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Editorial: An introduction to the Fields Medal and the work of Yu Deng and Hong Wang *

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1 The Fields Medal

The Fields Medal is an award presented by the International Mathematical Union at the International Congress of Mathematicians to young scholars who have made outstanding contributions to mathematics. To be eligible, a recipient must not have reached the age of forty on January 1 of the year in which the medal is awarded.

The medal was first awarded in 1936. Its presentation was interrupted by the Second World War and resumed in 1950. Since that year, it has been awarded every four years, with no more than four recipients on each occasion. The Fields Medal is regarded as one of the highest honors in mathematics.

Among the 2026's four recipients, two are Chinese citizens: Yu Deng and Hong Wang. Previously, two mathematicians of Chinese descent had received the medal—Shing-Tung Yau in 1982 and Terence Tao in 2006—but neither was a Chinese citizen, and both completed their university education in the West. Deng and Wang, by contrast, completed all their education before university in China, and both entered Peking University in 2007.

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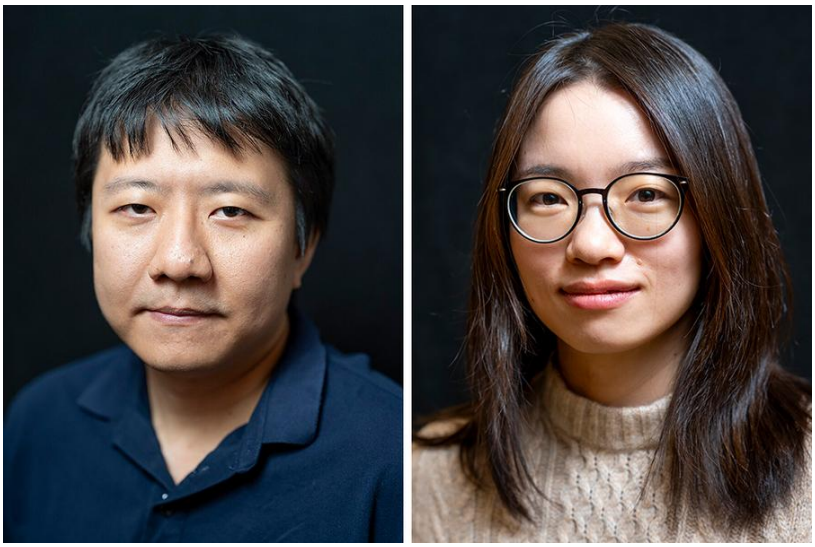


Figure 1. Yu Deng (left) and Hong Wang (right), recipients of the 2026 Fields Medal. Photographs courtesy of the International Mathematical Union (left) and the Simons Foundation (right).

The “Yu” in Yu Deng’s name is the same character as in the name of the great poet Li Yu, the last ruler of the Southern Tang dynasty. Deng entered Peking University and, after studying there for two years, transferred to the Massachusetts Institute of Technology in 2009. He received his bachelor’s degree in mathematics from MIT in 2011 and then entered the graduate program of Princeton University, where he received his doctoral degree in 2015.

Although Hong Wang entered Peking University in the same year as Deng, she initially enrolled in the School of Earth and Space Sciences. The following year she transferred to the School of Mathematical Sciences. Three years later, in 2011, she graduated from Peking University with a bachelor’s degree in mathematics. She then spent another three years studying in France and, in 2014, received an engineering degree from the École Polytechnique and a master’s degree from the University of Paris XI. Wang subsequently went to MIT to study mathematics and received her Ph.D. in mathematics there in 2019.

After receiving their doctoral degrees, both Deng and Wang distinguished themselves in mathematical research and received numerous honors. In 2026, they were both awarded the Fields Medal. Deng is now a full professor at the University of Chicago. Wang holds the Silver Professorship at the Courant Institute of New York University and is also a permanent professor at the Institut des Hautes Études Scientifiques (IHÉS) in France.

The prize-winning work of the two mathematicians will be introduced in turn below.

2 An introduction to Yu Deng’s prize-winning work

In its citation for Yu Deng’s Fields Medal, the International Mathematical Union stated that the award was given in recognition of his pioneering contributions to partial differential equations, including the rigorous derivation of the Boltzmann equation from the hard-sphere dynamics of rarefied gases, the derivation of wave kinetic equations from nonlinear dispersive systems, and probabilistic approaches to the dynamics of nonlinear Schrödinger equations.

This citation mentions several of Deng's outstanding contributions to mathematics. In particular, the rigorous derivation of the Boltzmann equation from the hard-sphere dynamics of rarefied gases has attracted special attention because of its close connection with Hilbert's Sixth Problem.

2.1 Hilbert's Sixth Problem

In 1900, at the Second International Congress of Mathematicians in Paris, the German mathematician David Hilbert presented his famous list of twenty-three mathematical problems [14]. These problems played an enormously important role in stimulating the development of mathematics during the twentieth century.

In 1974, the American Mathematical Society held a symposium devoted to the new mathematics inspired by Hilbert's problems. The purpose of the symposium was not primarily to discuss the known solutions to these problems, but rather the new concepts, theories, methods, and problems to which they had given rise. The proceedings filled two large volumes of more than six hundred pages [7].

Hilbert's Sixth Problem concerns the axiomatization of physical theories, especially those closely connected with mathematics. If a physical theory could be axiomatized in the manner of Euclidean geometry, we would know what its fundamental assumptions are and how every conclusion of the theory can be derived from them. The theory would then possess a rigorous logical structure, and our understanding of it would be correspondingly more complete.

The scope of this problem, however, is extremely broad. Hilbert therefore singled out several theories as suitable places to begin. Among them was the kinetic theory of gases.

At different observational scales, physics employs three different theories to describe the properties of gases.

From the microscopic point of view, every molecule in a gas may be regarded as a hard sphere moving continuously through space. The motion of these molecules and their mutual collisions obey Newtonian mechanics. For example, in the absence of external forces, every hard sphere moves in a straight line with constant velocity, and when two spheres collide, the laws of conservation of momentum and kinetic energy must be satisfied.

From the macroscopic point of view, the motion of gas may be regarded as the motion of a fluid. The fluid is governed by the Navier–Stokes equations, a system of partial differential equations arising from the application of Newton's second law to fluid mechanics.

Between the microscopic and macroscopic scales lies another scale of observation, called the mesoscopic scale. At this level, the relevant physical theory is the Boltzmann equation. This partial differential equation describes how the microscopic properties of a gas give rise to its macroscopic behavior.

Thus, for the kinetic theory of gases, Hilbert's Sixth Problem asks us to determine the relationship among these three physical theories: Can one derive the mesoscopic Boltzmann equation from the microscopic dynamics of colliding hard spheres governed by Newtonian mechanics? And can one then derive the macroscopic Navier–Stokes equations from the Boltzmann equation? After many years of work by numerous scholars (see, for example, [1, 18, 22]), the derivation of the macroscopic Navier–Stokes equations from the Boltzmann equation is

essentially understood. But the derivation of the mesoscopic Boltzmann equation from colliding hard spheres governed by Newtonian mechanics remained, for more than 120 years, an apparently insurmountable difficulty. The work of Deng and his two collaborators attracted so much attention precisely because they succeeded in overcoming this obstacle.

2.2 From the microscopic theory to the mesoscopic theory: Lanford's Theorem

Boltzmann himself attempted to derive his equation from microscopic hard-sphere collisions governed by Newtonian mechanics [4]. During his derivation, however, he made an assumption: before a pair of particles collides, their velocities are essentially independent. Here, "independent" means that the velocities have no statistical correlation, while "essentially" means that exceptional cases are statistically negligible.

