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An extension of Dedekind's arithmetic function

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Abstract: By analogy with the Schemmel function, a new arithmetic function is introduced. It is an extension of the well-known Dedekind arithmetic function. Some of its basic properties are discussed.

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In memory of my wife Lilija C. Atanassova
(8 July 1956 – 6 August 2026).
She was a very good mathematician.

1 Introduction

Leonard Euler's totient function and Julius Wilhelm Richard Dedekind's arithmetic function are well known. For the natural number

$$n = \prod_{i=1}^{k} p_i^{\alpha_i}, \tag{1}$$

where $k, \alpha_1, \ldots, \alpha_k, k \geq 1$ are natural numbers and $p_1 < \cdots < p_k$ are different primes, they are defined (see, e.g., [6–8, 10]) by:

$$\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1),\ \varphi(1) = 1,$$
$$\psi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i + 1),\ \psi(1) = 1.$$

The first one of these functions was extended by Victor Schemmel [11] to the form

$$\varphi^{(r)}(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - r) = n \prod_{i=1}^{k} \left(1 - \frac{r}{p_i}\right),\ \varphi^{(r)}(1) = 1. \tag{2}$$

This function is an object of research in a series of papers, e.g., [3–5, 9, 12–14].

Here, by analogy with Schemmel's function, we will introduce an extension of Dedekind's function and will study some of its properties. In addition, some properties of function $\varphi^{(r)}$ will be also discussed.

In the proofs we will use multiplicative induction on the number of divisors of n.

2 Definition and properties of an extension of Dedekind's function

Let everywhere below the natural number n be given by (1) . Let r be a natural number. Then we can define a new function that is an extension of Dedekind's function:

$$\psi^{(r)}(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i + r),\ \psi^{(r)}(1) = 1. \tag{3}$$

Obviously, it is a multiplicative function and

$$\psi(n) = \psi^{(1)}(n),$$

i.e., it is an extension of the standard Dedekind's function. When $r = 0$:

$$\varphi^{(0)}(n) = n = \psi^{(0)}.$$

Following [1], let us define

$$\underline{\text{mult}}(n) = \prod_{i=1}^{k} p_i,$$
$$\underline{\text{set}}(n) = \{p_1, \ldots, p_k\}.$$

If we suppose that the parameter r is an arbitrary integer, then we will see the deep connection between the functions $\varphi^{(r)}$ and $\psi^{(-r)}$ – it is evident from the following two equalities:

$$\varphi^{(-r)}(n) = \psi^{(r)}(n),$$
$$\psi^{(-r)}(n) = \varphi^{(r)}(n).$$

From (2) and (3) it is seen directly that for every two natural numbers r, s and $r > s$ they hold the inequalities:

$$n + 1 \leq \psi(n) \leq \psi^{(s)}(n) < \psi^{(r)}(n).$$

Below, we formulate and prove some assertions related to the two discussed functions that do not exist in cited above papers devoted to function $\varphi^{(r)}$.

Theorem 1. *For every two natural numbers r, s such that $r > s$:*

$$\left(\psi^{(r)}(n)\right)^2 > \psi^{(2r-s)}(n)\psi^{(s)}(n). \tag{4}$$

Proof. For the inequality (4) we obtain:

$$\begin{aligned}
&\left(\psi^{(r)}(n)\right)^2 - \psi^{(2r-s)}(n)\psi^{(s)}(n) \\
&= \left(\prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i + r)\right)^2 - \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i + 2r - s)\prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i + s) \\
&= \prod_{i=1}^{k} p_i^{2\alpha_i - 2}\left(\prod_{i=1}^{k}(p_i + r)^2 - \prod_{i=1}^{k}(p_i + 2r - s)(p_i + s)\right) > 0,
\end{aligned}$$

because for each i $(1 \leq i \leq k)$:

$$(p_i + r)^2 - (p_i + 2r - s)(p_i + s) = r^2 - 2rs + s^2 = (r - s)^2 > 0,$$

which proves the assertion. □

Let (see, e.g., [6, 7, 10])

$$\omega(n) = k.$$

Theorem 2. *For each natural number r that is less than or equal to the minimal element of $\underline{\text{set}}(n)$:*

$$\varphi^{(r)}(n)\psi^{(r)}(n) \leq n^2 - r^2\omega(n). \tag{5}$$

Proof. When n is a prime number, we obtain that $\omega(n) = 1$ and

$$n^2 - r^2\omega(n) - (n - r)(n + r) = 0,$$

i.e., (5) is valid. Let us assume that (5) is valid for n. Let $p \notin \underline{\text{set}}(n)$ and $p \geq r$. Then

$$\begin{aligned}
(np)^2 - r^2\omega(np) - \varphi^{(r)}(np)\psi^{(r)}(np) \quad &= n^2p^2 - r^2\omega(n) - r^2 - \varphi^{(r)}(n)\psi^{(r)}(n)(p^2 - r^2) \\
&\geq n^2p^2 - r^2\omega(n) - r^2 - (n^2 - r^2\omega(n))(p^2 - r^2) \\
&= r^2(p^2\omega(n) + n^2 - r^2\omega(n) - \omega(n) - 1) \\
&\geq r^2(n^2 - \omega(n) - 1) > 0.
\end{aligned}$$

