\quad - x^{15} + x^{14} - x^{13} + x^{12} + x^9 - x^8 \\ &\quad + 2x^7 - x^6 + x^5 + x^2 - x + 1, \\ \Phi_{1050}(x) &= x^{240} - x^{235} + x^{230} + x^{215} - x^{210} + 2x^{205} \\ &\quad - x^{200} + x^{195} + x^{180} - x^{175} + x^{170} - x^{165} \\ &\quad + x^{160} - x^{155} - x^{140} - x^{130} - x^{120} - x^{110} \\ &\quad - x^{100} - x^{85} + x^{80} - x^{75} + x^{70} - x^{65} + x^{60} \\ &\quad + x^{45} - x^{40} + 2x^{35} - x^{30} + x^{25} + x^{10} - x^5 + 1. \end{aligned}$$

Since $x \geqslant 2$, we have $2x^{n-1} \leqslant x^n \leqslant \frac{x^{n+1}}{2}$ for every $n \in \mathbb{N}$, and thus

$$x \leqslant \Phi_6(x) \leqslant x^2, \tag{30}$$

$$x^8 \leqslant \Phi_{30}(x) \leqslant 2x^8, \tag{31}$$

$$x^{12} \leqslant \Phi_{42}(x) \leqslant 2x^{12}, \tag{32}$$

$$x^{40} \leqslant \Phi_{150}(x) \leqslant 2x^{40}, \tag{33}$$

$$\frac{1}{2}x^{48} \leqslant \Phi_{210}(x) \leqslant x^{48}, \tag{34}$$

$$\frac{1}{2}x^{240} \leqslant \Phi_{1050}(x) \leqslant x^{240}. \tag{35}$$

By (30)–(34), we have

$$\Phi_6(x)\Phi_{30}(x)\Phi_{42}(x)\Phi_{150}(x)\Phi_{210}(x) \leqslant 8x^{110} \leqslant x^{113} \leqslant 2(x^{175}+1) \tag{36}$$

and

$$2(x^{175}+1)\Phi_6(x)\Phi_{30}(x)\Phi_{42}(x)\Phi_{150}(x)\Phi_{210}(x)$$

is practical by Lemma 2.1.

By (30)–(34), we have

$$2(x^{175}+1)\Phi_6(x)\Phi_{30}(x)\Phi_{42}(x)\Phi_{150}(x)\Phi_{210}(x) \geqslant x^{284}. \tag{37}$$

By (35) and (37), we have

$$\Phi_{1050}(x) \leqslant x^{240} \leqslant x^{284} \leqslant 2(x^{175}+1)\Phi_6(x)\Phi_{30}(x)\Phi_{42}(x)\Phi_{150}(x)\Phi_{210}(x) \tag{38}$$

and

$$2(x^{175}+1)\Phi_6(x)\Phi_{30}(x)\Phi_{42}(x)\Phi_{150}(x)\Phi_{210}(x)\Phi_{1050}(x)$$

is practical by Lemma 2.1.

Since m_0 is practical, we can show that m_k is practical for every $k = 0, 1, 2, \ldots$ by an inductive process. Now the proof of sequence $2^{175\times 3^k+1}+2$ is completed.

2.4 Sequence 4

We shall prove by induction that $2^{189\times 5^k+1}+2$ is practical for every $k \geqslant 0$.

Write $m_k = 2^{189\times 5^k+1}+2$ for $k = 0, 1, 2, \ldots$, then we only need to show that m_k is practical for every $k = 0, 1, 2, \ldots$. By a computer calculation we find that

$$\begin{aligned} m_0 &= 156927543384667019095894735580191660402558886111600862822 6 \\ &= 2 \times 3^4 \times 19 \times 43 \times 379 \times 5419 \times 87211 \times 119827 \times 77158673929 \\ &\quad \times 127391413339 \times 56202143607667. \end{aligned}$$

By Lemma 2.1, we know that m_0 is practical. Let $x = 2^{5^k}$, then we have $m_k = 2(x^{189}+1)$ and $m_{k+1} = 2(x^{945}+1)$. Assume that $2(x^{189}+1)$ is practical, then we only need to show that

$$2(x^{189}+1)\Phi_{10}(x)\Phi_{30}(x)\Phi_{70}(x)\Phi_{90}(x)\Phi_{210}(x)\Phi_{270}(x)\Phi_{630}(x)\Phi_{1890}(x)$$

is practical.