Many scholars found this assumption difficult to accept. There are two principal reasons. The first is that we cannot simply assume that particles whose velocities are correlated before collision occur so rarely that they may be neglected. Even if every pair of particles in a gas initially has independent velocities, after a collision the conservation of momentum and kinetic energy generally produces correlations between their velocities. From the point of view of kinetic theory, collisions among particles are unavoidable and may sometimes be quite frequent. How many particles that have collided once will collide again? Before colliding again, they may have undergone many further collisions with other particles. Can these intervening collisions cause the particles to "forget" their previous velocity correlations? These are not easy questions to answer.

The second difficulty concerns the problem of time reversibility and irreversibility. At the microscopic level, where individual hard spheres collide, the physical laws governing the system are time reversible. But at the mesoscopic level, where one considers the collective properties of a gas, the physical description becomes irreversible in time. For example, suppose a small amount of nitrogen is introduced into a body of oxygen. After repeated molecular collisions, the two gases will eventually become uniformly mixed. We cannot reverse time and cause the two gases to separate again. This is analogous to adding a spoonful of milk to a cup of coffee. After stirring, the milk and coffee become uniformly mixed. We cannot reverse the stirring process and make the milk gather again into a small separate region. Under the assumption that the velocities of particles are independent before collision, Boltzmann derived the mesoscopic Boltzmann equation from the microscopic properties of the gas. But in this derivation, it is difficult to see how a physical theory that is time-reversible at the microscopic level becomes time-irreversible at the mesoscopic level.

These difficulties remained unresolved for more than a century. During this period, one particularly important result was obtained by Oscar Lanford in 1975 [21]. Lanford used tree diagrams to represent the collision history of each particle with other particles. Each branch of the tree represents a collision. If two particles collide more than once, their collision trees contain at least two connections between corresponding branches, thereby forming a collision loop. Over a sufficiently short period of time, such collision loops occur so rarely in the relevant statistical calculations that they can be neglected. The difficulty was that Lanford's "short time" was extremely short—only about one-fifth of the mean free time between two collisions. This was far too short for a full derivation of the Boltzmann equation.

2.3 The work of Deng, Hani, and Ma

Since Lanford published his paper in 1975, many researchers have tried to deepen or generalize his results. Some investigated the relationship between Lanford's method and time reversibility; others developed series expansions valid over short time intervals or under perturbations of the gas. But all these results remained limited in time and could not be extended to long time scales. The difficulties associated with long time scales were not overcome until 2025, through the work of Yu Deng, Zaher Hani, and Xiao Ma [11]. The three authors' paper, subsequently accepted for publication, is extremely difficult and complex and runs to about two hundred pages. What follows is only a very elementary introduction.

Before 2021, while Deng was still at the University of Southern California and Hani was at the University of Michigan, he and Hani had already collaborated successfully on the derivation of wave kinetic equations [9, 10]. Wave kinetic equations and the Boltzmann equation occupy analogous positions in physical theory: one describes the interaction of waves, while the other describes the interaction of particles. Moreover, the wave kinetic equations derived by Deng and Hani were valid on long time scales. Beginning near the end of 2022, the two began to study the problem of deriving the Boltzmann equation over long time scales. They hoped that the methods they had developed for wave kinetic equations could also be applied to the Boltzmann equation.

They divided a long time interval into many short time layers and attempted to control the propagation of statistical information layer by layer. During this multilevel analysis, many complicated diagrams appeared. These diagrams recorded different possible collision histories of the particles. At this point they discovered that there were extraordinary diagrams that never arose in wave kinetics. The previous reduction algorithms could not be used. A new collection of methods for combinatorial decomposition and reduction had to be developed specifically for the Boltzmann problem.

Around this time, Deng met Xiao Ma, then a graduate student at Princeton, at a mathematical conference. Deng discovered that Ma's doctoral advisor was Alexandru Ionescu, who had himself been Deng's advisor at Princeton. The two discussed their respective work. Deng found that Ma not only understood the work of Deng and Hani very quickly but was also able to offer concrete ideas. He therefore invited Ma to join the research team.

Through the efforts of the three mathematicians, the situation gradually became clearer. They first simplified the problem. In one formulation of Hilbert's Sixth Problem, one considers a gas confined in a box, with periodic boundary conditions: whenever a particle strikes one face of the box, it reappears at the opposite face. The particles can therefore continue colliding indefinitely.

The researchers first considered a gas in an unbounded space rather than in a box. The particles in such a gas can collide in many ways, but collisions cause the gas to spread out. In an unbounded space, an expanding gas becomes increasingly dilute, and particle collisions consequently become less frequent. Eventually, re-collisions can be neglected. They also saw more clearly why individual collisions may be time-reversible while the collective behavior of the gas is irreversible: as the gas molecules continue to collide, the gas expands rather than contracts, and time effectively proceeds in one direction.

However, to extend Lanford's result to long time scales, they still had to show that re-collisions would not cause difficulties before the gas had become sufficiently dilute. They carefully examined

every possible type of collision-history diagram. They first eliminated some relatively simple cases. For example, situations in which two particles undergo many re-collisions within a short period occur very rarely and can therefore be neglected. They then turned to more difficult cases.

Deng was in Chicago, Hani in Michigan, and Ma in Princeton. Every day or two, the three held video conferences on Zoom. Deng and Ma were unmarried, while Hani had a young daughter. The three often began their discussions after Hani's daughter had gone to bed, and their conversations frequently lasted two or three hours.

During this period, they worked extremely hard. In some cases, a way of dealing with a particular difficulty became visible only after prolonged thought. Their basic method was to divide a long interval into many short intervals, or time layers. In each short interval, if a particle collided with other particles, they examined the previous collision histories of those particles. They hoped to show that, among the particles involved, those that had previously collided with the given particle were far fewer than those that had not. In other words, they wanted to prove that, at least within each short time interval, the motions of the particles were essentially independent, since only particles that had previously collided could possess statistical correlations in their velocities. But the number of particles with more than one collision in their histories was still too large. The researchers therefore classified them into many groups according to the depth of their statistical correlations. Each group was described in terms of a cumulant. They then examined the relationships among these cumulants.

They found that, although there might be many particles with multiple collisions in their histories, from a statistical point of view the number of particles with deeper and deeper correlations became increasingly negligible. This conclusion was fundamentally geometric. Roughly speaking, compared with freely moving particles, colliding particles must satisfy the conditions required for collision. These conditions force the particles to lie on certain surfaces or curves in the phase space. Such surfaces or curves occupy a negligible volume in the ambient space.

After controlling the individual short time intervals, the researchers then had to connect these intervals together. This was another formidable task, because they had to ensure that the relevant statistical quantities remained under control when the intervals were linked. In this way, step by step, they extended Lanford's conclusion to long time scales. They then returned to the gas confined in a box. They found that roughly eighty percent of the work developed for the gas in unbounded space could be transferred directly. The remaining twenty percent could also be adapted after suitable modifications. This included the problem of time irreversibility. Although a gas in a box cannot expand, repeated particle collisions cause quantities such as pressure and temperature to undergo irreversible macroscopic changes in one direction. In this way, they finally succeeded in rigorously deriving the mesoscopic Boltzmann equation from the microscopic dynamics of colliding hard spheres.

