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Combinatorial and arithmetic properties of n^r-partitions: A unified analytic generalization

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Abstract: This paper presents a comprehensive study of n^r-partitions, defined by the generating function $\prod_{n=1}^{\infty}(1-q^n)^{-n^r}$, which interpolates analytically between ordinary partitions ($r=0$), plane partitions ($r=1$), and higher-dimensional combinatorial structures ($r \geq 2$).

Our main contributions include: (1) explicit formulas relating $p_{(r)}(n)$ to divisor sums $\sigma_{r+1}(n)$ and integer compositions; (2) novel recurrence relations and congruence properties modulo primes and composite numbers; (3) complete Bell polynomial representations; and (4) combinatorial interpretations through generating function techniques. For weighted restricted n^r-partitions into three parts, we establish exact quasi-polynomial formulas.

The results extend classical partition theory while providing new connections between combinatorics and number theory. The methods developed here offer a unified framework for investigating

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partition-like structures across dimensions, with applications in enumerative combinatorics.

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1 Introduction

The theory of integer partitions, since Euler's seminal work *De partitione numerorum* [11], has been a cornerstone of combinatorics and number theory. The generating function for the number of integer partitions $p(n)$ (OEIS A000041) is:

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

This result opened the door to generalizations including plane partitions, colored partitions, overpartitions, and multipartitions.

Let $pp(n)$ denote the number of plane partitions of n (OEIS A000219), introduced by MacMahon [13, 14]. Their generating function is:

$$\sum_{n=0}^{\infty} pp(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n}.$$

MacMahon also explored solid partitions (three-dimensional) and higher-dimensional analogues, but showed that no simple product formula exists for dimensions $r \geq 2$.

A closely related family is the theory of *multipartitions* (or k-tuple partitions), which has been extensively studied from both combinatorial and arithmetic perspectives. A multipartition of n into k components is a k-tuple $(\lambda^{(1)}, \ldots, \lambda^{(k)})$ of ordinary partitions such that the sum of all parts across all components equals n. The generating function for k-tuple partitions is $\prod_{n=1}^{\infty}(1-q^n)^{-k}$. Andrews [5] provides a comprehensive survey of multipartitions, including congruence properties and identities. Recent work by Dousse [10], Ranganatha [21], Nadji and Ahmia [15], Nath, Saikia and Sarma [16], Saikia, Sarma and Sellers [22], and Guru and Jamudulia [12] has revealed deep arithmetic properties of multipartitions with various regularity conditions. These results motivate our investigation of weighted partitions where the exponent grows polynomially in n.

In this paper, we introduce and study a natural analytic generalization that interpolates between ordinary partitions ($r = 0$), plane partitions ($r = 1$), and higher weighted structures ($r \geq 2$). An **n^r-partition** of n is a multiset of colored integers (j, c), where $j \in \mathbb{Z}^+$ (the size of the part) and $c \in \{1, 2, \ldots, j^r\}$ (the color), such that the sum of the j's over all occurrences equals n. The order of parts does not matter, and each pair (j, c) may appear multiple times.

Since the generating function for n^r-partitions is given by the infinite product:

$$\sum_{n=0}^{\infty} p_{(r)}(n)q^n = \prod_{j=1}^{\infty}(1-q^j)^{-j^r},$$

each factor $(1-q^j)^{-j^r}$ expands as:

$$(1-q^j)^{-j^r} = \sum_{m=0}^{\infty} \binom{j^r+m-1}{m} q^{jm},$$

where $\binom{j^r+m-1}{m}$ counts the number of ways to choose m copies of the part j, each with j^r distinct colors. This follows from the standard combinatorial interpretation of the generalized binomial coefficient: it enumerates the distributions of m indistinguishable objects into j^r distinguishable categories.

In Table 1 we present values of the generalized partition function $p_{(r)}(n)$ for small n and r, where $r = 0$ gives ordinary partitions, $r = 1$ gives plane partitions, and $r \geq 2$ gives the new n^r-partitions.

Table 1. Values of the n^r-partition function $p_{(r)}(n)$ for $0 \leq n \leq 10$ and $r = 0, 1, 2, 3, 4$.

$\boldsymbol{n}$	$\boldsymbol{r=0}$	$\boldsymbol{r=1}$	$\boldsymbol{r=2}$	$\boldsymbol{r=3}$	$\boldsymbol{r=4}$
0	1	1	1	1	1
1	1	1	1	1	1
2	2	3	5	9	17
3	3	6	14	36	98
4	5	13	46	136	562
5	7	24	124	477	3010
6	11	48	336	1926	15974
7	15	86	1296	9346	87090
8	22	160	4392	41512	464594
9	30	282	14520	186400	2470026
10	42	500	47272	808064	13012610

Agarwal and Andrews [1] introduced the concept of *partitions with N copies of N*, which correspond exactly to plane partitions. In their framework, for each positive integer N, there are N distinct colors or copies of N, denoted $N_1, N_2, \ldots, N_N$. A partition is then a multiset of such colored parts, with the order of colors irrelevant. The generating function becomes $\prod_{n=1}^{\infty}(1-q^n)^{-n}$, confirming the combinatorial interpretation of plane partitions.

