

The Hosoya-type Pell r -lengths of certain 2-generator groups

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Abstract: In this paper, we give the lengths of the periods of the Hosoya-type Pell r -orbits of the dihedral group D_{2m} , the semidihedral group SD_{2m} and the generalized quaternion group $Q_{2^{m+1}}$ by using some special recurrence sequences.

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1 Introduction

It is well-known that the Fibonacci and Pell sequences are defined as follows, respectively:

$$F_n = \begin{cases} 0, & \text{if } n = 0, \\ 1, & \text{if } n = 1, \\ F_{n-1} + F_{n-2}, & \text{otherwise} \end{cases}$$



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and

$$P_n = \begin{cases} 0, & \text{if } n = 0, \\ 1, & \text{if } n = 1, \\ 2P_{n-1} + P_{n-2}, & \text{otherwise.} \end{cases}$$

Number theoretic properties such as these obtained from homogeneous linear recurrence relations relevant to this paper have been studied recently by many authors; see, for example, [6, 10, 15].

The study of the linear recurrence sequences in groups began with the earlier work of Wall [27] where the ordinary Fibonacci sequences in cyclic groups were investigated. Other work on the Fibonacci sequences in cyclic groups is discussed in, for example, [5, 20, 24–26]. In the mid-eighties, Wilcox studied the Fibonacci sequences in abelian groups in [28]. In [3], the theory was expanded to non-abelian groups by Campbell *et al.* There, they defined the Fibonacci orbit of a 2-generator group. In next process, some special linear recurrence sequences defined by the aid of group elements have been studied by many authors; for example, [1, 2, 4, 8, 9, 11, 12, 14, 17–19, 21, 22]. In [23], Peters *et al.* contributed to theory via the Hosoya-type Pell sequence. In their paper they defined this sequence by considering Hosoya triangle and extended it to groups:

Definition 1.1. Let r be a positive integer and let P_r be the r -th Pell number. For a generating pair $(x_1, x_2) \in X$, we define the Hosoya-type Pell r -orbits of the 2-generator group G as follows:

$$a_0^r = x_1, a_1^r = x_2, a_{n+2}^r = (a_n^r)^{-P_r} (a_{n+1}^r)^{P_{r+1}}, \quad (n \geq 0).$$

For the generating pair (x_1, x_2) , we denote the r -th Hosoya-type Pell orbit of G by $P_{(x_1, x_2)}^{h, r}(G)$.

A sequence of group elements is eventually periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. In addition, a sequence of group elements is simply (purely) periodic with period k if the first k elements in the sequence form a repeating subsequence. The number of elements in the repeating subsequence is called the length of the period (eventually or simply) of the sequence. For example, the sequence $a, b, c, d, b, c, d, b, c, d, b, c, d, \dots$ is eventually periodic after the initial element a and has the length period 3. On the other hand, the sequence $a, b, c, d, a, b, c, d, a, b, c, d, a, b, c, d, \dots$ is simply periodic with the length of period 4.

If the sequence $P_{(x_1, x_2)}^{h, r}(G)$ is periodic (eventually or simply), then the symbol $l_{(x_1, x_2)}^{h, r}(G)$ denotes the length of the period of the sequence $P_{(x_1, x_2)}^{h, r}(G)$ (cf. [23]).

In [23] it is shown that the sequence $\{P_n^{(r, a, b)}\}$ defined by

$$P_0^{(r, a, b)} = a, P_1^{(r, a, b)} = b, P_{n+2}^{(r, a, b)} = P_{r+1} \cdot P_{n+1}^{(r, a, b)} - P_r \cdot P_n^{(r, a, b)}, \quad (n \geq 0, r \geq 1),$$

where a and b are arbitrary integers and P_r is the r -th Pell number. The sequence $\{P_n^{(r, a, b)}\}$ in the integers modulo m can be written as $P_n^r(\mathbb{Z}_m; a, b)$ and the length of the period of the sequence $P_n^r(\mathbb{Z}_m; a, b)$ is denoted by $h^{(r, a, b)}(m)$ (cf. [23]).

This paper is in the spirit of [23]. In this paper, we examine the behaviors of the lengths of the periods of the Hosoya-type Pell r -sequences in certain 2-generator groups.

