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A note on some identities of degenerate Bernoulli numbers

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Abstract: In this note, we investigate the algebraic and combinatorial properties of degenerate special numbers and polynomials, focusing on their deep connections to classical sequences. By leveraging the framework of degenerate exponential and logarithm functions, we establish several explicit formulas and identities. Specifically, we provide an explicit expression for the Bessel numbers of the first kind and recall a previously established explicit formula for the degenerate Bernoulli numbers that is needed for the subsequent development. Finally, we establish an identity linking the Bessel numbers of the first kind, the degenerate Bernoulli numbers, and the



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degenerate Stirling numbers of the first kind.

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1 Introduction

In recent years, the study of degenerate versions of special polynomials and numbers has attracted significant interest from many mathematicians. This line of research, initiated by Carlitz in his pioneering work on degenerate Bernoulli and Eulerian numbers, has been revitalized and extensively developed. These degenerate variants not only possess elegant algebraic and combinatorial properties but also share deep connections with various fields of mathematics, including number theory, combinatorics, and mathematical physics. Central to the investigation of these structures are the degenerate exponential and degenerate logarithm functions, which serve as the foundation for defining degenerate versions of classical special numbers and polynomials. Among these, the degenerate Stirling numbers of both kinds, alongside their ordinary counterparts, play a pivotal role in combinatorial analysis and operator theory. Furthermore, the interplay between these structures and other classical sequences–such as the Bessel numbers, which arise naturally in the study of the Bessel polynomials and have important applications in network theory and statistics–offers a fertile ground for discovering new mathematical identities. The primary objective of this paper is to explore these connections further, particularly focusing on the relationships between degenerate Bernoulli polynomials, degenerate Stirling numbers, and Bessel numbers of the first kind. By leveraging the algebraic properties of degenerate polynomials, we establish several explicit formulas and identities that extend classical results in number theory and combinatorics.

This note is structured as follows. In Section 2, we recall the fundamental definitions and properties of the degenerate exponentials, the degenerate logarithm, the degenerate Stirling numbers of both kinds, and the ordinary Stirling numbers of both kinds. We also remind the reader of the essential background regarding the Bessel numbers and the degenerate Bernoulli polynomials and numbers. Section 3 contains the main results, including a degenerate analogue of the classical power-sum formula, an explicit expression for the Bessel numbers of the first kind, and a degenerate extension of the Bernoulli–Stirling–Bessel identity established in [1]. Section 4 concludes the paper.

2 Preliminaries

For any nonzero $\lambda \in \mathbb{R}$, the degenerate exponentials are defined by (see [4–6, 9])

$$e_{\lambda}^{x}(t) = \sum_{k=0}^{\infty} (x)_{k,\lambda} \frac{t^k}{k!} = (1+\lambda t)^{\frac{x}{\lambda}}, \quad e_{\lambda}(t) = e_{\lambda}^{1}(t), \tag{1}$$

where the degenerate falling factorial sequence is given by

$$(x)_{0,\lambda} = 1, \quad (x)_{n,\lambda} = x(x-\lambda)\cdots\big(x-(n-1)\lambda\big), \ (n \geq 1).$$

Let $\log_\lambda t$ be the degenerate logarithm which is the compositional inverse of $e_\lambda(t)$ so that

$$\log_\lambda\big(e_\lambda(t)\big) = e_\lambda\big(\log_\lambda t\big) = t.$$

Then we can show that (see [4, 5])

$$\log_\lambda(1+t) = \frac{1}{\lambda}\big((1+t)^\lambda - 1\big) = \sum_{k=1}^{\infty} \frac{\lambda^{k-1}(1)_{k,1/\lambda}}{k!} t^k. \tag{2}$$

The degenerate Stirling numbers of the first kind are determined by (see [4, 6, 7])

$$(x)_n = \sum_{k=0}^{n} S_{1,\lambda}(n,k)(x)_{k,\lambda}, \quad (n \geq 0). \tag{3}$$