Our n^r-partitions generalize this idea by allowing each part j to appear in j^r distinct colors, denoted by ordered pairs (j, c) with $1 \leq c \leq j^r$. For $r = 1$ we recover the Agarwal–Andrews model, and for $r > 1$ we obtain new combinatorial structures.

In this paper we study the analytic generalization:

$$\sum_{n=0}^{\infty} p_{(r)}(n) q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{n^r}}, \tag{1}$$

which interpolates:

- $r = 0$: ordinary partitions $p(n)$;

- $r = 1$: plane partitions $pp(n)$;
- $r > 1$: new functions extending the hierarchy.

While $p_{(r)}(n)$ for $r \geq 2$ does not enumerate geometric r-dimensional partitions in the sense of MacMahon, it provides a mathematically natural analytic continuation that reveals connections to divisor sums and modular forms. The parameter r serves as an analytic dimension that interpolates between different combinatorial structures.

This generalization allows us to derive explicit formulas, recurrences, and congruences that are not easily accessible in the geometric setting. Our first main result connects the generating function to divisor sums. As we will prove in Section 2 (Theorem 2.1),

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{n^r}} = 1 + \sum_{n=1}^{\infty} \left(\sum_{l=1}^{n} \frac{1}{l!} \sum_{(w_1,\ldots,w_l)\in C_n} \frac{\sigma_{r+1}(w_1)\cdots\sigma_{r+1}(w_l)}{w_1\cdots w_l} \right) q^n,$$

where $\sigma_k(n) = \sum_{d|n} d^k$ and C_n is the set of compositions of n (ordered tuples of positive integers summing to n).

We derive recurrences, congruences, and Bell polynomial formulas. For example:

$$\sum_{k=0}^{n-1} p_{(r)}(k)\sigma_{r+1}(n-k) = np_{(r)}(n). \tag{2}$$

Our work connects to recent developments in the arithmetic theory of partitions, including:

- Congruences for $p(n)$ and colored partitions modulo primes [2, 19];
- Generating function techniques and polynomial recurrences [24, 25];
- Recent results on multipartitions and k-tuple partitions [5, 10, 12, 15, 16, 21, 22].

While others use complex analysis, we focus on *explicit formulas* and *elementary congruences* for $p_{(r)}(n)$, offering a combinatorial perspective.

This paper is structured as follows. Section 2 introduces the theory of n^r-partitions, covering recurrence relations, congruence properties, explicit formulas, and
connections to Bell polynomials. Section 3 examines weighted restricted n^r-partitions into three parts, establishing their quasi-polynomial structure using Ehrhart theory [7, 8] and providing explicit formulas for the cases $r = 1$ and $r = 2$.

2 Results on n^r-partitions

2.1 Explicit formulas and generating functions

We begin with a logarithmic derivative approach to derive formulas and recurrences for $p_{(r)}(n)$. The connection to $\sigma_r(n)$ yields congruences, e.g.:

$$\sum_{k=0}^{n-1} p_{(l-1)}(k)\sigma(n-k) \equiv np_{(l-1)}(n) \pmod{2l}.$$

For a positive integer n, let C_n denote the set of *compositions* of n, i.e., ordered tuples $(w_1, \ldots, w_l)$ with $l \geq 1$, $w_i \geq 1$, and $\sum_{i=1}^{l} w_i = n$.

We start with the infinite product:

$$y = \prod_{n=1}^{\infty} \frac{1}{(1-wq^n)^{f(n)}}, \qquad |wq| < 1,$$

where $f : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 0}$. Taking logarithms and expanding the logarithm,

$$\log y = -\sum_{n=1}^{\infty} f(n) \log(1-wq^n) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{f(n) w^m q^{nm}}{m}.$$

All series converge absolutely for $|q| < 1$ and $|w| \leq 1$, so rearrangements of summations are justified. Collecting powers of q, write q^N with $N = nm$. Then $m = N/n$ is an integer only when $n \mid N$. Hence,

$$\log y = \sum_{N=1}^{\infty} \left(\sum_{d|N} \frac{f(d) w^{N/d}}{N/d} \right) q^N = \sum_{N=1}^{\infty} \left(\sum_{d|N} \frac{d f(d) w^{N/d}}{N} \right) q^N.$$

For convenience, define for $n \geq 1$:

$$a_n := \sum_{d|n} \frac{f(n/d) w^{n/d}}{d}.$$

Then $\log y = \sum_{n=1}^{\infty} a_n q^n$ (note that the constant term vanishes since $y(0) = 1$). Exponentiating,

$$y = \exp\left(\sum_{n=1}^{\infty} a_n q^n \right) = \sum_{l=0}^{\infty} \frac{1}{l!} \left(\sum_{n=1}^{\infty} a_n q^n \right)^l,$$

where absolute convergence justifies the expansion. Extracting the coefficient of q^n via the multinomial theorem yields

$$y = 1 + \sum_{n=1}^{\infty} \left(\sum_{l=1}^{n} \frac{1}{l!} \sum_{(w_1,\ldots,w_l) \in C_n} a_{w_1} \cdots a_{w_l} \right) q^n,$$

where C_n denotes the set of compositions of n (ordered tuples of positive integers summing to n).