2 Main results

There are various criteria used while discussing the algebraic properties of a group through a sequence defined by means of group elements. As it is known, Periodicity is one of the most important of these criteria. In this paper, we concentrate on periodicity of the Hosoya-type Pell r -orbits of the dihedral group D_{2m} , the semidihedral group SD_{2m} and the generalized quaternion group $Q_{2^{m+1}}$ with respect to the generating pairs (x, y) and (y, x) .

Consider the sequences

$$u_1^r = 1, u_2^r = 0 \text{ and } u_n^r = \begin{cases} -P_r u_{n-2}^r, & \text{if } n \text{ is odd,} \\ -P_{r+1} u_{n-1}^r + u_{n-2}^r, & \text{if } n \text{ is even} \end{cases}$$

and

$$v_1^r = 0, v_2^r = 1 \text{ and } v_n^r = \begin{cases} -P_{r+1} v_{n-1}^r + v_{n-2}^r, & \text{if } n \text{ is odd,} \\ -P_r v_{n-2}^r, & \text{if } n \text{ is even} \end{cases}$$

where r is a positive integer and P_r is the r -th term of the Pell sequence. If we reduce the sequences $\{u_n^r\}$ and $\{v_n^r\}$ by a modulus m taking least nonnegative residues, then we get the following recurrence sequences:

$$\{u_n^r(m)\} = \{u_1^r(m), u_2^r(m), u_3^r(m), \dots, u_i^r(m), \dots\}$$

and

$$\{v_n^r(m)\} = \{v_1^r(m), v_2^r(m), v_3^r(m), \dots, v_i^r(m), \dots\}$$

where $u_i^r(m)$ and $v_i^r(m)$ are used to mean the i -th term of the sequences $\{u_n^r\}$ and $\{v_n^r\}$ when read modulo m , respectively. It is easy to prove that the sequences $\{u_n^r(m)\}$ and $\{v_n^r(m)\}$ are periodic (eventually or simply). We denote the lengths of the periods of the sequences $\{u_n^r(m)\}$ and $\{v_n^r(m)\}$ by $h_{u_n}^r(m)$ and $h_{v_n}^r(m)$, respectively. From the recurrence relations in the definitions of the sequences $\{u_n^r\}$ and $\{v_n^r\}$, we may write the following companion matrices:

$$A_1^{(r)} = \begin{bmatrix} 0 & -P_r \\ 1 & 0 \end{bmatrix} \text{ and } A_2^{(r)} = \begin{bmatrix} -P_{r+1} & 1 \\ 1 & 0 \end{bmatrix}.$$

Let $B_1^{(r)} = A_2^{(r)} A_1^{(r)} = \begin{bmatrix} 1 & P_{r+1} P_r \\ 0 & -P_r \end{bmatrix}$ and let $B_2^{(r)} = A_1^{(r)} A_2^{(r)} = \begin{bmatrix} -P_r & 0 \\ -P_{r+1} & 1 \end{bmatrix}$. If we let

$$\left(M_1^{(r)}\right)^n = \begin{cases} A_1^{(r)} \left(B_1^{(r)}\right)^{\frac{n-1}{2}}, & \text{if } n \text{ is odd,} \\ \left(B_1^{(r)}\right)^{\frac{n}{2}}, & \text{if } n \text{ is even} \end{cases}$$

and

$$\left(M_2^{(r)}\right)^n = \begin{cases} A_2^{(r)} \left(B_2^{(r)}\right)^{\frac{n-1}{2}}, & \text{if } n \text{ is odd,} \\ \left(B_2^{(r)}\right)^{\frac{n}{2}}, & \text{if } n \text{ is even} \end{cases}$$

then by using the conventions found in [4] and [16], we can easily write the following vector relations:

$$\left(M_1^{(r)}\right)^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} u_{n+2}^r \\ u_{n+1}^r \end{bmatrix}$$

and

$$\left(M_2^{(r)}\right)^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} v_{n+2}^r \\ v_{n+1}^r \end{bmatrix}$$

for $n \geq 0$. From this result we immediately deduce that $h_{u_n}^r(m)$ and $h_{v_n}^r(m)$ are the smallest positive integers α and β that satisfy the following requirements, respectively:

$$\left(M_1^{(r)}\right)^\alpha \equiv I_2 \pmod{m}$$

and

$$\left(M_2^{(r)}\right)^\beta \equiv I_2 \pmod{m},$$

where I_2 is the 2×2 identity matrix and $\gcd(m, P_r) = 1$.