The three authors submitted their completed paper [11] to the *Annals of Mathematics*. After carefully studying the manuscript, the referees made many suggestions. Some parts needed clearer explanations, while others required greater rigor. After more than a year of revision, the paper grew from about 150 pages to approximately 200 pages. It was finally accepted by the Annals of Mathematics. During this period, the authors also wrote a simpler introduction to their work. Moreover, they gave a complete derivation in kinetic theory, beginning with microscopic hard-sphere dynamics, passing through the mesoscopic Boltzmann equation, and arriving at the macroscopic Navier–Stokes equations [12]. Thus, in the context of kinetic theory, Hilbert's Sixth

Problem received a complete solution. This is, of course, only a special case of Hilbert's Sixth Problem. But it was a special case explicitly mentioned by Hilbert himself when he discussed the problem, and it was also one on which Hilbert himself had made efforts [15].

The derivation in kinetic theory was accomplished through the collaboration of Deng, Hani, and Ma. Yet only Deng received the Fields Medal. The reason is that the Fields Medal recognizes a mathematician's overall body of contributions.

2.4 Yu Deng's Poetry

Yu Deng possesses talents in many areas. Not only did he show exceptional ability in mathematics from an early age, but he also became a highly skilled Go player and, in his youth, came close to being selected as a professional player. He is also an accomplished writer of classical Chinese poetry. Classical poems and lyrics written by him can readily be found online. The following are one poem and one lyric, included for readers' enjoyment.

Reflections on the New Year

The years pass swiftly as I gaze across the ends of sky;
The myriad worlds become one long song.
Blossoms in abundance adorn the fragrant trees;
Vast mirages float upon the waves of the sea.
I would seek a pearl emerging from chaos;
Once, within a tiny mote, I glimpsed the Milky Way.
When the brilliant lamps have all gone out and worldly sounds grow distant,
Only the moonlight upon the clear river remains abundant.

Linjiang Xian

Alone at the ends of the earth, I have long kept myself company;
When we parted then, it was all in such haste.
For the moment, with drunken eyes, I gaze toward the vault of heaven.
Where the Milky Way flickers,
Amid the being and nonbeing of black holes.
After a thousand years of iteration and chaos,
On what day may we still hope to meet again?
My sorrowful heart is layered with countless folds.
How much meaning lies in my thoughts of you?
They tend positively toward infinity.

In the first poem, a regulated verse in seven characters[1], the poet begins in a restrained and suggestive manner. Through a "long song[2] that might well be a lament," he expresses the loneliness and sense of drifting felt by a traveler far from home as the years pass swiftly. Unlike a traditional poet, however, the author does not dwell on this loneliness or continues merely to lament it. He intends to seek pearls within chaos, and he has once glimpsed a magnificent galaxy within a tiny particle. Thus, although he finds himself in the solitude of a place where the brilliant

[1] Seven as counted in Chinese.

[2] In classical Chinese poetry, long song is an expression of lament – 長歌當哭.

lights have gone out and the noise of the world has receded, he is not oppressed by loneliness. Instead, he can enjoy the quiet moonlight.

The second poem, *Linjiang Xian*, is very interesting. It uses several terms from mathematics and physics, yet they appear naturally and never seem forced. The parallelisms are also elegant, as in the pairing of "Milky Way" with "black hole.[3]"

The final two lines of both the upper and lower stanzas are especially interesting. Consider the last line of the upper stanza: "amid the being and nonbeing of black holes." People once wondered whether black holes truly existed. Although we cannot directly observe them, various indirect forms of evidence now seem to leave little doubt about their existence. In that sense, black holes presumably "are." Yet once matter or energy enters a black hole, it never emerges again. In this sense, a black hole may also be regarded as an extreme form of "nonbeing." The phrase "the being and nonbeing of black holes" is therefore worth contemplating.

The last line of the lower stanza, "they tend positively toward infinity," contains another interesting play on words. In ordinary language, the word "positively" may mean "undoubtedly" or "exactly." But in mathematics it also has the meaning of positive as opposed to negative. The poet seems to feel that distinction is necessary here: if longing were tending toward negative infinity, that would be highly inappropriate indeed. This was amusing since I thought perhaps only a mathematician would feel the need to clarify the sign of infinity in a poem.

3 A brief introduction to Hong Wang's work

In its citation for Hong Wang's Fields Medal, the International Mathematical Union stated that the award recognized her work in harmonic analysis and geometric measure theory. This includes, with different collaborators, the solution of the three-dimensional Kakeya conjecture and the Furstenberg set conjecture, as well as major breakthroughs in Fourier restriction, the Falconer distance problem, and local smoothing conjectures for wave equations.

Like Yu Deng, Hong Wang has made contributions in many areas of mathematics. A short article cannot cover all of them. We will therefore focus on the three-dimensional Kakeya conjecture mentioned in the prize citation. It had remained unsolved for many years, and Wang's work on it has attracted particular attention. Moreover, her work in this area is directly connected with several of the other problems mentioned in the citation. We will begin with the Kakeya problem itself, explain how its solution led to the Kakeya conjecture, and then discuss how Hong Wang and her collaborators proved the three-dimensional conjecture.

3.1 What is the *Kakeya Problem*?

To understand the Kakeya conjecture, we first need to understand the Kakeya problem, because the conjecture concerns a class of sets that arose from attempts to solve Kakeya's original problem. We begin in 1917, when the Japanese mathematician Sōichi Kakeya posed a problem in the *Tôhoku Mathematical Journal* [16] and the *Tôhoku Science Report* [19]: *Suppose we rotate a*

[3] In Chinese, not only the terms "Milky Way" and "black hole" are parallel, the corresponding Chinese characters also form parallel pairs.

needle of length 1 *and zero thickness through* 360° *in a plane. What is the smallest possible area swept out by the needle*?

We may first try a simple construction ourselves. If the midpoint of the needle is used as the center of a circle whose radius is half the length of the needle, then the needle can clearly rotate through 360° inside the circle. The area swept out is $\pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{4}$.

But this is not the configuration of smallest area. The reader may also observe that the needle can rotate through 360° inside an equilateral triangle of height 1. One may rotate the needle about one vertex from one side of the triangle toward another, then slide it along that side toward the next vertex and continue the rotation. The area of this triangle is only $\sqrt{3}/3$, smaller than $\pi/4$. Kakeya himself found a figure of still smaller area [16]. It was the three-cusped hypocycloid (see Figure 2). We know that among all tangent line segments of length 1 to this curve, one lies entirely inside the figure at every point. For example, in the figure, the tangent line *AB* at point *M* lies inside the figure. As *M* moves along the curve, the corresponding tangent line rotates smoothly. When *M* travels once around the curve, the needle represented by the tangent line *AB* has rotated through 360°. Kakeya did not know, however, whether a figure of still smaller area might exist.

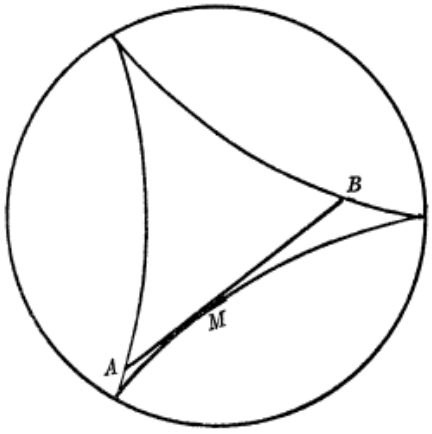


Figure 2. A three-cusped hypocycloid (deltoid) containing the moving unit segment *AB*, tangent to the curve at *M*.

Two years later, the problem was solved by the Russian mathematician Abram S. Besicovitch (the original paper is [2]; he later wrote a simpler and more accessible account in English [3]). Besicovitch's solution was highly surprising. He proved that *no figure of smallest area can exis*t. Given any positive number ε, however small, he could construct a figure with area smaller than ε in which the needle could rotate through 360°. At the same time, the area could not simply be zero, because any actual rotation of the needle sweeps out a positive area. Thus, the area can be made arbitrarily small but cannot be zero. Consequently, there is no region of minimum area.