For $f(n) = n^r$ and $w = 1$, we get:

$$y = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{n^r}},$$

and the coefficient of q^n is:

$$\sum_{l=1}^{n} \frac{1}{l!} \sum_{(w_1,\ldots,w_l) \in C_n} \frac{\sigma_{r+1}(w_1) \cdots \sigma_{r+1}(w_l)}{w_1 \cdots w_l}.$$

Thus:

Theorem 2.1. *For $r \geq 0$ and $|q| < 1$,*

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{n^r}} = 1 + \sum_{n=1}^{\infty} \left(\sum_{l=1}^{n} \frac{1}{l!} \sum_{(w_1,\ldots,w_l) \in C_n} \frac{\sigma_{r+1}(w_1) \cdots \sigma_{r+1}(w_l)}{w_1 \cdots w_l} \right) q^n. \tag{3}$$

For $r = 0$: $p_{(0)}(n) = p(n)$; for $r = 1$: $p_{(1)}(n) = pp(n)$. From Theorem 2.1, we write:

$$e^{\sum_{n=1}^{\infty} \frac{\sigma_{r+1}(n)}{n} q^n} = \sum_{m=0}^{\infty} p_{(r)}(m) q^m = F_r(q). \tag{4}$$

Taking logarithms:

$$\sum_{n=1}^{\infty} \frac{\sigma_{r+1}(n)}{n} q^n = \log F_r(q). \tag{5}$$

2.2 Recurrence relations and congruences

Differentiating the generating function, multiplying by q, and rearranging terms yields:

$$\left(\sum_{n=1}^{\infty} \sigma_{r+1}(n) q^n \right) \left(\sum_{k=0}^{\infty} p_{(r)}(k) q^k \right) = \sum_{m=1}^{\infty} m p_{(r)}(m) q^m. \tag{6}$$

Extracting coefficients gives the following recurrence.

Theorem 2.2. *For $n \geq 1$,*

$$\sum_{k=0}^{n-1} p_{(r)}(k) \sigma_{r+1}(n-k) = n p_{(r)}(n). \tag{7}$$

This extends Corollary 6 in Andrews et al. [6]. From this, we obtain the following parity result.

Theorem 2.3. *For $r \geq 1$,*

$$\sum_{k=0}^{n-1} p_{(r)}(k) \sigma(n-k) \equiv n p_{(r)}(n) \pmod 2. \tag{8}$$

Proof. Since $\sigma_r(n) \equiv \sigma(n) \pmod 2$ for all $r \geq 1$, reducing (7) modulo 2 gives the result. □

Before stating the next theorem, we recall Ramanujan's tau function $\tau(n)$, defined by the generating function

$$q \prod_{m=1}^{\infty} (1 - q^m)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n.$$

Ramanujan proved the classical congruence $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$.

Additionally, let $b_{10}(k)$ denote the coefficient of q^k in the infinite product $\prod_{m=1}^{\infty} (1 - q^m)^{m^{10}}$.

Theorem 2.4. *For $n \geq 1$,*

$$\sum_{k=0}^{n-1} b_{10}(k) \cdot (n-k) p_{(10)}(n-k) \equiv \sigma_{11}(n) \pmod{691}. \tag{9}$$

Proof. Setting $r = 10$ in Theorem 2.2 gives

$$\sum_{k=0}^{n-1} p_{(10)}(k) \sigma_{11}(n-k) = n p_{(10)}(n).$$

Reducing modulo 691 and using Ramanujan's congruence $\sigma_{11}(m) \equiv \tau(m) \pmod{691}$, we obtain

$$\sum_{k=0}^{n-1} p_{(10)}(k)\tau(n-k) \equiv np_{(10)}(n) \pmod{691}.$$

Multiplying both sides by the appropriate series and comparing coefficients yields the stated congruence with $b_{10}(k)$ (see [24] for further details). □

Theorem 2.5. *Let l be prime. Then:*

$$\sum_{k=0}^{n-1} b_{l-1}(k) \cdot (n-k)p_{(l-1)}(n-k) \equiv \sigma(n) \pmod{l}, \tag{10}$$

where $b_{l-1}(k)$ denotes the coefficient of q^k in $\prod_{m=1}^{\infty}(1-q^m)^{m^{l-1}}$.

Proof. From (6) with $r = l-1$,

$$\sum_{n=1}^{\infty} \sigma_l(n)q^n = \left(\prod_{m=1}^{\infty}(1-q^m)^{m^{l-1}}\right)\left(\sum_{m=1}^{\infty} mp_{(l-1)}(m)q^m\right).$$

By Fermat's little theorem, $\sigma_l(n) \equiv \sigma(n) \pmod{l}$. Reducing the identity modulo l and extracting coefficients gives the result. □

Corollary 2.1. *For $n \geq 1$,*

$$\sum_{k=0}^{n-1} b_{10}(k) \cdot (n-k)p_{(10)}(n-k) \equiv \sigma(n) \pmod{7601}.$$

Proof. The modulus $7601 = 11 \times 691$ follows from combining the $l = 11$ case of Theorem 2.5 with Theorem 2.4 via the Chinese remainder theorem. □

Theorem 2.6. *Let l be an odd prime. Then:*

$$\sum_{k=0}^{n-1} p_{(l-1)}(k)\sigma(n-k) \equiv np_{(l-1)}(n) \pmod{2l}. \tag{11}$$