In [23], the lengths of the periods of the Hosoya-type Pell r -orbits of the modular maximal-cyclic group M_n were obtained. Now we concentrate on the dihedral group D_{2m} , the semidihedral group SD_{2m} and the generalized quaternion group $Q_{2^{m+1}}$ in order to expand the concept.

We will deal with the following presentation of the dihedral group D_{2m} of order $2m$:

$$\langle x, y : x^m = y^2 = (xy)^2 = e \rangle.$$

We shall now address the periods of the r -orbits $P_{(x,y)}^{h,r}(D_{2m})$ and $P_{(y,x)}^{h,r}(D_{2m})$ by using the operators $h_{u_n}^r(m)$, $h_{v_n}^r(m)$ and $h^{(r,a,b)}(m)$.

Theorem 2.1. *Let m be an integer such that $m \geq 3$ and let r be a positive integer. Then the Hosoya-type Pell r -orbits of the dihedral group D_{2m} with respect to the generating pairs both (x, y) and (y, x) are periodic sequences. The lengths of the periods of the sequences $P_{(x,y)}^{h,r}(D_{2m})$ and $P_{(y,x)}^{h,r}(D_{2m})$ are as follows:*

$$i) \ l_{(x,y)}^{h,1}(D_{2m}) = l_{(y,x)}^{h,1}(D_{2m}) = 4.$$

$$ii) \text{ If } r \geq 2 \text{ is an odd integer, that is, } P_r \text{ is odd and } P_{r+1} \text{ is even, then } l_{(x,y)}^{h,r}(D_{2m}) = \text{lcm}(h_{u_n}^r(m), 2) \text{ and } l_{(y,x)}^{h,r}(D_{2m}) = \text{lcm}(h_{v_n}^r(m), 2).$$

$$iii) \text{ If } r \geq 2 \text{ is an even integer, that is, } P_r \text{ is even and } P_{r+1} \text{ is odd, then } l_{(x,y)}^{h,r}(D_{2m}) = 1 \text{ and } l_{(y,x)}^{h,r}(D_{2m}) = h^{(r,0,1)}(m).$$

Proof. i) From the definition of the Hosoya-type Pell r -orbit we can write terms of the sequences $P_{(x,y)}^{h,1}(D_{2m})$ and $P_{(y,x)}^{h,1}(D_{2m})$ in the following forms, respectively:

$$a_0^1 = x, \ a_1^1 = y, \ a_{n+2}^1 = (a_n^1)^{-1} (a_{n+1}^1)^2, \ (n \geq 0)$$

and

$$a_0^1 = y, \ a_1^1 = x, \ a_{n+2}^1 = (a_n^1)^{-1} (a_{n+1}^1)^2, \ (n \geq 0).$$

Using the fact that $xy = yx^{-1}$, $yx = x^{-1}y$, $(x^n y)^2 = e$ and $(yx^n)^2 = e$, we get the sequences $P_{(x,y)}^{h,1}(D_{2m})$ and $P_{(y,x)}^{h,1}(D_{2m})$ as follows, respectively:

$$\{x, y, x^{-1}, x^2 y, x, y, \dots\}$$

and

$$\{y, x, yx^2, x^{-1}, y, x, \dots\}.$$

Then, it is clear that the sequences $P_{(x,y)}^{h,1}(D_{2m})$ and $P_{(y,x)}^{h,1}(D_{2m})$ are *simply periodic* and whose periods are 4.

ii) Let $r \geq 2$ be a positive odd integer.

Firstly, we consider the Hosoya-type Pell r -orbit of the dihedral group D_{2m} with respect to the generating pair (x, y) . Let us calculate the terms of the sequence $P_{(x,y)}^{h,r}(D_{2m})$ depending on the recurrence relation $a_{n+2}^r = (a_n^r)^{-P_r} (a_{n+1}^r)^{P_{r+1}}$:

