

Rising factorials and hypergeometric functions

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Abstract: This note considers factorial binomial coefficients, focusing on both rising and falling factorial powers and their generalisations. We recall a factorial-binomial formulation of classical hypergeometric identities, including Saalschütz’s ${}_3F_2$ summation, and highlight the combinatorial structure underlying the coefficients. For Type 2 rising factorial binomial coefficients we establish a Pascal-type recurrence relation, identities for negative indices and some generating functions. We also discuss potential extensions to rising factorial Jordan coefficients and congruence properties, motivated by recent results of Kim and Kim. This unified approach offers a combinatorial and algebraic framework for analysing hypergeometric series and related number-theoretic properties.

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1 Introduction

The *rising factorial* (also known as the *Pochhammer symbol*) is defined by

$$n^{(a)} = n^{\bar{a}} = n(n+1) \cdots (n+a-1),$$

while the *falling factorial* (or *descending factorial*) is

$$n^{\underline{a}} = n(n-1) \cdots (n-a+1).$$

By convention, both factorials equal 1 when $a = 0$, corresponding to an empty product. Together, these symbols are referred to as *factorial powers* [6, p.50]. They are closely related to the usual binomial coefficients:

$$\binom{x}{n} = \frac{x^{\underline{n}}}{n!},$$

for a nonnegative integer n .

Moreover, the rising factorial plays a central role in hypergeometric functions, where coefficients of the power series are expressed in terms of Pochhammer symbols. Throughout the paper, in all hypergeometric formulas the Pochhammer symbol $(x)_k$ denotes the rising factorial,

$$(x)_k = x^{\bar{k}} = x(x+1) \cdots (x+k-1).$$

Motivated by these notions and by recent work of Kim and Kim [4], we consider *factorial binomial coefficients*. Two different kinds of rising factorial binomial coefficients appear in the literature [10–12], and it is important to distinguish them. The *rising factorial binomial coefficients of Type 1*, where the factorial power is taken with respect to the parameter a , are [14]

$$\left\| \begin{matrix} n \\ k \end{matrix} \right\|^a = \frac{a^{\bar{n}}}{a^{\bar{k}} a^{\bar{n-k}}}.$$

Since $1^{\bar{m}} = m!$, they satisfy

$$\left\| \begin{matrix} n \\ k \end{matrix} \right\|^1 = \binom{n}{k},$$

so they generalise the ordinary binomial coefficients. The *rising factorial binomial coefficients of Type 2*, which are the ones used in most of this paper, are

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle^a = \frac{n^{\bar{a}}}{k^{\bar{a}} (n-k)^{\bar{a}}}, \quad (1.1)$$

and, analogously, the *falling factorial binomial coefficients* are

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_a = \frac{n^{\underline{a}}}{k^{\underline{a}} (n-k)^{\underline{a}}}.$$

We stress that the Type 2 coefficients do *not* reduce to ordinary binomial coefficients when $a = 1$: indeed,

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle^1 = \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_1 = \frac{n}{k(n-k)},$$

which differs from $\binom{n}{k}$ in general. Nevertheless, they naturally arise in combinatorial identities and in the study of hypergeometric series.

According to Riordan [8], some of this notation stems from suggestions of Knuth at the 1967 Conference on Combinatorial Mathematics and its Applications. Knuth's proposal appears in the discussion following Riordan's paper. Further commentary on factorial notation and the Pochhammer symbol can be found in [5].

The aim of this paper is to investigate connections between factorial binomial coefficients and certain special functions of mathematical physics. This extends the work of [7], where several infinite series were analysed in terms of hypergeometric functions, integer structures, and modular rings [13, Appendix 1]. The resulting generalised functions were compared with what may be regarded as a "mother" series.

The paper is organised as follows. In Section 2 we recall, for completeness, the formulation of Saalschütz's theorem in terms of Type 2 coefficients, which is essentially contained in [11], and we connect it with Euler's transformation. The new contributions are in the subsequent sections: the Pascal-type recurrence of Theorem 3.1, the formula for negative indices in Proposition 3.2 and its relation with the falling factorial binomial coefficients, and the generating functions of Section 4. Some open directions are outlined in the concluding section.

2 A hypergeometric identity via factorial binomial coefficients

We begin with the classical Saalschütz summation [9]. In ${}_3F_2$ notation, the terminating balanced case reads

$${}_3F_2\left(\begin{matrix} -n, & a, & b \\ & c, & d \end{matrix}; 1\right) = \sum_{k=0}^n \frac{(-n)_k (a)_k (b)_k}{(c)_k (d)_k k!} = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n}, \quad (2.1)$$

with the balancing condition

$$c + d = a + b - n + 1,$$

where, as stated in the introduction, all Pochhammer symbols are rising factorials.