The sets constructed by Besicovitch were astonishing. It was natural to ask whether analogous phenomena existed in three-dimensional or higher-dimensional spaces. In three or more dimensions, however, a needle may rotate in infinitely many directions at every point. To avoid this complication, we simplify the problem. A set *S* in an *n*-dimensional space $\mathbb{R}^n$ is called a *Besicovitch set*, or a *Kakeya set*, if *S* contains a unit line segment pointing in every direction in $\mathbb{R}^n$. Clearly, every planar

figure within which a needle can rotate through 360° is a Kakeya set in the plane. Besicovitch's planar construction can also be generalized to *n*-dimensional space $\mathbb{R}^n$. Since we no longer need to describe an actual rotation, we can prove something even stronger: One can find Kakeya sets whose *n*-dimensional volume is arbitrarily small, and even equal to zero (see [29]).

3.2 Two-dimensional Kakeya sets

In this section, we first describe what a two-dimensional Kakeya set looks like. These planar sets were essentially created by Besicovitch in solving Kakeya's problem and were later somewhat simplified by the German mathematician Oskar Perron (see [23, 24]).

Incidentally, Isaac Schoenberg, who was a friend of Besicovitch and taught for many years at the University of Wisconsin, recalled that Besicovitch once told him about the circumstances under which he solved the Kakeya problem. At the time, the Soviet Revolution and civil war were taking place. Besicovitch was teaching at a small school near the Ural Mountains and was living in great hardship. During the cold winter, he kept his feet inside a box filled with straw to stay warm. Yet the work he accomplished during that year helped him obtain the qualifications necessary to become a professor at Leningrad University, now Saint Petersburg State University (see [24, p. 170]).

In the plane, it is easy to find a set containing a unit line segment in every direction. A circle centered at the origin, or a square, will do. But we seek a set whose area can be made arbitrarily small. Besicovitch began with a square centered at the origin and having side length 2 (see Figure 3). This set naturally contains unit line segments in every direction in the plane. He divided the square into many triangles having the origin as a common vertex. He reasoned that if the directions of these triangles were preserved while the triangles were translated horizontally or vertically, perhaps the total area of the union could be reduced.

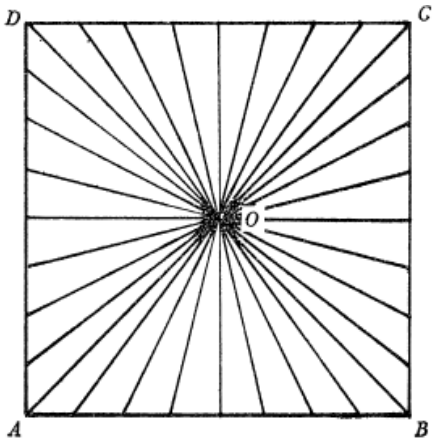


Figure 3. Subdivision of a square into triangles with a common vertex at its centre, preserving unit line segments in every direction.

Suppose two adjacent small triangles in the square are made partially to overlap, as in the first row of the Figure 4. The line segments contained in the two triangles remain present, but because of the overlap the total area is reduced. If the resulting overlapping figures are themselves

overlapped again, the area can be reduced further while the original collection of line segments in different directions is preserved, as illustrated in the second and third rows of the figure.

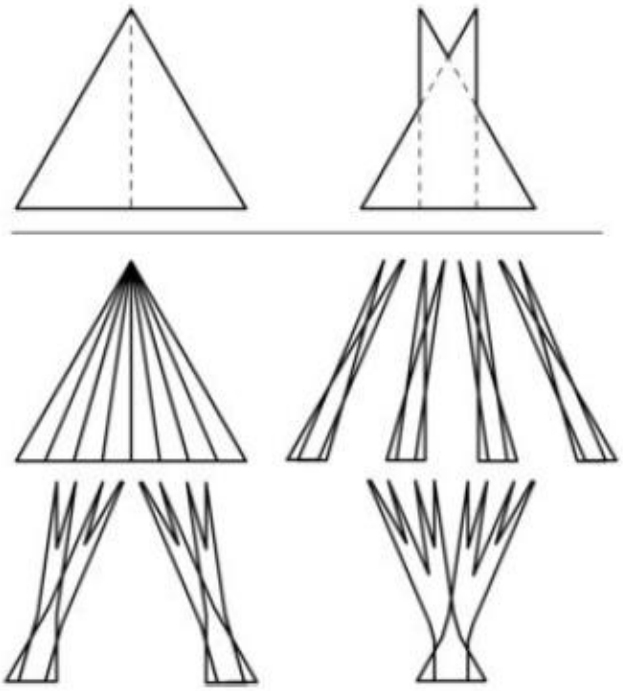


Figure 4. Successive translations and overlaps of triangular pieces in the Perron-tree construction

Besicovitch therefore divided the square into four large 45°–90°–45° triangles, each having the center of the square as a vertex and one side of the square as its base. Each large triangle was then divided into 2^m smaller triangles. He partially overlapped adjacent small triangles along each side. This reduced the number of distinct pieces in each large triangle by half. He then overlapped the resulting figures again. The number of pieces in each large triangle was again reduced by half. After repeating this process m times, only one figure remained within each large triangle. This figure is called a *Perron tree*. The square now contains four Perron trees: two vertical trees facing one another and two horizontal trees facing one another. Together, these four trees still contain line segments pointing in every direction in the plane.

We now use Oskar Perron's method to calculate the area occupied by a Perron tree. Instead of following the overlapping directly, Perron viewed the process as the growth of "sprouts" on a tree. At each stage he also enlarged the figure so that the change in area could be seen more clearly. He began with an isosceles right triangle of height 2—the triangle *ABC* in the accompanying Figure 5. This triangle is the zeroth-stage tree T_0. Its base is 4 and its height is 2, so its area is 4.

We now allow T_0 to "*sprout*." First extend the two sides *DC* and *EC* upward to the height $y = 3$, obtaining the points *G* and *F*. Then connect *F* with the corresponding point *D* on $y = 1$, and connect *G* with *E*. The triangles *FCD* and *GCE* are the two new sprouts. Together with the original triangle *ABC*, they form the first-stage tree T_1. We can see that T_1 is obtained by cutting an isosceles right triangle of height 3—the triangle Δ_1 in the figure—into two halves and then translating them horizontally so that they overlap over two-thirds of their bases.