$$\begin{aligned} & x, \\ & y, \\ & x^{-P_r} y^{P_{r+1}} = x^{-P_r}, \\ & y^{-P_r} (x^{-P_r})^{P_{r+1}} = yx^{-P_r P_{r+1}} = x^{P_r P_{r+1}} y, \\ & (x^{-P_r})^{-P_r} (x^{P_r P_{r+1}} y)^{P_{r+1}} = x^{(P_r)^2}, \\ & (x^{P_r P_{r+1}} y)^{-P_r} (x^{(P_r)^2})^{P_{r+1}} = yx^{-P_r P_{r+1} + (P_r)^2 P_{r+1}} = x^{-(P_r)^2 P_{r+1} + P_r P_{r+1}} y, \\ & (x^{(P_r)^2})^{-P_r} (x^{-(P_r)^2 P_{r+1} + P_r P_{r+1}} y)^{P_{r+1}} = x^{-(P_r)^3}, \\ & (x^{-(P_r)^2 P_{r+1} + P_r P_{r+1}} y)^{-P_r} (x^{-(P_r)^3})^{P_{r+1}} = yx^{(P_r)^2 P_{r+1} - P_r P_{r+1} - (P_r)^3 P_{r+1}} \\ & = x^{(P_r)^3 P_{r+1} - (P_r)^2 P_{r+1} + P_r P_{r+1}} y, \\ & \vdots \end{aligned}$$

For $n \geq 1$ we can see that the sequence $P_{(x,y)}^{h,r}(D_{2m})$ will separate into some natural layers and each layer will be of the form

$$\begin{aligned} a_{2n}^r &= x^{(-P_r)^n}, \\ a_{2n+1}^r &= x^{P_r P_{r+1}} ((-1)^{n+1} (P_r)^{n-1} + (-1)^n (P_r)^{n-2} + (-1)^{n-1} (P_r)^{n-3} + \dots - P_r + 1) y. \end{aligned}$$

It is interesting to note that when $(m, P_r) = 1$ the sequence $P_{(x,y)}^{h,r}(D_{2m})$ is *simply periodic* and when $(m, P_r) \neq 1$ the sequence is *periodic*. Considering the sequence $\{u_n^r\}$, we may write the terms of $P_{(x,y)}^{h,r}(D_{2m})$ as follows:

$$a_{2n}^r = x^{u_{2n+1}^r}$$

and

$$a_{2n+1}^r = x^{u_{2n+2}^r} y$$

for $n \geq 0$. Obviously, the last relations imply that the length of the period of $P_{(x,y)}^{h,r}(D_{2m})$ is $\text{lcm}(h_{u_n}^r(m), 2)$ for $m \geq 3$.

Now we concentrate on determining the period of the sequence $P_{(y,x)}^{h,r}(D_{2m})$ by the aid of the operator $h_{v_n}^r(m)$. Using an argument similar to that given above, we see that the sequence $P_{(y,x)}^{h,r}(D_{2m})$ is of the form:

$$\begin{aligned} a_0^r &= y, \quad a_1^r = x, \dots, \\ a_{2n}^r &= x^{P_{r+1}((-1)^n(P_r)^{n-1} + (-1)^{n-1}(P_r)^{n-2} + (-1)^{n-2}(P_r)^{n-3} + \dots + P_r - 1)} y, \quad a_{2n+1}^r = x^{(-P_r)^n} \end{aligned}$$

for $n \geq 1$. Thus we can easily derive

$$a_{2n}^r = x^{v_{2n+1}^r} y, \quad a_{2n+1}^r = x^{v_{2n+2}^r}, \quad (n \geq 0)$$

and so $l_{(y,x)}^{h,r}(D_{2m}) = \text{lcm}(h_{v_n}^r(m), 2)$. Also note that if $(m, P_r) = 1$, then $P_{(y,x)}^{h,r}(D_{2m})$ is a *simply periodic* sequence; if $(m, P_r) \neq 1$, then the sequence is *eventually periodic*.

iii) Let $r \geq 2$ be a positive even integer.

By direct calculation, it is easy to see that the sequence $P_{(x,y)}^{h,r}(D_{2m})$ is

$$\{x, y, x^{-P_r}y, x^{-P_r}y, \dots, x^{-P_r}y, x^{-P_r}y, \dots\}.$$

Thus it is verified that the sequence is *eventually periodic* and $l_{(x,y)}^{h,r}(D_{2m}) = 1$.

Now we examine the length of the period of the Hosoya-type Pell r -orbit of the dihedral group D_{2m} with respect to the generating pair (y, x) by considering the period of the sequence $\{P_n^{(r,0,1)}\}$. Here the start of the sequence $P_{(y,x)}^{h,r}(D_{2m})$ is

$$\{y, x, x^{P_{r+1}}, \dots\}.$$

More precisely,

$$a_0^r = y, \quad a_n^r = x^{P_n^{(r,0,1)}} \quad (n \geq 1).$$

Then, it is easy to see that the sequence is *eventually periodic* and $l_{(y,x)}^{h,r}(D_{2m}) = h^{(r,0,1)}(m)$. $\square$

It is clear that $\det(B_1^{(r)}) = \det(B_2^{(r)}) = -P_r$. With this in mind, we consider the lengths $l_{(a,b)}^{h,r}(D_{2m})$ and $l_{(b,a)}^{h,r}(D_{2m})$ when $m \nmid P_r$.