This formula can be naturally recast using rising factorial binomial coefficients of Type 2. The following reformulation is essentially contained in [11]. We recall it, with a short proof, for completeness.

Proposition 2.1. *Let n be a nonnegative integer and let a, c be complex numbers. Then, under the Saalschütz balancing condition with $b = n - 1$ and $d = a - c$ (so that $c + d = a + (n - 1) - n + 1$), we have*

$$\sum_{k=0}^n (-1)^k \binom{n}{k} (n-1)^{\bar{k}} \left\langle \begin{matrix} a \\ c \end{matrix} \right\rangle^k = \frac{(c-a)_n (c-n+1)_n}{(c)_n (c-a-n+1)_n}, \quad (2.2)$$

provided that none of the denominator Pochhammer symbols vanishes; elsewhere the identity may be interpreted by analytic continuation. Here $(x)_n = x^{\bar{n}}$ is the rising factorial and $\left\langle \begin{matrix} a \\ c \end{matrix} \right\rangle^k$ is the Type 2 rising factorial binomial coefficient (1.1).

Proof. With $b = n - 1$, the balancing condition $c + d = a + b - n + 1$ gives $d = a - c$. Substituting into (2.1) yields

$$\sum_{k=0}^n \frac{(-n)_k (a)_k (n-1)_k}{(c)_k (a-c)_k k!}.$$

Using the definition of the rising factorial binomial coefficient,

$$\left\langle \begin{matrix} a \\ c \end{matrix} \right\rangle^k = \frac{a^{\bar{k}}}{c^{\bar{k}} (a-c)^{\bar{k}}},$$

and noting that $(x)_k = x^{\bar{k}}$ for every x , this becomes

$$\sum_{k=0}^n \frac{(-n)_k}{k!} (n-1)^{\bar{k}} \left\langle \begin{matrix} a \\ c \end{matrix} \right\rangle^k.$$

Finally, for a nonnegative integer n , we have the well-known identity

$$(-n)_k = (-1)^k \frac{n!}{(n-k)!} = (-1)^k k! \binom{n}{k},$$

which gives

$$\sum_{k=0}^n (-1)^k \binom{n}{k} (n-1)^{\bar{k}} \left\langle \begin{matrix} a \\ c \end{matrix} \right\rangle^k.$$

The right-hand side follows directly from (2.1) with $b = n - 1$ and $d = a - c$, yielding

$$\frac{(c-a)_n (c-n+1)_n}{(c)_n (c-a-n+1)_n}.$$

□

Replacing c by $a + n - c$ in (2.2) produces

$$\sum_{k=0}^n (-1)^k (n-1)^{\bar{k}} \binom{n}{k} \left\langle \begin{matrix} a \\ a+n-c \end{matrix} \right\rangle^k = \frac{(n-c)^{\bar{n}} (a-c+1)^{\bar{n}}}{(a+n-c)^{\bar{n}} (1-c)^{\bar{n}}},$$

again provided that none of the denominator Pochhammer symbols vanishes. (Since $\left\langle \begin{matrix} a \\ c \end{matrix} \right\rangle^k = \left\langle \begin{matrix} a \\ a-c \end{matrix} \right\rangle^k$, the coefficient on the left can equivalently be written as $\left\langle \begin{matrix} a \\ c-n \end{matrix} \right\rangle^k$.) The parameters on the right-hand side are precisely those of Euler's classical hypergeometric transformation

$${}_2F_1 \left(\begin{matrix} a, n-1 \\ a+n-c \end{matrix}; x \right) = (1-x)^{1-c} {}_2F_1 \left(\begin{matrix} n-c, a-c+1 \\ a+n-c \end{matrix}; x \right),$$

where ${}_2F_1$ is the Gauss hypergeometric function. Indeed, the two identities are closely linked: equating the coefficients of x^m on both sides of Euler's transformation leads to a terminating balanced ${}_3F_2$ series, which is summed by Saalschütz's theorem [13]. The rising factorial binomial formulation highlights the combinatorial structure of the coefficients.