Proof. From Theorem 2.2 with $r = l-1$ and the congruence $\sigma_l(n) \equiv \sigma(n) \pmod{l}$, we obtain the result modulo l. Theorem 2.3 provides the congruence modulo 2. Since l is odd, 2 and l are coprime, and the Chinese remainder theorem yields the congruence modulo $2l$. □

From the logarithmic identity (5), we also obtain:

Theorem 2.7. *For $r \geq 0$ and $n \geq 1$,*

$$\sigma_{r+1}(n) = n \sum_{\substack{a_1+\cdots+a_k=n \\ 1\leq k\leq n}} \frac{(-1)^{k+1}}{k} p_{(r)}(a_1)\cdots p_{(r)}(a_k). \tag{12}$$

Proof. From (5),

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\sigma_{r+1}(n)}{n}q^n &= \log\left(1+\sum_{m=1}^{\infty}p_{(r)}(m)q^m\right)\\ &= \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\left(\sum_{m=1}^{\infty}p_{(r)}(m)q^m\right)^k\\ &= \sum_{n=1}^{\infty}\left(\sum_{\substack{a_1+\cdots+a_k=n\\1\le k\le n}}\frac{(-1)^{k+1}}{k}p_{(r)}(a_1)\cdots p_{(r)}(a_k)\right)q^n.\end{aligned}$$

Equating coefficients and multiplying by n gives the result. □

The following corollaries follow directly from Theorem 2.7 by specializing r and applying known divisibility properties of $\sigma(n)$.

Corollary 2.2. *If l is prime, $l \mid \sigma(n)$, and $l \nmid n$, then:*

$$\sum_{\substack{a_1+\cdots+a_k=n\\1\le k\le n}}\frac{(-1)^k}{k}p_{(l-1)}(a_1)\cdots p_{(l-1)}(a_k)\equiv 0 \pmod{l}.$$

Corollary 2.3. *If n is an odd non-square, then:*

$$\sum_{\substack{a_1+\cdots+a_k=n\\1\le k\le n}}\frac{(-1)^k}{k}p_{(1)}(a_1)\cdots p_{(1)}(a_k)\equiv 0 \pmod{2}.$$

Corollary 2.4. *For odd prime l,*

$$\sum_{\substack{a_1+\cdots+a_k=2^{l-2}\\1\le k\le 2^{l-2}}}\frac{(-1)^k}{k}p_{(l-1)}(a_1)\cdots p_{(l-1)}(a_k)\equiv 0 \pmod{l}.$$

Lemma 2.1. *Let $A(q)=\sum_{n=0}^{\infty}a_nq^n$ be a formal power series with $a_0=1$, converging absolutely for $|q|<R$ with $R>0$. Define the logarithmic series*

$$B(q)=\log A(q)=\sum_{n=1}^{\infty}b_nq^n,$$

where the principal branch of the logarithm is taken. Then the coefficients b_n are given by

$$b_n=\sum_{m=1}^{n}\frac{(-1)^{m+1}}{m}\sum_{(w_1,\ldots,w_m)\in C_n}a_{w_1}\cdots a_{w_m},\qquad n\ge 1,$$

where C_n denotes the set of compositions of n into positive integers. Conversely, the coefficients a_n are given by the exponential formula

$$a_n=\sum_{m=1}^{n}\frac{1}{m!}\sum_{(w_1,\ldots,w_m)\in C_n}b_{w_1}\cdots b_{w_m},\qquad n\ge 1.$$

Proof. Since $A(q) = 1 + \sum_{n\geq 1} a_n q^n$, we may write $B(q) = \log(1 + u(q))$ with $u(q) = \sum_{n\geq 1} a_n q^n$. For sufficiently small $|q|$, $|u(q)| < 1$, we may expand using the Mercator series:

$$B(q) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} u(q)^m.$$

Expanding $u(q)^m$ via the multinomial theorem and collecting powers of q gives the first formula. Conversely, exponentiating $B(q)$ gives $A(q) = \exp(B(q)) = \sum_{m=0}^{\infty} \frac{1}{m!} B(q)^m$. Expanding $B(q)^m$ and extracting the coefficient of q^n yields the second formula. Absolute convergence justifies all rearrangements for $|q|$ sufficiently small. □

From the logarithmic identity (5), we have

$$\sum_{n=1}^{\infty} \frac{\sigma_{r+1}(n)}{n} q^n = \log\left(\sum_{n=0}^{\infty} p_{(r)}(n) q^n\right).$$

Applying Lemma 2.1 with $a_n = p_{(r)}(n)$ and $b_n = \sigma_{r+1}(n)/n$ yields the following inversion formula.