Corollary 2.2. *Let $p \geq 3$ be a prime and t be a positive integer and let (a, b) be one of the generating pairs (x, y) and (y, x) . If $r \geq 2$ is an odd integer with $p \nmid P_r$, then $l_{(a,b)}^{h,r}(D_{2p^{t+1}}) = l_{(a,b)}^{h,r}(D_{2p^t})$ or $l_{(a,b)}^{h,r}(D_{2p^{t+1}}) = p \cdot l_{(a,b)}^{h,r}(D_{2p^t})$.*

Proof. Let us consider the sequence $P_{(x,y)}^{h,r}(D_{2m})$. Then, by the first condition above, it is known that $h_{u_n}^r(p^{t+1})$ is the smallest positive integer satisfying $(M_1^{(r)})^{h_{u_n}^r(p^{t+1})} \equiv I_2 \pmod{p^{t+1}}$. So we can derive $(M_1^{(r)})^{h_{u_n}^r(p^{t+1})} \equiv I_2 \pmod{p^t}$, which means that $h_{u_n}^r(p^t)$ divides $h_{u_n}^r(p^{t+1})$. Moreover, we may write

$$(M_1^{(r)})^{h_{u_n}^r(p^t)} = \begin{bmatrix} k_1 \cdot p^t + 1 & k_2 \cdot p^t \\ k_3 \cdot p^t & k_4 \cdot p^t + 1 \end{bmatrix},$$

where k_1, k_2, k_3 and k_4 are positive integers. Then, by the binomial theorem, we have

$$\left(M_1^{(r)}\right)^{p \cdot h_{u_n}^r(p^t)} \equiv I_2 \pmod{p^{t+1}},$$

which implies that $h_{u_n}^r(p^{t+1})$ divides $p \cdot h_{u_n}^r(p^t)$. Therefore, it can be clearly seen that $h_{u_n}^r(p^{t+1}) = h_{u_n}^r(p^t)$ or $h_{u_n}^r(p^{t+1}) = p \cdot h_{u_n}^r(p^t)$. Since $l_{(x,y)}^{h,r}(D_{2m}) = h_{u_n}^r(m)$ when $r \geq 2$ is an odd integer, we have the conclusion.

The proof for the sequence $P_{(y,x)}^{h,r}(D_{2m})$ is directly similar to the above and is omitted. $\square$

Corollary 2.3. *Let m_1 and m_2 be integers greater than or equal to 3 and let (a, b) be one of the generating pairs (x, y) and (y, x) . If $r \geq 2$ is an odd integer with $m_1 \nmid P_r$ and $m_2 \nmid P_r$, then $l_{(a,b)}^{h,r}(D_{2\text{lcm}(m_1, m_2)}) = \text{lcm}\left(l_{(a,b)}^{h,r}(D_{2m_1}), l_{(a,b)}^{h,r}(D_{2m_2})\right)$.*

Proof. We now deal with the sequence $P_{(y,x)}^{h,r}(D_{2m})$. Let $\text{lcm}(m_1, m_2) = m$. Clearly, $\left(M_2^{(r)}\right)^{h_{v_n}^r(m_1)} \equiv I_2 \pmod{m_1}$ and $\left(M_2^{(r)}\right)^{h_{v_n}^r(m_2)} \equiv I_2 \pmod{m_2}$. By means of the least common multiple operation, this implies that

$$\left(M_2^{(r)}\right)^{h_{v_n}^r(m)} \equiv I_2 \pmod{m_1}$$

and

$$\left(M_2^{(r)}\right)^{h_{v_n}^r(m)} \equiv I_2 \pmod{m_2}.$$

Thus we immediately deduce $h_{v_n}^r(m_1) \mid h_{v_n}^r(m)$ and $h_{v_n}^r(m_2) \mid h_{v_n}^r(m)$, which means that $\text{lcm}(h_{v_n}^r(m_1), h_{v_n}^r(m_2))$ divides $h_{v_n}^r(\text{lcm}(m_1, m_2))$.