Note that

$$\begin{aligned} \Phi_{10}(x) &= x^4 - x^3 + x^2 - x + 1, \\ \Phi_{30}(x) &= x^8 + x^7 - x^5 - x^4 - x^3 + x + 1, \\ \Phi_{70}(x) &= x^{24} + x^{23} - x^{19} - x^{18} - x^{17} - x^{16} + x^{14} + x^{13} \\ &\quad + x^{12} + x^{11} + x^{10} - x^8 - x^7 - x^6 - x^5 + x + 1, \\ \Phi_{90}(x) &= x^{24} + x^{21} - x^{15} - x^{12} - x^9 + x^3 + 1, \\ \Phi_{210}(x) &= x^{48} - x^{47} + x^{46} + x^{43} - x^{42} + 2x^{41} - x^{40} \\ &\quad + x^{39} + x^{36} - x^{35} + x^{34} - x^{33} + x^{32} - x^{31} \\ &\quad - x^{28} - x^{26} - x^{24} - x^{22} - x^{20} - x^{17} + x^{16} \\ &\quad - x^{15} + x^{14} - x^{13} + x^{12} + x^9 - x^8 \\ &\quad + 2x^7 - x^6 + x^5 + x^2 - x + 1, \\ \Phi_{270}(x) &= x^{72} + x^{63} - x^{45} - x^{36} - x^{27} + x^9 + 1, \end{aligned}$$

$$\begin{aligned}\Phi_{630}(x) &= x^{144} - x^{141} + x^{138} + x^{129} - x^{126} + 2x^{123} - x^{120}\\ &\quad + x^{117} + x^{108} - x^{105} + x^{102} - x^{99} + x^{96} - x^{93} - x^{84}\\ &\quad - x^{78} - x^{72} - x^{66} - x^{60} - x^{51} + x^{48} - x^{45} + x^{42}\\ &\quad - x^{39} + x^{36} + x^{27} - x^{24} + 2x^{21} - x^{18} + x^{15} + x^{6} - x^{3} + 1,\\ \Phi_{1890}(x) &= x^{432} - x^{423} + x^{414} + x^{387} - x^{378} + 2x^{369} - x^{360} + x^{351}\\ &\quad + x^{324} - x^{315} + x^{306} - x^{297} + x^{288} - x^{279} - x^{252} - x^{234}\\ &\quad - x^{216} - x^{198} - x^{180} - x^{153} + x^{144} - x^{135} + x^{126} - x^{117}\\ &\quad + x^{108} + x^{81} - x^{72} + 2x^{63} - x^{54} + x^{45} + x^{18} - x^{9} + 1.\end{aligned}$$

Since $x \geqslant 2$, we have $2x^{n-1} \leqslant x^n \leqslant \frac{x^{n+1}}{2}$ for every $n \in \mathbb{N}$, and thus

$$\frac{1}{2}x^4 \leqslant \Phi_{10}(x) \leqslant x^4, \tag{39}$$

$$x^8 \leqslant \Phi_{30}(x) \leqslant 2x^8, \tag{40}$$

$$x^{24} \leqslant \Phi_{70}(x) \leqslant 2x^{24}, \tag{41}$$

$$x^{24} \leqslant \Phi_{90}(x) \leqslant 2x^{24}, \tag{42}$$

$$\frac{1}{2}x^{48} \leqslant \Phi_{210}(x) \leqslant x^{48}, \tag{43}$$

$$x^{72} \leqslant \Phi_{270}(x) \leqslant 2x^{72}, \tag{44}$$

$$\frac{1}{2}x^{144} \leqslant \Phi_{630}(x) \leqslant x^{144}, \tag{45}$$

$$\frac{1}{2}x^{432} \leqslant \Phi_{1890}(x) \leqslant x^{432}. \tag{46}$$

By (39)–(44), we have

$$\Phi_{10}(x)\Phi_{30}(x)\Phi_{70}(x)\Phi_{90}(x)\Phi_{210}(x)\Phi_{270}(x) \leqslant 16x^{180} \leqslant x^{184} \leqslant 2(x^{189}+1) \tag{47}$$

and

$$2(x^{189}+1)\Phi_{10}(x)\Phi_{30}(x)\Phi_{70}(x)\Phi_{90}(x)\Phi_{210}(x)\Phi_{270}(x)$$

is practical by Lemma 2.1.

By (39)–(44), we have

$$2(x^{189}+1)\Phi_{10}(x)\Phi_{30}(x)\Phi_{70}(x)\Phi_{90}(x)\Phi_{210}(x)\Phi_{270}(x) \geqslant \frac{1}{2}x^{369} \geqslant x^{368}. \tag{48}$$

By (45) and (48), we have

$$\Phi_{630}(x) \leqslant x^{144} \leqslant x^{368} \leqslant 2(x^{189}+1)\Phi_{10}(x)\Phi_{30}(x)\Phi_{70}(x)\Phi_{90}(x)\Phi_{210}(x)\Phi_{270}(x) \tag{49}$$

and

$$2(x^{189}+1)\Phi_{10}(x)\Phi_{30}(x)\Phi_{70}(x)\Phi_{90}(x)\Phi_{210}(x)\Phi_{270}(x)\Phi_{630}(x)$$

is practical by Lemma 2.1.