3 Type 2 rising factorial binomial coefficients

We now examine the Type 2 rising factorial binomial coefficients (1.1) in more detail. They share some features with the classical binomial coefficients, such as the symmetry $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle^a = \left\langle \begin{matrix} n \\ n-k \end{matrix} \right\rangle^a$,

but display modified recurrences and series expansions useful in hypergeometric contexts. For instance, for b and $a - b$ different from $0, -1, -2, \dots$,

$${}_1F_2\left(\begin{matrix} a \\ b, a - b \end{matrix}; x\right) = \sum_{r=0}^{\infty} \left\langle \frac{a}{b} \right\rangle^r \frac{x^r}{r!};$$

for positive integers a, n, k with $n \geq 2a$ and $a \leq k \leq n - a$,

$$\left\langle \begin{matrix} -n \\ -k \end{matrix} \right\rangle^a = \frac{(-1)^a \binom{n}{k}}{\binom{n-a}{k} \binom{k}{k-a} a!};$$

and, for positive integers n and nonnegative integers r ,

$$\left\langle \begin{matrix} n+1 \\ 1 \end{matrix} \right\rangle^r = \frac{(n+1)^{\bar{r}}}{n^{\bar{r}} r!} = \frac{n+r}{n r!}.$$

Theorem 3.1. For $n \geq 4$, $2 \leq k \leq n - 2$, and a positive integer a ,

$$\left\langle \begin{matrix} n-1 \\ k \end{matrix} \right\rangle^a + \left\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rangle^a = \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle^a \frac{(n-1)(2nk - 2k^2 - 2a - 2n + 2 + an)}{(n+a-1)(k-1)(n-k-1)}. \quad (3.1)$$

Proof. Under the stated hypotheses, $k-1$, $n-k-1$ and $n-1$ are positive integers, so all the rising factorials involved are nonzero. For any m and positive integer a we have

$$\frac{(m+1)^{\bar{a}}}{m^{\bar{a}}} = \frac{(m+1)(m+2) \cdots (m+a)}{m(m+1) \cdots (m+a-1)} = \frac{m+a}{m},$$

whenever $m \neq 0$. Hence, from (1.1),

$$\frac{\left\langle \begin{matrix} n-1 \\ k \end{matrix} \right\rangle^a}{\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle^a} = \frac{(n-1)^{\bar{a}}}{n^{\bar{a}}} \cdot \frac{(n-k)^{\bar{a}}}{(n-k-1)^{\bar{a}}} = \frac{n-1}{n+a-1} \cdot \frac{n-k+a-1}{n-k-1},$$

and

$$\frac{\left\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rangle^a}{\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle^a} = \frac{(n-1)^{\bar{a}}}{n^{\bar{a}}} \cdot \frac{k^{\bar{a}}}{(k-1)^{\bar{a}}} = \frac{n-1}{n+a-1} \cdot \frac{k+a-1}{k-1}.$$

Adding these two ratios, we obtain

$$\frac{\left\langle \begin{matrix} n-1 \\ k \end{matrix} \right\rangle^a + \left\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rangle^a}{\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle^a} = \frac{n-1}{n+a-1} \cdot \frac{(n-k-1+a)(k-1) + (k-1+a)(n-k-1)}{(k-1)(n-k-1)}.$$

The numerator of the last fraction equals

$$2(k-1)(n-k-1) + a(k-1) + a(n-k-1) = 2nk - 2k^2 - 2n + 2 + a(n-2),$$

which is $2nk - 2k^2 - 2a - 2n + 2 + an$. This proves (3.1). $\square$

For $a = 1$, (3.1) reads

$$\left\langle \begin{matrix} n-1 \\ k \end{matrix} \right\rangle^1 + \left\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rangle^1 = \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle^1 \frac{(n-1)(2nk - 2k^2 - n)}{n(k-1)(n-k-1)},$$

which illustrates once more that the Type 2 coefficients do not obey Pascal's rule.

Proposition 3.2. For positive integers a, n, k with $n \geq 2a$ and $a \leq k \leq n - a$,

$$\left\langle \begin{matrix} -n \\ -k \end{matrix} \right\rangle^a = \frac{(-1)^a}{(n - k - a + 1)^{\bar{a}}} \frac{\binom{n}{k}}{\binom{n-a}{k-a}}. \quad (3.2)$$

Proof. From the definition,

$$\left\langle \begin{matrix} -n \\ -k \end{matrix} \right\rangle^a = \frac{(-n)^{\bar{a}}}{(-k)^{\bar{a}}(-n-k)^{\bar{a}}}.$$

For any x we have

$$(-x)^{\bar{a}} = (-x)(-x+1) \cdots (-x+a-1) = (-1)^a x(x-1) \cdots (x-a+1) = (-1)^a x^{\bar{a}} = (-1)^a (x-a+1)^{\bar{a}}.$$