Theorem 2.8. *For $r \geq 0$ and $n > 0$,*

$$\frac{\sigma_{r+1}(n)}{n} = \sum_{m=1}^{n} \frac{(-1)^{m+1}}{m} \sum_{(w_1,\ldots,w_m)\in C_n} p_{(r)}(w_1)\cdots p_{(r)}(w_m). \tag{13}$$

Equivalently,

$$p_{(r)}(n) = \sum_{m=1}^{n} \frac{1}{m!} \sum_{(w_1,\ldots,w_m)\in C_n} \frac{\sigma_{r+1}(w_1)}{w_1}\cdots\frac{\sigma_{r+1}(w_m)}{w_m}. \tag{14}$$

Proof. The first formula follows directly from Lemma 2.1 with $a_n = p_{(r)}(n)$ and $b_n = \sigma_{r+1}(n)/n$. The second formula is the converse relation from the same lemma. □

2.3 Bell polynomial representations

We also express $p_{(r)}(n)$ via Bell polynomials:

Theorem 2.9. *For $r \geq 0$, $n \geq 1$,*

$$p_{(r)}(n) = \frac{1}{n!} B_n(0!\sigma_{r+1}(1), 1!\sigma_{r+1}(2), \ldots, (n-1)!\sigma_{r+1}(n)).$$

Proof. The exponential Bell polynomial generating function is:

$$\sum_{n=0}^{\infty} B_n(x_1,\ldots,x_n)\frac{t^n}{n!} = \exp\left(\sum_{j=1}^{\infty} x_j \frac{t^j}{j!}\right) = 1 + \sum_{n=1}^{\infty}\left(\sum_{l=1}^{n} \frac{1}{l!} \sum_{(w_1,\ldots,w_l)\in C_n} \frac{x_{w_1}\cdots x_{w_l}}{w_1!\cdots w_l!}\right) t^n.$$

Set $x_i = (i-1)!\sigma_{r+1}(i)$. Then:

$$\frac{1}{n!} B_n(x_1,\ldots,x_n) = \sum_{l=1}^{n} \frac{1}{l!} \sum_{(w_1,\ldots,w_l)\in C_n} \frac{\sigma_{r+1}(w_1)\cdots\sigma_{r+1}(w_l)}{w_1\cdots w_l} = p_{(r)}(n). \tag{15}$$

□

Corollary 2.5. *For $n > 0$,*

$$p_{(r)}(n) = \sum_{\substack{k_1+2k_2+\cdots+nk_n=n \\ k_i \geq 0}} \frac{(\sigma_{r+1}(1))^{k_1}(\sigma_{r+1}(2))^{k_2}\cdots(\sigma_{r+1}(n))^{k_n}}{k_1!k_2!\cdots k_n!(2!)^{k_2-k_3}\cdots((n-1)!)^{k_{n-1}-k_n}(n!)^{k_n}}.$$

Using the binomial expansion, we obtain an explicit formula in terms of partitions.

Theorem 2.10. *For $r \geq 0$ and $n \geq 1$,*

$$p_{(r)}(n) = \sum_{\substack{f_1a_1+\cdots+f_la_l=n \\ a_1>a_2>\cdots>a_l\geq 1 \\ f_i\geq 1}} \prod_{i=1}^{l} \binom{a_i^r + f_i - 1}{f_i},$$

where the sum runs over all partitions of n written with distinct part sizes a_i and multiplicities f_i.

Proof. Expanding the generating function as a product of binomial series gives

$$\prod_{m=1}^{\infty}(1-q^m)^{-m^r} = \prod_{m=1}^{\infty}\sum_{f=0}^{\infty}\binom{m^r+f-1}{f}q^{mf}.$$

When expanding the product, the coefficient of q^n is obtained by summing over all sequences of nonnegative integers $(f_1, f_2, \dots)$ with finite support such that $\sum_{m=1}^{\infty} mf_m = n$. Each such sequence contributes $\prod_{m=1}^{\infty}\binom{m^r+f_m-1}{f_m}$. Grouping together the indices m with $f_m > 0$ and writing them as distinct part sizes $a_1, \dots, a_l$ with corresponding multiplicities f_i yields the stated formula. □

3 Weighted restricted partitions and quasi-polynomial structure

This section examines specific classes of weighted restricted n^r-partitions, focusing on partitions with a fixed number of parts and their combinatorial properties.

3.1 Partitions into two parts

For n^r-partitions into 2 parts, the total weighted count is

$$p_{r,2}(N) = \sum_{b=1}^{\lfloor N/2 \rfloor}(N-b)^r b^r,$$

or equivalently,

$$p_{r,2}(N) = \frac{1}{2}\left(\sum_{a=1}^{N-1} a^r(N-a)^r + \mathbf{1}_{2|N}\left(\frac{N}{2}\right)^{2r}\right),$$

where $\mathbf{1}_{2|N}$ is 1 if N is even and 0 otherwise. Expanding via the binomial theorem,

$$p_{r,2}(N) = \sum_{j=0}^{r}(-1)^j\binom{r}{j}N^{r-j}\sum_{a=1}^{\lfloor N/2 \rfloor} a^{r+j} + (\text{correction term}).$$

Using Faulhaber's formula for power sums, each inner sum is a polynomial in N of degree $r+j+1$. Consequently, $p_{r,2}(N)$ is a quasi-polynomial in N of degree $2r+1$ with period dividing 2.

3.2 Partitions into three parts: General framework

We now derive formulas for weighted n^r-partitions into exactly three parts, focusing on the cases $r = 1$ and $r = 2$. We define the counting functions $p_{r,3}(N)$ as weighted sums over ordered triples $a \geq b \geq c \geq 1$ with $a + b + c = N$, where each part k contributes weight k^r. We prove that $p_{r,3}(N)$ are quasi-polynomials of degree $3r + 2$ with period 6, and provide explicit formulas for each residue class modulo 6.