Note that

$$\lim_{\lambda\to 0} S_{1,\lambda}(n,k) = S_1(n,k),$$

where $S_1(n,k)$ are the ordinary Stirling numbers of the first kind characterized by (see [3, 11])

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k, \quad (n \geq 0). \tag{4}$$

As the inversion formula of (3), the degenerate Stirling numbers of the second kind are specified as (see [4, 6, 7])

$$(x)_{n,\lambda} = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_\lambda (x)_k, \quad (n \geq 0). \tag{5}$$

From (4), we note that

$$\lim_{\lambda\to 0} \left\{ {n \atop k} \right\}_\lambda = \left\{ {n \atop k} \right\},$$

where $\left\{ {n \atop k} \right\}$ are the classical Stirling numbers of the second kind given by (see [3, 11])

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (x)_k, \quad (n \geq 0).$$

Recall that the Bessel numbers of the first kind are represented by (see [1, 12])

$$x(x-2)(x-4)\cdots\big(x-2(n-1)\big) = \sum_{k=0}^{n} b(n,k)(x)_k. \tag{6}$$

In [2], Carlitz introduced the degenerate Bernoulli polynomials as (see [8, 10])

$$\frac{t}{e_\lambda(t)-1} e_\lambda^x(t) = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x)\frac{t^n}{n!}. \tag{7}$$

When $x = 0$, $\beta_{n,\lambda} = \beta_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers.

From (1) and (7), it is immediate to see that

$$\beta_{n,\lambda}(x) = \sum_{k=0}^{n} \binom{n}{k} \beta_{k,\lambda}(x)_{n-k,\lambda}, \quad (n \geq 0). \tag{8}$$

3 Some identities of degenerate Bernoulli numbers

For $N \in \mathbb{N}$ and from (1) and (7), we derive that

$$\begin{aligned}\sum_{k=0}^{N-1} e_\lambda^k(t) &= \frac{1}{e_\lambda(t)-1}\big(e_\lambda^N(t)-1\big) \\ &= \frac{1}{t}\left(\frac{t}{e_\lambda(t)-1}e_\lambda^N(t) - \frac{t}{e_\lambda(t)-1}\right) \\ &= \frac{1}{t}\sum_{n=0}^{\infty}\big(\beta_{n,\lambda}(N)-\beta_{n,\lambda}\big)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{1}{n+1}\big(\beta_{n+1,\lambda}(N)-\beta_{n+1,\lambda}\big)\frac{t^n}{n!}.\end{aligned} \tag{9}$$

On the other hand, by (1), we observe that

$$\sum_{k=0}^{N-1} e_\lambda^k(t) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{N-1}(k)_{n,\lambda}\right)\frac{t^n}{n!}. \tag{10}$$

Thus, by (8), (9) and (10), we conclude that

$$\begin{aligned}\sum_{k=0}^{N-1}(k)_{n,\lambda} &= \frac{1}{n+1}\big(\beta_{n+1,\lambda}(N)-\beta_{n+1,\lambda}\big) \\ &= \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}\beta_{k,\lambda}(N)_{n+1-k,\lambda}.\end{aligned} \tag{11}$$

By (11), we obtain the following result which reduces to the well known formula on the sums of powers of consecutive integers as $\lambda \to 0$.

Lemma 3.1. *For $n \geq 0$, we have*

$$\sum_{k=0}^{N-1}(k)_{n,\lambda} = \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}\beta_{k,\lambda}(N)_{n+1-k,\lambda}.$$

A slight modification of (6) yields the following:

$$\begin{aligned}\sum_{k=0}^{n} b(n,k)(x)_k &= x(x-2)(x-4)\big(x-2(n-1)\big) \\ &= 2^n\frac{x}{2}\left(\frac{x}{2}-1\right)\left(\frac{x}{2}-2\right)\cdots\left(\frac{x}{2}-(n-1)\right) \\ &= 2^n\left(\frac{x}{2}\right)_n.\end{aligned} \tag{12}$$