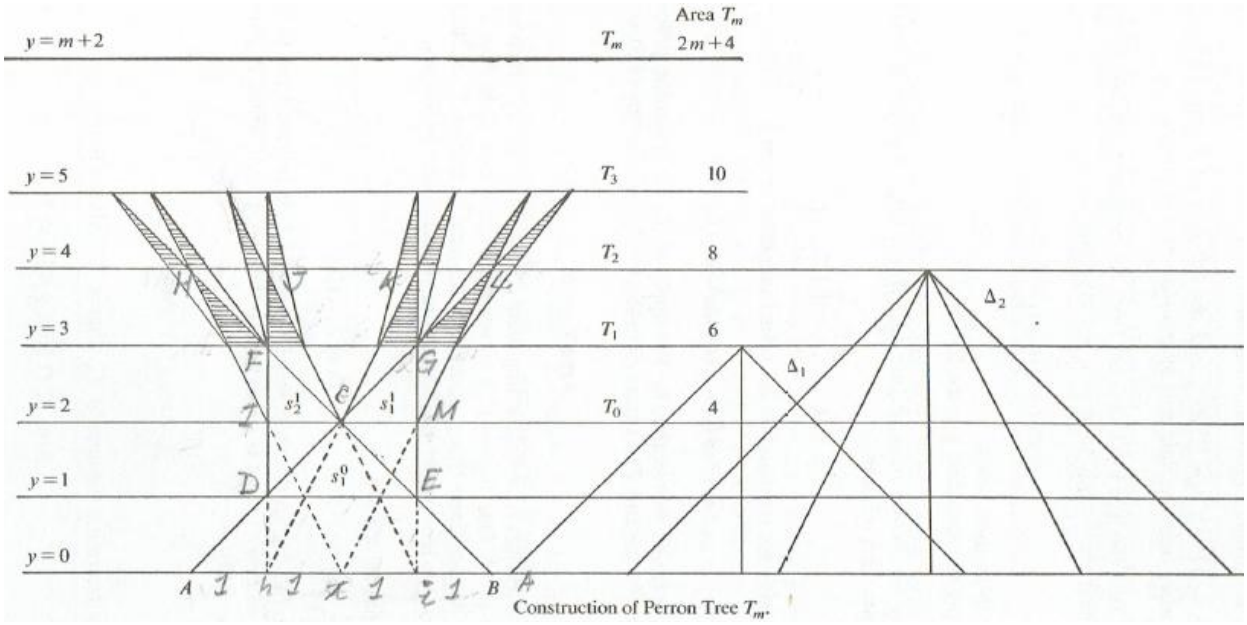


Figure 5. Construction of the Perron trees T_0, T_1, T_2, T_3, … by successive sprouting, showing their heights and area

The two new sprouts of T_1 can be divided into four triangles of equal area: two lying between $y = 2$ and $y = 3$, and two more lying between $y = 1$ and $y = 2$. The combined area of the upper two triangles, s_2^1 and s_1^1, equals the area of triangle s_2^0, because all three triangles have height 1 and the sum of the bases of s_2^1 and s_1^1 equals the base of s_2^0. Since the area of s_2^0 is 1, and the four small triangles in the two new sprouts have equal areas, the passage from T_0 to T_1 increases the area by 2 units.

We can develop T_k into T_{k+1} in the same manner. For example, T_1 has two top vertices, F and G. At each vertex a pair of sprouts can grow, so T_2 has four top vertices: H, J, K, and L. In general, T_k has 2^k top vertices. These 2^k top vertices are the vertices of 2^k small triangles of height $k + 2$ contained in T_k. When a large 45°–90°–45° triangle of height $k + 2$ is divided into 2^k small triangles and then repeatedly overlapped, the resulting figure is T_k. The small triangles appearing in T_k are precisely those originating from the original large triangle.

What is the area of T_k? Just as in the passage from T_0 to T_1, each increase of k by 1 increases the area of T_k by 2 units. This is because, in passing from T_k to T_{k+1}, the total area of the upper halves of every pair of new sprouts always equals the upper half of the previous sprouts—the shaded triangles in the figure. These triangles all have height 1, and the combined bases of each pair of new triangles have the same total width as the base of the previous triangle. At T_1, the upper halves of the two sprouts, namely s_2^1 and s_1^1, have total area 1. Therefore, at every stage of sprouting, the total area of the upper halves of all the new sprouts remains 1. Since each sprout also has a lower half of equal area, each stage from T_k to T_{k+1} increases the total area by 2 units. Because the area of T_0 is 4, the area of T_k is $4+2k$. In the Kakeya construction, each large 45°–90°–45° triangle is divided into 2^m small triangles. We therefore need to perform m horizontal translations and overlaps, producing a Perron Tree T_m of height $m + 2$ and area $4 + 2m$.

In the preceding calculation, we enlarged the figure at every stage so that the sprouting process could be seen clearly. In the original Kakeya problem, we begin with a square centered at the

origin and having side length 2. It is divided into four 45°–90°–45° triangles of height 1. Each triangle is then divided into 2^m small triangles, which are repeatedly translated and overlapped to form a Perron tree. The heights of all the small triangles in these trees should be 1, and the heights of the trees should also be 1. After our repeated enlargements, however, we obtained T_m of height $m + 2$. Since each enlargement was performed by the same scale factor, we must divide both the height and the width of T_m by $m+2$ to recover the actual dimensions. Consequently, the actual area of T_m should be

$$\frac{4+2m}{(m+2)^2} = \frac{2}{m+2}.$$

Since there are four Perron trees in the Kakeya construction, the area of the resulting two-dimensional Kakeya set is $\frac{8}{m+2}$. From this formula we see that, simply by increasing the number m of small triangles, the area of the set can be made arbitrarily small. According to the construction of the Perron trees, the bases of the four T_m trees all have length 4 after enlargement. Removing the enlargement factor reduces their bases to $\frac{4}{m+2}$. Moreover, according to the construction, all four trees open towards the origin. Since translating part of the set does not affect the Kakeya property, we may translate the upper and lower trees vertically and the left and right trees horizontally, obtaining four small trees whose bases lie near the origin, and which open in the four outward directions.

3.3 Solution of the two-dimensional Kakeya Problem

The Kakeya sets described in the preceding section merely contain line segments pointing in every direction in the plane. We have not yet considered how actually to rotate a needle through 360°. In this section we describe how Besicovitch showed that, by enlarging such Kakeya sets only slightly, a unit line segment can be rotated through a full 360° inside them. Since the Kakeya conjecture itself does not require the actual rotation of a needle, readers interested only in the conjecture may skip this section. In the next section we return to the Kakeya conjecture and to the work of Roy Davies, Hong Wang, Joshua Zahl, and others.

Before discussing Besicovitch's proof, we need the following lemma: A line segment can be translated to any parallel position in the plane using an arbitrarily small additional area. For example, in the accompanying Figure 6, suppose we wish to move the line segment *HG* to the parallel position *FD*. First rotate *HG* through a very small angle to the position *KG*. Now *KG* and *FD* are no longer parallel, and their extended lines must intersect. We may slide *KG* along its extended line to the position *LN*, then rotate *LN* through the same angle to *LM*, and finally slide it along *LD* to the position *FD*. Since translating a line segment along a straight line requires no area, the only additional area required by this motion consists of two small triangles, whose total area can be made arbitrarily small.

We now return to the Kakeya problem. Besicovitch pointed out that a unit line segment can rotate through 360° inside a square of side length 2 centered at the origin. If we divide the square into many triangles, as shown in Figure 3, the line segment can rotate successively inside one triangle after another. After completing its rotation within one triangle, it moves to the next



Figure 6. Moving a line segment to a parallel position
while sweeping out an arbitrarily small additional area

triangle and continues. When these triangles are repeatedly overlapped to form the four Perron trees T_m, the triangles themselves have not disappeared; they have merely overlapped. Therefore, after a line segment rotates within each triangle of the four Perron trees, the triangle that was originally adjacent to it is still present within the same collection of trees. Moreover, the common side of the two originally adjacent triangles remains parallel to itself under the overlapping translations. We may therefore use the lemma above to move the line segment to the next triangle, using only an arbitrarily small additional area, and continue the rotation.

Let us now make this argument precise. If ε is a positive number, choose a Kakeya set whose total area is $\frac{8}{m+2} < \frac{\varepsilon}{2}$. The four Perron trees contain altogether 42^m triangles, and we need to perform 4×2^m parallel translations. If we require the sum of the areas of the two small triangles needed for each parallel translation to be less than $\frac{\varepsilon}{8\times 2^m}$, then, after adding all the line segments and small triangles required for the translations, a unit line segment can rotate through 360°, while the total area of the enlarged set remains less than ε. Thus Kakeya's original problem is solved.