3.3 Ehrhart theory background

To analyze $p_{r,k}(N)$ systematically, we recall some key concepts from Ehrhart theory (see [7, 8]). For a rational polytope $P \subset \mathbb{R}^d$ (a convex polytope with vertices in $\mathbb{Q}^d$), the *Ehrhart function* $L_P(N) = \#(NP \cap \mathbb{Z}^d)$ counts integer points in the N-th dilate of P. Ehrhart's theorem states that $L_P(N)$ is a quasi-polynomial in N whose period divides the least common multiple of the denominators of the vertices of P.

More generally, for a polynomial weight $f(x)$ on P, the *weighted Ehrhart sum*

$$S_{P,f}(N) = \sum_{m \in NP \cap \mathbb{Z}^d} f(m)$$

is also a quasi-polynomial in N, of degree $\deg f + \dim P$, provided P is rational and f is a polynomial (see [8] for the local Euler–Maclaurin formula). The leading coefficient of this quasi-polynomial is given by the continuous integral

$$\lim_{N \to \infty} \frac{S_{P,f}(N)}{N^{\deg f + \dim P}} = \int_P f(x)\, dx,$$

where the integral is taken with respect to the standard Lebesgue measure normalized so that the fundamental parallelepiped of $\mathbb{Z}^d$ has volume 1.

3.4 General quasi-polynomial structure

We now apply this framework to our weighted restricted partition function.

Definition 3.1. *For integers $N \geq k \geq 1$ and $r \geq 0$, define*

$$p_{r,k}(N) = \sum_{\substack{a_1 \geq a_2 \geq \cdots \geq a_k \geq 1 \\ a_1 + \cdots + a_k = N}} a_1^r a_2^r \cdots a_k^r.$$

This counts weighted restricted partitions*: each ordinary partition $(a_1, \ldots, a_k)$ is assigned the multiplicative weight $a_1^r \cdots a_k^r$. This weight is interpreted combinatorially as a product of part weights, not as a number of available colors for each part.*

Theorem 3.1 (General structure)**.** *Fix integers $k \geq 1$ and $r \geq 0$. The function $p_{r,k}(N)$ is a quasi-polynomial in N of degree*

$$\deg p_{r,k} = kr + (k - 1),$$

and its period divides $\operatorname{lcm}(1,2,\dots,k)$*. The leading coefficient (coefficient of* N^{kr+k-1}*) equals*

$$\frac{(r!)^k}{k!\,(kr+k-1)!}.$$

Proof. We place the problem in the Ehrhart framework. Let

$$\Delta^{(k-1)} = \left\{x \in \mathbb{R}^k_{\geq 0} : x_1 + \cdots + x_k = 1\right\}$$

be the standard $(k-1)$-simplex, and let

$$P = \left\{x \in \Delta^{(k-1)} : x_1 \geq x_2 \geq \cdots \geq x_k\right\}$$

be the ordered chamber. The vertices of P are points of the form

$$\left(\frac{1}{j}, \frac{1}{j}, \dots, \frac{1}{j}, 0, \dots, 0\right)$$

with denominators at most k, so P is a rational polytope.

For an integer $N \geq 1$, consider the dilated polytope $NP = \{Nx : x \in P\}$. The integer points in NP correspond to k-tuples $(a_1, \dots, a_k)$ of nonnegative integers satisfying $a_1 \geq \cdots \geq a_k \geq 0$ and $\sum a_i = N$. To obtain strictly positive parts $a_i \geq 1$, we shift variables by setting $b_i = a_i - 1$. Then $(b_1, \dots, b_k)$ satisfies

$$b_1 \geq \cdots \geq b_k \geq 0, \qquad b_1 + \cdots + b_k = N - k.$$

Thus the exact weighted Ehrhart representation is

$$p_{r,k}(N) = \sum_{b \in (N-k)P \cap \mathbb{Z}^k} \prod_{i=1}^{k} (b_i + 1)^r.$$

This correction does not change the stated degree, period bound, or leading coefficient.

The weighted Ehrhart theorem (see [7, 8]) states that for a rational polytope P of dimension d and a polynomial f of total degree D, the sum $\sum_{m \in NP \cap \mathbb{Z}^d} f(m)$ is a quasi-polynomial in N of degree $D + d$, with period dividing the least common multiple of the denominators of the vertices of P. In our case, $d = \dim P = k - 1$ and $D = \deg f_r = kr$. Hence the degree is $kr + (k-1)$, and the period divides $\operatorname{lcm}(1, 2, \dots, k)$.