In turn, (3) gives another expression:

$$\begin{aligned}2^n\left(\frac{x}{2}\right)_n &= 2^n\sum_{k=0}^{n} S_{1,\lambda}(n,k)\left(\frac{x}{2}\right)_{k,\lambda} \\ &= 2^n\sum_{k=0}^{n} S_{1,\lambda}(n,k)\left(\frac{1}{2}\right)^k(x)_{k,2\lambda} = \sum_{k=0}^{n} 2^{n-k}S_{1,\lambda}(n,k)(x)_{k,2\lambda}.\end{aligned} \tag{13}$$

From (5), (12) and (13), it follows that

$$\begin{aligned}\sum_{k=0}^{n} b(n,k)(x)_k &= \sum_{k=0}^{n} 2^{n-k} S_{1,\lambda}(n,k)(x)_{k,2\lambda} \\ &= \sum_{k=0}^{n} 2^{n-k} S_{1,\lambda}(n,k) \sum_{j=0}^{k} \left\{ {k \atop j} \right\}_{2\lambda} (x)_j \\ &= \sum_{j=0}^{n} (x)_j \sum_{k=j}^{n} 2^{n-k} S_{1,\lambda}(n,k) \left\{ {k \atop j} \right\}_{2\lambda}.\end{aligned} \tag{14}$$

The equation (14) gives us an explicit expression for the Bessel numbers of the first kind.

Theorem 3.2. *For $n \geq j \geq 0$, we have*

$$b(n,j) = \sum_{k=j}^{n} 2^{n-k} S_{1,\lambda}(n,k) \left\{ {k \atop j} \right\}_{2\lambda}.$$

From (2) and (7), we deduce that

$$\begin{aligned}&\sum_{n=0}^{\infty} \beta_{n,\lambda} \frac{t^n}{n!} = \frac{1}{e_\lambda(t)-1} \Big(\log_\lambda \big(e_\lambda(t) - 1 + 1 \big) \Big) \\ &= \frac{1}{e_\lambda(t)-1} \sum_{m=1}^{\infty} \frac{\lambda^{m-1}(1)_{m,1/\lambda}}{m!} \big(e_\lambda(t) - 1 \big)^m \\ &= \sum_{m=1}^{\infty} \frac{\lambda^{m-1}(1)_{m,1/\lambda}}{m!} \big(e_\lambda(t) - 1 \big)^{m-1} = \sum_{m=0}^{\infty} \frac{\lambda^m (1)_{m+1,1/\lambda}}{(m+1)!} \big(e_\lambda(t) - 1 \big)^m \\ &= \sum_{m=0}^{\infty} \frac{\lambda^m (1)_{m+1,1/\lambda}}{m+1} \sum_{n=m}^{\infty} \left\{ {n \atop m} \right\}_\lambda \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{\lambda^m (1)_{m+1,1/\lambda}}{m+1} \left\{ {n \atop m} \right\}_\lambda \frac{t^n}{n!}.\end{aligned} \tag{15}$$

Thus, from (15), we find an explicit expression for the degenerate Bernoulli numbers. Here we mention that this explicit formula for the degenerate Bernoulli numbers appeared already in our earlier work in [8]. The formula is recalled here because it is needed for the subsequent development.

Theorem 3.3. *For $n \geq 0$, we have*

$$\beta_{n,\lambda} = \sum_{m=0}^{n} \left\{ {n \atop m} \right\}_\lambda \frac{\lambda^m}{m+1} (1)_{m+1,1/\lambda} = \sum_{m=0}^{n} \left\{ {n \atop m} \right\}_\lambda (-1)^m m! \frac{\binom{m-\lambda}{m}}{m+1}. \tag{16}$$

From (14), we note that

$$\sum_{k=0}^{n} b(n,k) \sum_{x=0}^{N-1} (x)_k = \sum_{k=0}^{n} 2^{n-k} S_{1,\lambda}(n,k) \sum_{x=0}^{N-1} (x)_{k,2\lambda}, \tag{17}$$

where N is a positive integer.