Now assume that $\text{lcm}(h_{v_n}^r(m_1), h_{v_n}^r(m_2)) = \alpha$. Then we may write $\left(M_2^{(r)}\right)^\alpha \equiv I_2 \pmod{m_1}$ and $\left(M_2^{(r)}\right)^\alpha \equiv I_2 \pmod{m_2}$, which implies that $\left(M_2^{(r)}\right)^\alpha \equiv I_2 \pmod{m}$. So we see that $\text{lcm}(h_{v_n}^r(m_1), h_{v_n}^r(m_2))$ is divisible by $h_{v_n}^r(\text{lcm}(m_1, m_2))$. Thus we obtain

$$\text{lcm}(h_{v_n}^r(m_1), h_{v_n}^r(m_2)) = h_{v_n}^r(\text{lcm}(m_1, m_2)).$$

Since $l_{(y,x)}^{h,r}(D_{2m}) = h_{v_n}^r(m)$ when $r \geq 2$ is an odd integer, we have the conclusion. $\square$

The semidihedral (quasidihedral) group of order 2^m , SD_{2^m} is defined by the presentation

$$\langle x, y : x^{2^{m-1}} = y^2 = e, yxy = x^{2^{m-2}-1} \rangle.$$

For more information on this group see [13].

Now we consider the lengths of the periods of the Hosoya-type Pell r -orbits of the semidihedral group SD_{2^m} when $m \geq 4$. We give the following Theorem.

Theorem 2.4. *Let m be an integer such that $m \geq 4$ and let r be a positive integer. Then the lengths of the periods of the sequences $P_{(x,y)}^{h,r}(SD_{2^m})$ and $P_{(y,x)}^{h,r}(SD_{2^m})$ are as follows:*

$$i) \ l_{(x,y)}^{h,1}(SD_{2^m}) = l_{(y,x)}^{h,1}(SD_{2^m}) = 4.$$

$$ii) \text{ If } r \geq 2 \text{ is an odd integer, that is, } P_r \text{ is odd and } P_{r+1} \text{ is even, then } l_{(x,y)}^{h,r}(SD_{2^m}) = \text{lcm}(h_{u_n}^r(2^{m-1}), 2) \text{ and } l_{(y,x)}^{h,r}(SD_{2^m}) = \text{lcm}(h_{v_n}^r(2^{m-1}), 2).$$

iii) If $r \geq 2$ is an even integer, that is, P_r is even and P_{r+1} is odd, then $l_{(x,y)}^{h,r}(SD_{2^m}) = 1$ and $l_{(y,x)}^{h,r}(SD_{2^m}) = h^{(r,0,1)}(2^{m-1})$.

Proof. The proof is similar to the above and is omitted. $\square$

A generalized quaternion (dicyclic) group $Q_{2^{m+1}}$ of order 2^{m+1} is defined by the presentation

$$\langle x, y : x^{2^m} = y^4 = e, x^{2^{m-1}} = y^2, y^{-1}xy = x^{-1} \rangle.$$

For more information on this group see [7].

We give the following Theorem for the computing the lengths $l_{(x,y)}^{h,r}(Q_{2^{m+1}})$ and $l_{(y,x)}^{h,r}(Q_{2^{m+1}})$ by the aid of the operators $h_{u_n}^r(m)$, $h_{v_n}^r(m)$ and $h^{(r,a,b)}(m)$.

Theorem 2.5. *Let m be an integer such that $m \geq 2$ and let r be a positive integer. Then the lengths of the periods of the sequences $P_{(x,y)}^{h,r}(Q_{2^{m+1}})$ and $P_{(y,x)}^{h,r}(Q_{2^{m+1}})$ are as follows:*

i) If $m = 2$, then

$$l_{(x,y)}^{h,r}(Q_8) = l_{(y,x)}^{h,r}(Q_8) = \begin{cases} 2, & \text{if } r \equiv 1, 2 \pmod{4}, \\ 4, & \text{if } r \equiv 3 \pmod{4}, \\ 1, & \text{if } r \equiv 0 \pmod{4} \end{cases}$$

for $r \geq 1$.