The leading coefficient of the quasi-polynomial is the continuous integral of f_r over P:

$$c_{\text{lead}} = \int_P f_r(x)\, dx.$$

The full simplex $\Delta^{(k-1)}$ is partitioned into $k!$ ordered chambers, each congruent to P under permutation of coordinates. Therefore,

$$\int_P f_r(x)\, dx = \frac{1}{k!} \int_{\Delta^{(k-1)}} x_1^r \cdots x_k^r\, dx.$$

The Dirichlet (multinomial Beta) integral gives

$$\int_{\Delta^{(k-1)}} x_1^\alpha \cdots x_k^\alpha\, dx = \frac{(\alpha!)^k}{(k\alpha + k - 1)!},$$

for α a nonnegative integer, where we interpret $\alpha! = \Gamma(\alpha + 1)$. Setting $\alpha = r$ yields

$$c_{\text{lead}} = \frac{1}{k!} \cdot \frac{(r!)^k}{(kr + k - 1)!} = \frac{(r!)^k}{k!\,(kr + k - 1)!}.$$

This completes the proof. □

Remark 3.1. *The period bound* $\operatorname{lcm}(1, 2, \ldots, k)$ *is generally sharp: denominators* $q \leq k$ *occurring in vertex coordinates typically produce dependence on* $N \bmod q$ *through the cone contributions in Brion's theorem* [9]. *The lower-order coefficients may be computed in principle via the local Euler–Maclaurin expansion* [8]*: they are determined by integrals of* f_r *and its contractions over faces of* P*, plus periodic contributions coming from tangent cones at rational vertices.*

3.5 Explicit formulas for three parts

We now determine explicit formulas for $p_{1,3}(N)$ and $p_{2,3}(N)$ using the quasi-polynomial structure established in Theorem 3.1. For $k = 3$, the period divides $\operatorname{lcm}(1, 2, 3) = 6$, so $p_{r,3}(N)$ is a quasi-polynomial of degree $3r+2$ with six residue classes modulo 6. We compute the coefficients by:

1. Using the leading coefficient from Theorem 3.1:

$$\text{lead}(p_{r,3}) = \frac{(r!)^3}{6\,(3r + 2)!}.$$

2. Using the weighted Burnside decomposition for the action of S_3 on ordered triples, which gives the exact functional form including the 3-cycle residue condition.
3. Verifying the resulting formulas against direct computations for $N \leq 30$ (and beyond).

The following theorems present the resulting formulas.

Theorem 3.2 (Case $r = 1$, $k = 3$)**.** *The function* $p_{1,3}(N)$ *is a quasi-polynomial of degree* 5 *and period* 6*. The explicit formulas by residue class modulo* 6 *are:*

$$p_{1,3}(N) = \begin{cases} \dfrac{N^5}{720} + \dfrac{N^4}{192} + \dfrac{7N^3}{1296} - \dfrac{N^2}{48} + \dfrac{N}{180}, & N \equiv 0 \pmod 6, \\ \dfrac{N^5}{720} + \dfrac{N^4}{192} + \dfrac{7N^3}{1296} + \dfrac{N^2}{96} + \dfrac{N}{180} - \dfrac{1}{64}, & N \equiv 3 \pmod 6, \\ \dfrac{N^5}{720} + \dfrac{N^4}{192} - \dfrac{N^3}{144} + \dfrac{N^2}{96} + \dfrac{N}{180} - \dfrac{1}{64}, & N \equiv 1, 5 \pmod 6, \\ \dfrac{N^5}{720} + \dfrac{N^4}{192} - \dfrac{N^3}{144} - \dfrac{N^2}{48} + \dfrac{N}{180}, & N \equiv 2, 4 \pmod 6. \end{cases}$$

Proof. We use weighted Burnside counting for the action of the symmetric group S_3 on ordered positive triples (a, b, c) with $a + b + c = N$. The counting function satisfies

$$p_{r,3}(N) = \frac{1}{6}\left(S_{\text{id}}(N) + 3S_{\text{trans}}(N) + 2S_{\text{cyc}}(N)\right),$$

where:

- $S_{\mathrm{id}}(N) = \sum_{a,b,c\geq 1, a+b+c=N} a^r b^r c^r$ is the identity contribution;
- $S_{\mathrm{trans}}(N) = \sum_{a,b\geq 1, 2a+b=N} a^{2r} b^r$ is the transposition contribution (fixing one coordinate);
- $S_{\mathrm{cyc}}(N) = \sum_{a\geq 1, 3a=N} a^{3r}$ is the 3-cycle contribution.

The crucial observation is that the 3-cycle contribution $S_{\mathrm{cyc}}(N)$ is nonzero if and only if $3 \mid N$. Hence,

$$S_{\mathrm{cyc}}(N) = \mathbf{1}_{3|N} \left(\frac{N}{3}\right)^{3r}.$$

The sums S_{id} and S_{trans} are evaluated exactly using Faulhaber's formulas for power sums (or standard polynomial summation), yielding the polynomial parts in N for each congruence class modulo 6. The identity contribution for $r = 1$ is

$$S_{\mathrm{id}}(N) = \binom{N-1}{5} + 3\binom{N-1}{4} + 3\binom{N-1}{3} + \binom{N-1}{2},$$

and the transposition contribution splits by parity:

$$S_{\mathrm{trans}}(N) = \begin{cases} \dfrac{N^4 - 4N^2}{96}, & N \text{ even}, \\ \dfrac{(N^2-1)(N^2+3)}{96}, & N \text{ odd}. \end{cases}$$

Adding the periodic 3-cycle term $\mathbf{1}_{3|N} N^3/81$ and simplifying by residue classes modulo 6 produces the four cases above. The formula has been independently verified by direct computation for $N \leq 30$ (with values not used in the derivation), confirming the exact minimal period and coefficients. □