3.4 From the Kakeya Problem to the Kakeya Conjecture

What kind of object is a Kakeya set in an n-dimensional space $\mathbb{R}^n$? Its n-dimensional volume can be made arbitrarily small and may even be zero. Yet such a set must contain a substantial amount of geometric "content," because it contains unit line segments pointing in every direction in space. Ordinarily, we use volume to measure the size of a set. But volume is unsuitable for Kakeya sets, since their volume can be arbitrarily small. During the 1970s, fractal theory became increasingly influential. Mathematicians came to understand that the dimension of a set need not be an integer. Moreover, in an n-dimensional space, a set of zero volume may have dimension greater than $n - 1$. This suggested that dimension might measure the geometric "substance" of a set better than volume does. People therefore began to investigate how large the dimension of a Kakeya set in $\mathbb{R}^n$ must be.

In fractal theory, two commonly used notions of dimension are the **Hausdorff dimension** $\dim_H$ and the **Minkowski dimension** $\dim_M$. Let us briefly examine how these two dimensions are calculated.

The Hausdorff dimension is related to the scaling behavior of a figure S. Suppose we enlarge a figure S by a scale factor p, and the enlarged figure consists of p^n copies of the original figure.

Then $\dim_H(S) = n$. For example, if *S* is a square and we enlarge it by the factor $p = 2$, so that every side becomes twice as long, then four copies of *S* are needed to fill the enlarged square. Since $4 = 2^2$, the square is two-dimensional.

If *S* is a cube and we enlarge it by the factor $p = 2$, the enlarged cube has two layers, each requiring four copies of *S*. Thus, the enlarged cube consists of eight copies of *S*. Since $8 = 2^3$, the cube is three-dimensional.

Another example is the Sierpiński triangle (see the accompanying Figure 7). Again, enlarge the figure by the factor $p = 2$. The original figure may be regarded as one of the three triangles in the enlarged figure. Thus, the enlarged figure consists of three copies of the original figure. We therefore write 3 in the form $3 = 2^n$, because the scaling factor is $p = 2$. In this example

$$\dim_H(S) = n = \frac{\log 3}{\log 2} \approx 1.58.$$

Thus, the Hausdorff dimension of the Sierpiński triangle is 1.58.

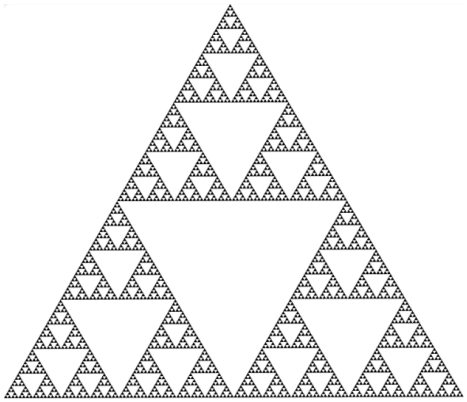


Figure 7. Sierpiński triangle

The definition of the Minkowski dimension is somewhat more complicated, but its computation can be more convenient for certain sets. Let ε be a positive number. By an "ε-box" in $\mathbb{R}^n$ we mean a cube in $\mathbb{R}^n$ whose side length is ε. Suppose $S \subset \mathbb{R}^n$ is a set. We cover *S* with the smallest possible number of ε-boxes. Then we let ε tend to zero step by step and observe how the number of boxes required increases. This also reveals the dimension of *S*. For example, suppose *S* is a square of side length 1 in the plane. If $\varepsilon = 1, \frac{1}{2}, \frac{1}{3}, \ldots$, the number of ε-boxes required is $N(\varepsilon) = 1, 2^2, 3^2, \ldots$. But if S is a unit cube in three-dimensional space, then when $\varepsilon = 1, \frac{1}{2}, \frac{1}{3}, \ldots$, the minimum number of boxes required becomes $N(\varepsilon) = 1, 2^2, 3^2, \ldots$.

We can easily see that, for a square, the number of boxes increases quadratically, while for a cube it increases cubically. This is because a square is a two-dimensional object, requiring the number of boxes to increase along two coordinate directions, whereas a cube requires increases along three coordinate directions. Thus, it is natural to expect $N(\varepsilon)$ to be a power of $1/\varepsilon$. The Minkowski dimension is therefore defined by

$$\dim_M\left(S\right) = \lim_{\varepsilon \to 0}\left(\frac{\log N(\varepsilon)}{\log\left(\frac{1}{\varepsilon}\right)}\right)$$

In the definition we take the limit because the boxes used to cover a set may be replaced by balls, rectangular boxes, and other shapes. When the limit is taken, the irregularities introduced by such choices can be eliminated. For example, when considering the dimension of a three-dimensional Kakeya set, the "boxes" often used are tubes containing the relevant line segments.

For the Sierpiński triangle, we see that when $1/\varepsilon$. increases in the manner of 2^n, $N(\varepsilon)$ increases in the manner of 3^n. Thus $\dim_M(S) = \dim_H(S)$. In general, however, the two dimensions need not coincide. We do know that $\dim_H(S) \le \dim_M(S)$ always holds.

Let us now return to the question raised at the beginning of this section: How large can the dimension of a Kakeya set in $\mathbb{R}^n$ be? In 1971, the British mathematician Roy O. Davies proved that if S is a Kakeya set in the plane, then $\dim_H(S) = \dim_M(S) = 2$ (see [8]). Thus, although such sets may have arbitrarily small area, their geometric "substance" cannot be ignored: they are genuinely two-dimensional sets. The transition from two dimensions to three dimensions, however, turned out to be extraordinarily difficult. If S is a Kakeya set in three-dimensional space, Thomas Wolff proved in 1995 that $2.5 \le \dim_H(S) = \dim_M(S) \le 3$ (see [28]). For Minkowski dimension, a further improvement was obtained in 2000 by three of the world's leading mathematicians, including Terence Tao, who later received the Fields Medal in 2006. They proved

$$2.500,000,000,1 \le \dim_M(S) \le 3\,.$$

Three mathematicians worked for five years and improved only the tenth decimal digit! Their sixty-three-page paper was nevertheless regarded as an important contribution and was published in the *Annals of Mathematics*, one of the most authoritative mathematical journals (see [20]). Of course, the paper also removed several obstacles for later researchers and provided useful new tools.

The ***n*-dimensional Kakeya conjecture** is the following:

If S is a Kakeya set in n-dimensional space, then $\dim_H(S) = \dim_M(S) = n.$

From 2000 until 2025, although several partial discoveries were made, there was essentially no major breakthrough, especially concerning the three-dimensional conjecture. This situation continued until February 2025, when Hong Wang and Joshua Zahl announced their work (see [26] and [27]). They raised 2.500,000,000,1 all the way to 3. In other words, they completely proved the three-dimensional Kakeya conjecture: If S is a Kakeya set in three-dimensional space, then $\dim_H(S) = \dim_M(S) = 3$. This was an extraordinary achievement.

Why are mathematicians so interested in the Kakeya conjecture? At first sight, Kakeya sets appear to be isolated and peculiar objects. Their dimensions might not seem to deserve so much attention. But during the past half-century, mathematicians have discovered that more and more problems relate to the dimension of Kakeya sets. For example, [5] and [29] discuss how Kakeya-

type problems arise in number theory and combinatorics. Reference [13] discusses how the truth or falsehood of the Kakeya conjecture is directly connected with problems involving Fourier transforms, partial differential equations, oscillatory integrals, and related subjects.