Therefore,

$$\left\langle \begin{matrix} -n \\ -k \end{matrix} \right\rangle^a = \frac{(-1)^a n^{\bar{a}}}{(-1)^a k^{\bar{a}} (-1)^a (n-k)^{\bar{a}}} = (-1)^a \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_a. \quad (3.3)$$

Under our hypotheses $k \geq a$ and $n - k \geq a$, so the falling factorials in the denominator are nonzero. Moreover,

$$\frac{n^{\bar{a}}}{k^{\bar{a}}} = \frac{n! (k-a)!}{(n-a)! k!} = \frac{\binom{n}{k}}{\binom{n-a}{k-a}} \quad \text{and} \quad (n-k)^{\bar{a}} = (n-k-a+1)^{\bar{a}},$$

and (3.2) follows. $\square$

Relation (3.3) shows that, up to the sign $(-1)^a$, the Type 2 rising factorial binomial coefficients with negative indices are precisely the falling factorial binomial coefficients introduced in the first section.

4 Generating functions for rising factorial binomials

Generating functions provide a natural framework to study sequences and combinatorial identities. In this section we work with the Type 2 rising factorial binomial coefficients (1.1), with a a positive integer. Since $0^{\bar{a}} = 0$, for $a \geq 1$, definition (1.1) is meaningless for $k = 0$ and $k = n$, and we adopt the natural convention

$$\left\langle \begin{matrix} n \\ 0 \end{matrix} \right\rangle^a = \left\langle \begin{matrix} n \\ n \end{matrix} \right\rangle^a = 1, \quad n \geq 0.$$

Consider the ordinary generating function in k for fixed n and a :

$$G_n^a(x) = \sum_{k=0}^n \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle^a x^k. \quad (4.1)$$

From the definition,

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle^a = \frac{n^{\bar{a}}}{k^{\bar{a}}(n-k)^{\bar{a}}}, \quad 0 < k < n,$$

we may rewrite (4.1), for $n \geq 1$, as

$$G_n^a(x) = 1 + x^n + n^{\bar{a}} \sum_{k=1}^{n-1} \frac{x^k}{k^{\bar{a}}(n-k)^{\bar{a}}}.$$

The last sum is a convolution of reciprocals of rising factorial powers: if

$$f_a(t) = \sum_{j=1}^{\infty} \frac{t^j}{j^{\bar{a}}} = \frac{t}{a!} {}_2F_1\left(\begin{matrix} 1, 1 \\ a+1 \end{matrix}; t\right),$$

then

$$G_n^a(x) = 1 + x^n + n^{\bar{a}} [t^n] f_a(t) f_a(xt), \quad n \geq 1,$$

where $[t^n]$ denotes the coefficient of t^n .

Example 4.1. For small n , using $1^{\bar{a}} = a!$, the generating functions are explicitly:

$$\begin{aligned} G_2^a(x) &= \left\langle \begin{matrix} 2 \\ 0 \end{matrix} \right\rangle^a + \left\langle \begin{matrix} 2 \\ 1 \end{matrix} \right\rangle^a x + \left\langle \begin{matrix} 2 \\ 2 \end{matrix} \right\rangle^a x^2 \\ &= 1 + \frac{2^{\bar{a}}}{1^{\bar{a}} 1^{\bar{a}}} x + 1 \cdot x^2 \\ &= 1 + \frac{2^{\bar{a}}}{(a!)^2} x + x^2, \\ G_3^a(x) &= 1 + \frac{3^{\bar{a}}}{1^{\bar{a}} 2^{\bar{a}}} x + \frac{3^{\bar{a}}}{2^{\bar{a}} 1^{\bar{a}}} x^2 + x^3 \\ &= 1 + \frac{3^{\bar{a}}}{a! 2^{\bar{a}}} x + \frac{3^{\bar{a}}}{a! 2^{\bar{a}}} x^2 + x^3. \end{aligned}$$

Since $2^{\bar{a}} = (a+1)!$ and $3^{\bar{a}} = (a+2)!/2$, these can also be written as $G_2^a(x) = 1 + \frac{a+1}{a!} x + x^2$ and $G_3^a(x) = 1 + \frac{a+2}{2a!} (x + x^2) + x^3$.

These examples illustrate how the rising factorial powers weight the coefficients compared to ordinary binomial generating functions. Notice that, in contrast with the classical case, for $a = 1$ we do not recover $(1+x)^n$: for instance, $G_3^1(x) = 1 + \frac{3}{2}x + \frac{3}{2}x^2 + x^3$. The classical binomial generating function $(1+x)^n$ is instead recovered from the analogous generating function of the Type 1 coefficients,

$$\sum_{k=0}^n \left\| \begin{matrix} n \\ k \end{matrix} \right\|^a x^k,$$

when $a = 1$.