Theorem 3.3 (Case $r = 2$, $k = 3$)**.** *The function $p_{2,3}(N)$ is a quasi-polynomial of degree 8 and period 6. The explicit formulas by residue class modulo 6 are:*

$$p_{2,3}(N) = \begin{cases} \dfrac{N^8}{30240} + \dfrac{N^7}{6720} + \dfrac{N^6}{2187} - \dfrac{N^4}{1440} - \dfrac{N^3}{120} + \dfrac{N^2}{1512} + \dfrac{N}{42}, & N \equiv 0 \pmod 6, \\ \dfrac{N^8}{30240} + \dfrac{N^7}{6720} + \dfrac{N^6}{2187} - \dfrac{N^4}{1440} + \dfrac{7N^3}{960} + \dfrac{N^2}{1512} - \dfrac{5N}{672}, & N \equiv 3 \pmod 6, \\ \dfrac{N^8}{30240} + \dfrac{N^7}{6720} - \dfrac{N^4}{1440} + \dfrac{7N^3}{960} + \dfrac{N^2}{1512} - \dfrac{5N}{672}, & N \equiv 1, 5 \pmod 6, \\ \dfrac{N^8}{30240} + \dfrac{N^7}{6720} - \dfrac{N^4}{1440} - \dfrac{N^3}{120} + \dfrac{N^2}{1512} + \dfrac{N}{42}, & N \equiv 2, 4 \pmod 6. \end{cases}$$

Proof. The proof follows the same Burnside decomposition as in Theorem 3.2, with $r = 2$. The 3-cycle contribution is nonzero only when $3 \mid N$, contributing $\mathbf{1}_{3|N} N^6/3^7 = \mathbf{1}_{3|N} N^6/2187$. This term is present in the residue classes $N \equiv 0, 3 \pmod 6$ and absent for $N \equiv 1, 2, 4, 5 \pmod 6$.

The identity contribution S_{id} and transposition contribution S_{trans} are evaluated exactly via Faulhaber's formulas (with separate treatments for N even and N odd), yielding the polynomial parts for each parity class. The leading coefficient matches $\frac{(2!)^3}{6 \cdot 8!} = \frac{1}{30240}$.

Adding the periodic 3-cycle term and simplifying by residue class modulo 6 produces the four cases above. The formulas have been independently verified by direct computation for $N \leq 74$ (with values not used in the derivation), confirming the exact minimal period and coefficients. □

Remark 3.2. *The coefficient* $\frac{1}{2187}$ *in the* N^6 *term for* $r = 2$ *arises from the* 3*-cycle contribution* $N^6/3^7$ *and appears only when* $3 \mid N$*. This explains conceptually the previously surprising coefficient* $1/2187$*.*

3.6 Comparison of properties

Table 2. Comparison of n^r-partition properties for small k

Property	$\boldsymbol{k=1}$	$\boldsymbol{k=2}$	$\boldsymbol{k=3}$	**General** $\boldsymbol{k}$
Degree	r	$2r+1$	$3r+2$	$kr+(k-1)$
Period	1	divides 2	6	$\mathrm{lcm}(1,2,\ldots,k)$
Leading coefficient	$\frac{1}{r!}$	$\frac{1}{2(2r+1)!}$	$\frac{(r!)^3}{6(3r+2)!}$	$\frac{(r!)^k}{k!(kr+k-1)!}$
Structure	Polynomial	Quasi-polynomial	Quasi-polynomial	Quasi-polynomial

4 Concluding remarks and future directions

Our work on n^r-partitions reveals deep connections between partition theory, divisor sums, and combinatorial identities. The results provide a framework for understanding these generalized partitions via generating functions, recurrences, explicit formulas, and congruences.

The study of n^r-partitions is significant as an analytic extension beyond two dimensions. Although not geometric for $r \geq 2$, they are mathematically rich and offer a unified approach to studying partitions across dimensions.

4.1 Open problems and research directions

1. **Bijective proofs:** Construct explicit bijections for the identities. This would provide deeper combinatorial insight and connect to bijective combinatorics.
2. **Asymptotic analysis:** Study asymptotics of $p_{(r)}(n)$ for $r > 1$. This could involve saddle-point methods or connections to modular forms.
3. **Modular forms:** Investigate connections with modular forms. The generating functions for $r = 0$ and $r = 1$ are known to be related to classical modular forms, while cases with higher r may involve quasimodular or mock modular forms. Furthermore, various identities analogous to those in Slater [23] can also be established for $r \geq 2$.
4. **Higher-dimensional interpretations:** Find combinatorial interpretations for $p_{(r)}(n)$ when $r \geq 2$. Although not geometric, they may count certain labeled structures or sequences.
5. **Generalized congruences:** Systematically study congruences for these functions. Ramanujan-type congruences for $r \geq 2$ are largely unexplored.
6. **Extensions to other weights:** Consider more general weight functions $f(n)$ beyond n^r and study their combinatorial and arithmetic properties.
7. **Computational aspects:** Develop efficient algorithms for computing $p_{(r)}(n)$ for large n and various r, potentially using the recurrence relations derived in this work.

These directions promise further advances. The unity of methods here highlights the richness of partition theory and demonstrates how classical techniques can be extended to study new combinatorial structures.

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