By (11), we have the following expression:

$$\sum_{x=0}^{N-1}(x)_{k,2\lambda} = \frac{1}{k+1}\sum_{j=0}^{k}\binom{k+1}{j}\beta_{j,2\lambda}(N)_{k+1-j,2\lambda} \tag{18}$$
$$= \frac{N}{k+1}\sum_{j=0}^{k}\binom{k+1}{j}\beta_{j,2\lambda}(N-2\lambda)_{k-j,2\lambda}.$$

In addition, using the binomial formula, we obtain

$$\sum_{x=0}^{N-1}(x)_k = k!\sum_{x=0}^{N-1}\binom{x}{k} = k!\sum_{x=0}^{N-1}\left(\binom{x+1}{k+1} - \binom{x}{k+1}\right) \tag{19}$$
$$= k!\binom{N}{k+1} = \frac{N(N-1)\cdots(N-k)}{k+1}.$$

Combining (17), (18) and (19) altogether yields

$$\sum_{k=0}^{n} b(n,k)\frac{(N-1)(N-2)\cdots(N-k)}{k+1} \tag{20}$$
$$= \sum_{k=0}^{n} 2^{n-k}S_{1,\lambda}(n,k)\frac{1}{k+1}\sum_{j=0}^{k}\binom{k+1}{j}\beta_{j,2\lambda}(N-2\lambda)_{k-j,2\lambda}.$$

The equation (20), after simplification, is a polynomial identity in N; since it holds for all positive integers N, it holds identically and can therefore be evaluated at $N = 2\lambda$. Consequently, by (20), we have

$$\sum_{k=0}^{n} b(n,k)\frac{k!}{k+1}\binom{2\lambda-1}{k} = \lim_{N\to 2\lambda}\sum_{k=0}^{n} b(n,k)\frac{(N-1)\cdots(N-k)}{k+1} \tag{21}$$
$$= \lim_{N\to 2\lambda}\sum_{k=0}^{n} 2^{n-k}S_{1,\lambda}(n,k)\frac{1}{k+1}\sum_{j=0}^{k}\binom{k+1}{j}\beta_{j,2\lambda}(N-2\lambda)_{k-j,2\lambda}$$
$$= \sum_{k=0}^{n} 2^{n-k}S_{1,\lambda}(n,k)\frac{1}{k+1}\sum_{j=0}^{k}\binom{k+1}{j}\beta_{j,2\lambda}(0)_{k-j,2\lambda}$$
$$= \sum_{k=0}^{n} 2^{n-k}S_{1,\lambda}(n,k)\beta_{k,2\lambda}.$$

Therefore, by (21), we establish an identity involving the Bessel numbers of the first kind, the degenerate Bernoulli numbers and the degenerate Stirling numbers of the first kind.

Theorem 3.4. *For $n \geq 0$, we have*

$$\sum_{k=0}^{n} b(n,k)\frac{k!}{k+1}\binom{2\lambda-1}{k} = \sum_{k=0}^{n} 2^{n-k}S_{1,\lambda}(n,k)\beta_{k,2\lambda}. \tag{22}$$

As $\lambda \to 0$, identity (22) reduces to the Bernoulli–Stirling–Bessel identity obtained by Benmoussa [1]. Hence Theorem 3.4 may be regarded as a degenerate extension of that result.

4 Conclusion

In conclusion, this paper successfully bridges the gap between degenerate variants of special polynomials and classical sequences like the Bessel numbers. By utilizing the foundational properties of degenerate exponentials and logarithms, we extended classical combinatorial formulas into the degenerate domain. Our main results—including a degenerate analogue for the sums of powers of consecutive integers, an explicit formula for the Bessel numbers of the first kind, and a degenerate extension of the Bernoulli–Stirling–Bessel identity—offer further insight into their underlying algebraic structures. These findings open up potential avenues for future research, particularly in exploring how these degenerate identities might apply to broader problems in operator theory, network theory, and mathematical physics.

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