In fact, it is not difficult to understand why Kakeya-type ideas arise in applications. Let us give a simple intuitive explanation. Today electromagnetic waves are commonly used to transmit information. For convenience in transmission and storage, we often compress information into a small region of space. A very practical question is: how much information can be compressed into a given space? Since waves may point in any direction in space, one naturally thinks of Kakeya sets, which contain line segments in every possible direction and yet can arrange these segments so densely that the total volume of the set can even be zero. But waves must oscillate, and it is difficult simply to replace a wave with a line segment. We therefore place each line segment inside a tube and allow the wave to oscillate within the tube. Recalling the preceding discussion of $\dim_M(S)$, we see that as the diameter of the tubes changes, the number of tubes required is precisely related to the computation of the dimension of the set. Thus, it is quite natural that the Kakeya conjecture appears in various applied problems.

In its Fields Medal citation for Hong Wang, the International Mathematical Union mentioned, in addition to the Kakeya conjecture, four other areas of her work. It cited her work with Shukun Wu in substantially improving Fourier restriction estimates; her work with Kaiyuan Ren in solving the two-dimensional Furstenberg set conjecture and obtaining sharp or near-sharp estimates in three-dimensional settings—a major development in connection with the famous restriction conjectures initiated by Elias Stein in the 1960s. The citation also mentioned her work with Larry Guth and Ruizhe Zhang on the local smoothing conjecture for wave equations in two and three dimensions, and her collaboration with Larry Guth, Alex Iosevich, and Yumeng Ou in obtaining major breakthroughs on the Falconer distance set conjecture.

These mathematical problems belong to harmonic analysis, metric geometry, partial differential equations, and combinatorics. Yet they are all directly related to the Kakeya conjecture (see [13, 25, 29]). Wang's work in these different fields undoubtedly deepened her understanding of the Kakeya conjecture from several different perspectives. At the same time, her work on the Kakeya conjecture has naturally helped her make major advances in these other areas. Hong Wang's career therefore provides a clear example of why so many mathematicians have followed the progress of the Kakeya conjecture with such interest.

3.5 Hong Wang, Joshua Zahl, and the Kakeya Conjecture

We now finally turn to the work of Hong Wang and Joshua Zahl on the three-dimensional Kakeya conjecture. Before doing so, however, it is useful first to describe briefly how Roy Davies proved the two-dimensional Kakeya conjecture [8]. This will help us understand why, after the two-dimensional problem was solved in 1971, it took more than half a century—until 2025—for the three-dimensional conjecture to be solved. What is fundamentally different between two and three dimensions? And how did Wang and Zahl overcome these difficulties?

As explained in the preceding section, the n-dimensional Kakeya conjecture asserts that a Kakeya set in n-dimensional space has dimension n. We also discussed two different notions of dimension: Hausdorff dimension and Minkowski dimension. Hong Wang, Joshua Zahl, and Roy

Davies all considered both notions. Since our purpose here is only to introduce the main ideas of their proofs, we shall discuss only the Minkowski dimension.

Consider a Kakeya set *K* in the plane. The set *K* contains a unit line segment in every direction in the plane. As explained above, to calculate the dimension of *K*, we cover it with squares of side length δ. Let $N(\delta)$ be the minimum number of such squares required to cover *K*. As δ decreases, the way $N(\delta)$ increases reveals the dimension of *K*. Now cover each line segment in *K* by a row of squares of side length δ. Since every line segment has length 1, each one requires $1/\delta$ squares. If the line segments were independent—for example, if they were uniformly distributed as parallel vertical segments in a square—then $N(\delta) = \left(\frac{1}{\delta}\right)\left(\frac{1}{\delta}\right) = \delta^{-2}$, and the square would have dimension 2. But the line segments are not independent. They may overlap with one another. Davies began by estimating how such overlaps reduce the number of squares required.

He first extended the line segments. Suppose two lines intersect at an angle $\geq \delta$. At a distance r from the point of intersection, the distance between the two lines is already $\geq r\delta$. When $r > 1$, it is therefore difficult for the two lines to be covered by the same δ-square. Based on this observation, Davies estimated, for a given line and a possible intersection point, how many lines in different directions could intersect it and how many δ-squares the two lines could share for each r. Since after two intersecting lines continue in different directions, the distance between them rapidly increases, the amount of overlapping cannot be comparable to r, and he showed that the loss caused by overlaps can be controlled by a logarithmic factor. Thus, the number $N(\delta)$ of δ-squares required to cover a Kakeya set *K* satisfies

$$N(\delta) \geq \frac{\delta^{-2}}{\log\left(\frac{1}{\delta}\right)}.$$

This is a very elegant conclusion. On the one hand, if we calculate the total area required to cover *K*, we obtain

$$N(\delta)\delta^2 \geq \frac{1}{\log\left(\frac{1}{\delta}\right)} \to 0 \text{ as } \delta \to 0.$$

This agrees with Besicovitch's conclusion that Kakeya sets may have arbitrarily small area.

On the other hand, if the dimension *d* of *K* were less than 2, the area required for the covering would be

$$N(\delta)\delta^d \geq \frac{\delta^{d-2}}{\log\left(\frac{1}{\delta}\right)} \to \infty \text{ as } \delta \to 0$$

This is impossible. Therefore, the dimension of *K* must be 2.

From an intuitive point of view, the two-dimensional case is much simpler than the three-dimensional one for the following reason: A Kakeya set *K* contains line segments pointing in different directions. In the plane, two nonparallel lines must intersect. After they intersect and continue in different directions, they rapidly separate. Thus, in essence, each line eventually requires its own covering boxes, much as independent parallel line segments in the plane do. Consequently, the dimension is 2.

To prove the three-dimensional Kakeya conjecture, however, we must show that the number of boxes required to cover all the line segments in a Kakeya set is essentially comparable with the number required to cover a three-dimensional cube. In other words, there cannot be too many line segments tightly bundled together and sharing the same covering boxes. If there were, the number of boxes would be too small to yield dimension 3. Although the line segments in a three-dimensional Kakeya set point in every possible direction, three-dimensional space allows many ways for infinitely many nonparallel lines to remain bundled together and share the same rows of covering boxes. These difficulties prevent the methods used in two dimensions from being applied directly in three dimensions.

For example, within a slab of thickness *d*, it is easy to find a family of line segments that have the same direction in the plane but differ only slightly in the direction of the thickness. All of them may fit within the same row of covering boxes. Wang and Zahl had to prove that, although configurations of this type reduce the number of boxes needed to cover the entire set, they do not reduce it enough to affect the dimension. Their paper proving the three-dimensional Kakeya conjecture [27] is 127 pages long. It is extraordinarily difficult, and it constantly draws upon the work of other authors, including papers such as [20] and [26], which are themselves highly technical. The following discussion can therefore provide only a few examples of the difficulties they faced.

The Heisenberg group model

To introduce additional mathematical tools, N. H. Katz, I. Łaba, and T. Tao formulated a version of the Kakeya problem in three-dimensional complex space $\mathbb{C}^n$ [20]. They discovered a counter-example to the Kakeya conjecture in $\mathbb{C}^n$. In this space there is a two-dimensional surface which is invariant under transformations of the group $U(2,1)$ and is thus called a Heisenberg group model. At every point on this surface there is a family of complex lines from the Kakeya set. When these lines are thickened into tubes, the amount of overlapping volume exceeds what would be permitted by the Kakeya conjecture. For a long time, many mathematicians feared that this counterexample might give rise to a counterexample to the Kakeya conjecture in the real space. This became one of the obstacles that Wang and Zahl had to overcome. They succeeded in doing so by proving that the complex counterexample cannot occur in the corresponding way in real space.