By (39)–(45), we have

$$2(x^{189}+1)\Phi_{10}(x)\Phi_{30}(x)\Phi_{70}(x)\Phi_{90}(x)\Phi_{210}(x)\Phi_{270}(x)\Phi_{630}(x) \geqslant \frac{1}{4}x^{513} \geqslant x^{511}. \tag{50}$$

By (46) and (50), we have

$$\Phi_{1890}(x) \leqslant x^{432} \leqslant x^{511} \leqslant 2(x^{189}+1)\Phi_{10}(x)\Phi_{30}(x)\Phi_{70}(x)\Phi_{90}(x)\Phi_{210}(x)\Phi_{270}(x)\Phi_{630}(x) \quad (51)$$

and

$$2(x^{189}+1)\Phi_{10}(x)\Phi_{30}(x)\Phi_{70}(x)\Phi_{90}(x)\Phi_{210}(x)\Phi_{270}(x)\Phi_{630}(x)\Phi_{1890}(x)$$

is practical by Lemma 2.1.

Since m_0 is practical, we can show that m_k is practical for every $k = 0, 1, 2, \ldots$ by an inductive process. Now the proof of sequence $2^{189\times 5^k+1}+2$ is completed.

Finally, we get Theorem 1.1 by combining the above four cases.

3 Proof of Theorem 1.2 and its corollaries

Let p be a prime with $7 \leqslant p < 100$. In order to prove Theorem 1.2, we need to show that at least one of the following five sequences generate infinitely many multiples of p:

$$35\times 3^k+1,\ 135\times 7^k+1,\ 165\times 3^k+1,\ 175\times 3^k+1,\ 189\times 5^k+1.$$

By Fermat's little theorem, the following identities hold for any positive integer m:

$$\begin{aligned}
165\times 3^{5+6m}+1 &\equiv 0\ (\mathrm{mod}7),\\
35\times 3^{3+10m}+1 &\equiv 0\ (\mathrm{mod}11),\\
135\times 7^{3+12m}+1 &\equiv 0\ (\mathrm{mod}13),\\
35\times 3^{8+16m}+1 &\equiv 0\ (\mathrm{mod}17),\\
35\times 3^{17+18m}+1 &\equiv 0\ (\mathrm{mod}19),\\
135\times 7^{20+22m}+1 &\equiv 0\ (\mathrm{mod}23),\\
35\times 3^{24+28m}+1 &\equiv 0\ (\mathrm{mod}29),\\
35\times 3^{27+30m}+1 &\equiv 0\ (\mathrm{mod}31),\\
189\times 5^{32+36m}+1 &\equiv 0\ (\mathrm{mod}37),\\
135\times 7^{7+40m}+1 &\equiv 0\ (\mathrm{mod}41),\\
35\times 3^{3+42m}+1 &\equiv 0\ (\mathrm{mod}43),\\
35\times 3^{11+46m}+1 &\equiv 0\ (\mathrm{mod}47),\\
35\times 3^{1+52m}+1 &\equiv 0\ (\mathrm{mod}53),\\
165\times 3^{21+58m}+1 &\equiv 0\ (\mathrm{mod}59),\\
135\times 7^{50+60m}+1 &\equiv 0\ (\mathrm{mod}61),\\
135\times 7^{33+66m}+1 &\equiv 0\ (\mathrm{mod}67),\\
35\times 3^{11+70m}+1 &\equiv 0\ (\mathrm{mod}71),
\end{aligned}$$

$$189 \times 5^{57+72m} + 1 \equiv 0 \pmod{73},$$
$$35 \times 3^{2+78m} + 1 \equiv 0 \pmod{79},$$
$$35 \times 3^{24+82m} + 1 \equiv 0 \pmod{83},$$
$$35 \times 3^{69+88m} + 1 \equiv 0 \pmod{89},$$
$$35 \times 3^{40+96m} + 1 \equiv 0 \pmod{97}.$$

For a family $2^{a\times b^k+1}+2$ satisfying $p \mid a \times b^k + 1$, we define $q = 2^{(a\times b^k+1)/p}$. Then q and q^p are powers of 2, hence practical, while $q^p + 2$ is practical by Theorem 1.1. Now the proof of Theorem 1.2 is completed. By combining those identities, we can also show that Conjecture 1.1 holds true for many other integers.

We can prove Corollary 1.1 by taking the sequences

$$4 \times \left(2^{35\times 3^k+1} + 2\right) = 2^{35\times 3^k+3} + 8$$

and

$$16 \times \left(2^{35\times 3^k+1} + 2\right) = 2^{35\times 3^k+5} + 32.$$

Lemma 2.1 ensures that we can multiply a practical number by any powers of 2 to get new practical numbers. Similarly, for Corollary 1.2 we can take the sequence

$$2^{h-1} \times \left(2^{35\times 3^k+1} + 2\right) = 2^{35\times 3^k+h} + 2^h$$

with a suitable h such that $35 \times 3^k + h$ is a multiple of C.

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