ii) If $m \geq 3$, then three subcases occur:

$$i') l_{(x,y)}^{h,1}(Q_{2^{m+1}}) = l_{(y,x)}^{h,1}(Q_{2^{m+1}}) = 4.$$

$$ii') \text{ If } P_r \equiv 1 \pmod{4} \text{ and } P_{r+1} \text{ is even, then } l_{(x,y)}^{h,r}(Q_{2^{m+1}}) = \text{lcm}(h_{u_n}^r(2^m), 4) \text{ and } l_{(y,x)}^{h,r}(Q_{2^{m+1}}) = \text{lcm}(h_{v_n}^r(2^m), 4).$$

iii') If $P_{r+1} \equiv 1 \pmod{4}$ and P_r is even, then

$$l_{(x,y)}^{h,r}(Q_{2^{m+1}}) = \begin{cases} 2, & \text{if } P_r \equiv 2 \pmod{4}, \\ 1, & \text{if } P_r \equiv 0 \pmod{4} \end{cases}$$

and

$$l_{(y,x)}^{h,r}(Q_{2^{m+1}}) = \begin{cases} h^{(r,1,P_{r+1}+2^{m-1})}(2^m), & \text{if } P_r \equiv 2 \pmod{4}, \\ h^{(r,0,1)}(2^m), & \text{if } P_r \equiv 0 \pmod{4}. \end{cases}$$

Proof. We first note that $xy = yx^{-1}$, $yx = x^{-1}y$, $(x^n y)^2 = y^2$ and $(yx^n)^2 = y^2$. The proof of every statements follow from an analogous argument as the above. For all values of r , we obtain the Hosoya-type Pell r -orbits of the group $Q_{2^{m+1}}$ as follows:

When $m = 2$ and $r \equiv 1 \pmod{4}$, for the generating pair (x, y)

$$a_0^r = x, a_1^r = y, a_2^r = x, a_3^r = y, \dots,$$

and so it is *simply periodic* with period 2.

When $m = 2$ and $r \equiv 1 \pmod{4}$, for the generating pair (y, x)

$$a_0^r = y, a_1^r = x, a_2^r = y, a_3^r = x, \dots,$$

and so it is *simply periodic* with period 2.

When $m = 2$ and $r \equiv 2 \pmod{4}$, for the generating pair (x, y)

$$a_0^r = x, a_1^r = y, a_2^r = y^{-1}, a_3^r = y, a_4^r = y^{-1}, \dots,$$

and so it is *eventually periodic* with period 2.

When $m = 2$ and $r \equiv 2 \pmod{4}$, for the generating pair (y, x)

$$a_0^r = y, a_1^r = x, a_2^r = x^{-1}, a_3^r = x, a_4^r = x^{-1}, \dots,$$

and so it is *eventually periodic* with period 2.

When $m = 2$ and $r \equiv 3 \pmod{4}$, for the generating pair (x, y)

$$a_0^r = x, a_1^r = y, a_2^r = x^{-1}, a_3^r = y^{-1}, a_4^r = x, a_5^r = y, \dots,$$

and so it is *simply periodic* with period 4.

When $m = 2$ and $r \equiv 3 \pmod{4}$, for the generating pair (y, x)

$$a_0^r = y, a_1^r = x, a_2^r = y^{-1}, a_3^r = x^{-1}, a_4^r = y, a_5^r = x, \dots,$$

and so it is *simply periodic* with period 4.

When $m = 2$ and $r \equiv 0 \pmod{4}$, for the generating pair (x, y)

$$a_0^r = x, a_1^r = y, a_2^r = y, a_3^r = y, a_4^r = y, \dots,$$

and so it is *eventually periodic* with period 1.

When $m = 2$ and $r \equiv 0 \pmod{4}$, for the generating pair (y, x)

$$a_0^r = y, a_1^r = x, a_2^r = x, a_3^r = x, a_4^r = x, \dots,$$

and so it is *eventually periodic* with period 1.

When $r = 1$, for the generating pair (x, y)

$$a_0^1 = x, a_1^1 = y, a_2^1 = x^{-1}y^2, a_3^1 = x^2y^3, a_4^1 = x, a_5^1 = y, \dots,$$

and so it is *simply periodic* with period 4.

Now we consider the case $m \geq 3$. Then we have the following results:

When $r = 1$, for the generating pair (y, x)

$$a_0^1 = y, a_1^1 = x, a_2^1 = y^{-1}x^2, a_3^1 = y^2x^{-1}, a_4^1 = y, a_5^1 = x, \dots,$$

and so it is *simply periodic* with period 4.