The sticky sets

Suppose a tube contains infinitely many line segments belonging to a Kakeya set. These segments can themselves be distributed among a densely packed family of smaller tubes inside the original tube. Then, within each smaller tube, the line segments can again be distributed in a similar manner among still smaller tubes. If this process can be repeated indefinitely, with each level containing densely packed smaller tubes arranged in a similar fashion, the resulting family of line segments is called a **sticky set**. This concept was also introduced by N. H. Katz, I. Łaba, and T. Tao in their paper [20]. They regarded such configurations as among the worst possible cases that would have to be understood in solving the three-dimensional conjecture. Because these line configurations possess a self-similar structure reminiscent of fractal geometry, Jean Bourgain

introduced additive and multiplicative operations into their study. He conjectured that genuinely fractal sticky sets could not occur in the Kakeya problem [6]. Wang and Zahl proved Bourgain's conjecture. In doing so, they eliminated one of the most difficult potential obstacles to proving the Kakeya conjecture. They also showed that other, simpler types of sticky configurations, even if they occur, do not cause serious difficulties.

The non-sticky case

After the sticky case had been brought under control, Wang and Zahl turned their attention to the non-sticky case. For a time, they felt that sticky sets might not actually represent the principal obstruction to proving the Kakeya conjecture. The situation became clearer after Wang and Kaiyuan Ren solved the two-dimensional Furstenberg set conjecture. Wang realized that Fourier-analytic methods could be used to decompose the general situation into two types of sets: one consisting of sticky configurations, and another consisting of configurations that are, in an important sense, almost the opposite of sticky. The two cases could then be treated separately.

The examples above represent only a few of the difficulties that Wang and Zahl had to overcome. Their work involved much more. Interested readers may consult Larry Guth's article introducing the work of Wang and Zahl [17]. Guth was Wang's doctoral advisor at MIT. His article is detailed, reliable, and relatively accessible.

Davies, Wang, and Zahl solved the Kakeya conjecture in two and three dimensions. In higher-dimensional spaces, however, comparatively little is known, and much of the territory remains almost completely unexplored. There is therefore still a great deal of work to be done in the future.

References

[1] Bardos, C., & Ukai, S. (1991). The classical incompressible Navier–Stokes limit of the Boltzmann equation. *Mathematical Models and Methods in Applied Sciences*, 1, 235–257.

[2] Besicovitch, A. S. (1919). Sur deux questions d'integrabilite des fonctions. *Journal of the Perm Physics and Mathematics Society*, 2, 105–123.

[3] Besicovitch, A. S. (1963). The Kakeya problem. *The American Mathematical Monthly*, 70, 697–706.

[4] Boltzmann, L. (1872). Weitere Studien uber das Warme gleichgenicht unfer Gasmolakular. Sitzungsberichte der Akademie der Wissenschaften, 2, 275–370. Translation: Further studies on the thermal equilibrium of gas molecules. In: *S. G. Brush (Ed.), Kinetic Theory, Vol. 2*, Pergamon, Oxford, 1966, 88–174.

[5] Bourgain, J. (2000). Harmonic analysis and combinatorics: How much may they contribute to each other? *Mathematics: Frontiers and Perspectives*, IMU/American Mathematical Society, 13–32.

[6] Bourgain, J. (2003). On the Erdős–Volkmann and Katz–Tao ring conjectures. *Geometric and Functional Analysis*, 13, 334–365.

[7] Browder, F. E. (Ed.). (1976). *Mathematical Developments Arising from Hilbert Problems*. Proceedings of Symposia in Pure Mathematics (Vol. 28), American Mathematical Society.

[8] Davies, R. O. (1971). Some remarks on the Kakeya problem. *Proceedings of the Cambridge Philosophical Society*, 69, 417–421.

[9] Deng, Y., & Hani, Z. (2026). Derivation of the wave kinetic equation: Full range of scaling laws. *Memoirs of the American Mathematical Society*, Vol. 320(1631), American Mathematical Society.

[10] Deng, Y., & Hani, Z. (2023). Full derivation of the wave kinetic equation. *Inventiones Mathematicae*, 233(2), 543–724.

[11] Deng, Y., Hani, Z., & Ma, X. (2025). Long time derivation of Boltzmann equation from hard sphere dynamics. *Annals of Mathematics* (to appear).

[12] Deng, Y., Hani, Z., & Ma, X. Hilbert's sixth problem: Derivation of fluid equations via Boltzmann's kinetic theory. arXiv. Available online at: https://arxiv.org/abs/2503.01800.

[13] Hickman, J. The Kakeya conjecture: Where does it come from and why is it important? arXiv. Available online at: https://arxiv.org/abs/2512.09842

[14] Hilbert, D. (1902). Mathematical problems. *Bulletin of the American Mathematical Society*, 8, 437–479.

[15] Hilbert, D. (1912). Begründung der kinetischen Gastheorie. *Mathematische Annalen*, 72, 562–577.

[16] Fujiwara, M., & Kakeya, S. (1917). On some problems of maxima and minima for the curve of constant breadth and the in-revolvable curve of the equilateral triangle. *Tôhoku Mathematical Journal*, 11, 92–110.

[17] Guth, L. Introduction to the proof of the Kakeya conjecture. Preprint, arXiv:2505.07695.

[18] Kawashima, S., Matsumura, A., & Nishida, T. (1979). On the fluid-dynamical approximation to the Boltzmann equation at the level of the Navier–Stokes equation. *Communications in Mathematical Physics*, 70(2), 97–124.

[19] Kakeya, S. (1917). Some problems on minima and maxima regarding ovals. *Science Reports of the Tôhoku Imperial University*, 6, 71–88.

[20] Katz, N. H., Łaba, I., & Tao, T. (2000). An improved bound on the Minkowski dimension of Besicovitch sets in $\mathbb{R}^3$. *Annals of Mathematics*, 152, 383–446.

[21] Lanford, O. E. (1975). Time evolution of large classical systems. *Lecture Notes in Physics*, Vol. 38, Springer Verlag, Berlin–New York, 1–111.

[22] Liu, S., Yang, T., & Zhao, H. (2014). Compressible Navier–Stokes approximation to the Boltzmann equation. *Journal of Differential Equations*, 256(11), 3770–3816.

[23] Perron, O. (1928). Über einen Satz von Besicovitch. *Mathematische Zeitschrift*, 28, 383–386.

[24] Schoenberg, I. J. (1982). On the Kakeya–Besicovitch problem. In: *Mathematical Time Exposures* (pp. 168–184). Mathematical Association of America.

[25] Tao, T. (2001). From rotating needles to stability of waves: Emerging connections between combinatorics, analysis, and PDE. Notices *of the American Mathematical Society*, 48(3) , 294–303.

[26] Wang, H., & Zahl, J. (2026). Sticky Kakeya sets and the sticky Kakeya conjecture. *Journal of the American Mathematical Society*, 39, 515–585.

[27] Wang, H., & Zahl, J. *Volume estimates for unions of convex sets, and the Kakeya set conjecture in three dimensions*. arXiv. Available online at: https://arxiv.org/abs/2502.17655

[28] Wolff, T. (1995). An improved bound for Kakeya type maximal functions. *Revista Matematica Iberoamericana*, 11(3), 651–674.

[29] Wolff, T. (1999). Recent work connected with the Kakeya problem. In: *Prospects in Mathematics* (Princeton, NJ, 1996), American Mathematical Society, Providence, RI, 129–162.