We can also consider the exponential generating function in n :

$$\mathcal{G}^a(x, t) = \sum_{n=0}^{\infty} \sum_{k=0}^n \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle^a \frac{x^k t^n}{n!} = \sum_{n=0}^{\infty} G_n^a(x) \frac{t^n}{n!}. \quad (4.2)$$

Preliminary analysis suggests that $\mathcal{G}^a(x, t)$ can be expressed in terms of generalized hypergeometric functions or Pochhammer symbols, in a manner reminiscent of the transformations in Section 2. This perspective provides a unifying view linking the combinatorial structure of the coefficients with the analytic properties of hypergeometric series.

Beyond generating functions, these constructions may lead to combinatorial or graph-theoretic interpretations. For instance, one may look for an interpretation of $\langle n \rangle_k^a$ as a weighted count of paths in a lattice, or of networks of size n with k distinguished nodes, where the weights are governed by the rising factorial power a . Such connections suggest that further exploration may reveal interesting enumerative identities and combinatorial models for Type 2 rising factorial binomials.

Table 1 highlights how the rising factorial power a modifies the coefficients, producing a “weighted” analogue of $(1+x)^n$. The case $n = 1$ is determined by the convention above, $G_1^a(x) = \langle 1 \rangle_0^a + \langle 1 \rangle_1^a x = 1+x$, for all a . For $a = 1$, the coefficients are $n/(k(n-k))$, $0 < k < n$, rather than the ordinary binomial numbers; for example, $G_4^1(x) = 1 + \frac{4}{3}x + x^2 + \frac{4}{3}x^3 + x^4$.

Table 1. Ordinary generating functions $G_n^a(x)$ of the Type 2 coefficients, for small n .

n	$G_n^a(x)$
1	$1 + x$
2	$1 + \frac{2^{\bar{a}}}{(a!)^2}x + x^2$
3	$1 + \frac{3^{\bar{a}}}{a! 2^{\bar{a}}}x + \frac{3^{\bar{a}}}{a! 2^{\bar{a}}}x^2 + x^3$
4	$1 + \frac{4^{\bar{a}}}{a! 3^{\bar{a}}}x + \frac{4^{\bar{a}}}{(2^{\bar{a}})^2}x^2 + \frac{4^{\bar{a}}}{a! 3^{\bar{a}}}x^3 + x^4$

5 Concluding comments

Type 2 rising factorial binomial coefficients are not a straightforward generalisation of ordinary binomials. In this regard, Jordan [3, p. 70] considered generalised binomial coefficients of the form

$$\binom{x}{n}_h = \frac{x(x-h)(x-2h)\cdots(x-nh+h)}{n!},$$

so that $\binom{x}{n}_1 = \binom{x}{n}$.

It would be of interest to extend these to a rising factorial Jordan coefficient. Denoting by

$$a^{\bar{m}|h} = a(a+h)(a+2h)\cdots(a+(m-1)h)$$

the rising factorial with increment h (so that $a^{\bar{m}|1} = a^{\bar{m}}$), such a coefficient could perhaps be defined by

$$\binom{x}{n}_{h,a} = \frac{a^{\overline{x/h}|h}}{a^{\overline{(x-n)/h}|h}} = \frac{a(a+h)(a+2h)\cdots(a+(x/h-1)h)}{a(a+h)(a+2h)\cdots(a+((x-n)/h-1)h)},$$

when x/h and $(x-n)/h$ are nonnegative integers, and it would be natural to investigate its congruence properties. This idea has been motivated recently by the work of Kim and Kim [4].

Carlitz's note [1] on a theorem of Glaisher [2] provides a means to study congruences for rising factorials. Let $p \geq 7$ be a prime and write

$$(x+1)^{\overline{p-1}} = (x+1)(x+2)\cdots(x+p-1) = x^{p-1} + A_1x^{p-2} + \cdots + A_{p-1}.$$

Then, for odd r with $5 \leq r \leq p-2$,

$$\frac{1}{2}p(p-r)A_{r-1} \equiv A_r \pmod{p^4},$$

which Carlitz [1] extended further to $\pmod{p^5}$ as

$$\frac{1}{2}p(p-r)A_{r-1} \equiv A_r + \frac{r(r-1)(r-2)B_{r-3}p^4}{24(r-3)} \pmod{p^5},$$

where B_r is a Bernoulli number. This naturally suggests investigating congruences modulo higher powers of a prime p for the coefficients introduced in this paper.

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