When $P_r \equiv 1 \pmod{4}$ and $P_{r+1} \equiv 0 \pmod{4}$, for the generating pair (x, y)

$$a_0^r = x, a_1^r = y, a_2^r = x^{-P_r}, a_3^r = x^{P_r P_{r+1}} y^3, \dots, \\ a_{4n}^r = x^{u_{4n+1}^r}, a_{4n+1}^r = x^{u_{4n+2}^r} y, a_{4n+2}^r = x^{u_{4n+3}^r}, a_{4n+3}^r = x^{u_{4n+4}^r} y^3, \dots,$$

which has period $\text{lcm}(h_{u_n}^r(2^m), 4)$.

When $P_r \equiv 1 \pmod{4}$ and $P_{r+1} \equiv 2 \pmod{4}$, for the generating pair (x, y)

$$a_0^r = x, a_1^r = y, a_2^r = x^{-P_r} y^2, a_3^r = x^{P_r P_{r+1}} y^3, \dots, \\ a_{4n}^r = x^{u_{4n+1}^r}, a_{4n+1}^r = x^{u_{4n+2}^r} y, a_{4n+2}^r = x^{u_{4n+3}^r} y^2, a_{4n+3}^r = x^{u_{4n+4}^r} y^3, \dots,$$

which has period $\text{lcm}(h_{u_n}^r(2^m), 4)$.

When $P_r \equiv 1 \pmod{4}$ and $P_{r+1} \equiv 0 \pmod{4}$, for the generating pair (y, x)

$$a_0^r = y, a_1^r = x, a_2^r = x^{-P_{r+1}} y^3, a_3^r = x^{-P_r}, \dots, \\ a_{4n}^r = x^{v_{4n+1}^r} y, a_{4n+1}^r = x^{v_{4n+2}^r}, a_{4n+2}^r = x^{v_{4n+3}^r} y^3, a_{4n+3}^r = x^{v_{4n+4}^r}, \dots,$$

which has period $\text{lcm}(h_{v_n}^r(2^m), 4)$.

When $P_r \equiv 1 \pmod{4}$ and $P_{r+1} \equiv 2 \pmod{4}$, for the generating pair (y, x)

$$a_0^r = y, a_1^r = x, a_2^r = x^{-P_{r+1}} y^3, a_3^r = y^2 x^{-P_r}, \dots, \\ a_{4n}^r = x^{v_{4n+1}^r} y, a_{4n+1}^r = x^{v_{4n+2}^r}, a_{4n+2}^r = x^{v_{4n+3}^r} y^3, a_{4n+3}^r = y^2 x^{v_{4n+4}^r}, \dots,$$

which has period $\text{lcm}(h_{v_n}^r(2^m), 4)$.

Note that the last four sequences are *eventually periodic*.

When $P_r \equiv 2 \pmod{4}$ and $P_{r+1} \equiv 1 \pmod{4}$, for the generating pair (x, y)

$$a_0^r = x, a_1^r = y, a_2^r = x^{-P_r} y, a_3^r = x^{-P_r} y^3, a_4^r = x^{-P_r} y, a_5^r = x^{-P_r} y^3, \dots,$$

and so it is *eventually periodic* with period 2.

When $P_r \equiv 0 \pmod{4}$ and $P_{r+1} \equiv 1 \pmod{4}$, for the generating pair (x, y)

$$a_0^r = x, a_1^r = y, a_2^r = x^{-P_r} y, a_3^r = x^{-P_r} y, a_4^r = x^{-P_r} y, a_5^r = x^{-P_r} y, \dots,$$

and so it is *eventually periodic* with period 1.

When $P_r \equiv 2 \pmod{4}$ and $P_{r+1} \equiv 1 \pmod{4}$, for the generating pair (y, x)

$$a_0^r = y, a_1^r = x, a_2^r = x^{P_{r+1}+2^{m-1}}, \dots,$$

and so it is *eventually periodic* with period $h^{(r,1,P_{r+1}+2^{m-1})}(2^m)$.

When $P_r \equiv 0 \pmod{4}$ and $P_{r+1} \equiv 1 \pmod{4}$, for the generating pair (y, x)

$$a_0^r = y, a_1^r = x, a_2^r = x^{P_{r+1}}, \dots,$$

and so it is *eventually periodic* with period $h^{(r,0,1)}(2^m)$.

Thus we have the conclusion. □

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