Let $p \in \underline{\text{set}}(n)$. Then $\omega(np) = \omega(n)$ and by the condition $p \geq r$

$$\begin{aligned}
(np)^2 - r^2\omega(np) - \varphi^{(r)}(np)\psi^{(r)}(np) \quad &= n^2p^2 - r^2\omega(n) - \varphi^{(r)}(n)\psi^{(r)}(n)p^2 \\
&\geq n^2p^2 - r^2\omega(n) - (n^2 - r^2\omega(n))p^2 \\
&= \omega(n)(p^2r^2 - r^2) > 0,
\end{aligned}$$

with which (5) is proved. □

Theorem 3. *For each odd number n and for each natural number r that is less than the minimal element of $\underline{\text{set}}(n)$:*

$$\varphi^{(r)}(n) + \psi^{(r)}(n) \geq 2n + (\omega(n) - 1)r. \tag{6}$$

Proof. Let $n \geq 3$ be a prime number. Then $r \leq n - 1$ and

$$\varphi^{(r)}(n) + \psi^{(r)}(n) = n - r + n + r = 2n + 0.r,$$

i.e., (6) is valid. Let us assume that (6) is valid for the natural numbers less than or equal to n. Let $p \notin \underline{\text{set}}(n) \cup \{r\}$, $p > 3$ and $p \geq r$. Then

$$\begin{aligned}
&\varphi^{(r)}(np) + \psi^{(r)}(np) - 2np - (\omega(np) - 1)r \\
&= \varphi^{(r)}(n)(p - r) + \psi^{(r)}(n)(p + r) - 2np - \omega(n)r \\
&= p(\varphi^{(r)}(n) + \psi^{(r)}(n) - 2n) + r(\psi^{(r)}(n) - \varphi^{(r)}(n) - \omega(n)) \\
&\geq (\omega(n) - 1)pr + r(\psi^{(r)}(n) - \varphi^{(r)}(n) - \omega(n)) \\
&= r(\psi^{(r)}(n) - \varphi^{(r)}(n) + \omega(n)p - p - \omega(n)) \equiv X.
\end{aligned}$$

If $\omega(n) = 1$, then

$$X = r(\psi^{(r)}(n) - \varphi^{(r)}(n) - 1) > 0.$$

If $\omega(n) > 1$, then from $p \geq 2$

$$\begin{aligned}
X &= r(\psi^{(r)}(n) - \varphi^{(r)}(n) + \omega(n)p - p - \omega(n)) \\
&\geq r(p(\omega(n) - 1) - \omega(n)) \geq r(2\omega(n) - 2 - \omega(n)) = r(\omega(n) - 2) \geq 0.
\end{aligned}$$

Let $p \in \underline{\text{set}}(n)$. Then

$$\begin{aligned}
&\varphi^{(r)}(np) + \psi^{(r)}(np) - 2np - (\omega(np) - 1)r \\
&= \varphi^{(r)}(n)p + \psi^{(r)}(n)p - 2np - (\omega(n) - 1)r \\
&= (\varphi^{(r)}(n) + \psi^{(r)}(n) - 2n)p - (\omega(n) - 1)r \\
&\geq (\omega(n) - 1)pr - (\omega(n) - 1)r \\
&= (\omega(n) - 1)(p - 1)r \geq 0.
\end{aligned}$$

Therefore, the Theorem is proved. $\square$

3 Special cases

Below, we will discuss the following special case of functions $\varphi^{(r)}$ and $\psi^{(r)}$. Let

$$\varphi^*(n) = \varphi^{(2)}, \tag{7}$$
$$\psi^*(n) = \psi^{(2)}. \tag{8}$$

Obviously, for each even number n,

$$\begin{aligned}
&\varphi^*(n) = 0, \\
&\psi^*(n) \geq n + 2^{\omega(n)}.
\end{aligned}$$

The first 20 values of both functions are given in the following Table 1.

Table 1. Values of $\varphi^*(n)$ and $\psi^*(n)$ for $1 \le n \le 20$

n	$\varphi^*(n)$	$\psi^*(n)$	n	$\varphi^*(n)$	$\psi^*(n)$
1	1	1	11	9	13
2	0	4	12	0	40
3	1	5	13	11	15
4	0	8	14	0	36
5	3	7	15	3	35
6	0	20	16	0	32
7	5	9	17	15	19
8	0	16	18	0	60
9	3	15	19	17	21
10	0	28	20	0	56

From the above three theorems we obtain as particular cases for $r = 2$ the following inequalities:

$$(\psi^*(n))^2 \ge \psi^{(4-s)}(n)\psi^{(s)}(n),$$

(with equality for $s = 2$ and strict inequality for $s = 1$),

$$\varphi^*(n)\psi^*(n) \le n^2 - 4,$$
$$\varphi^*(n) + \psi^*(n) \ge 2(n + \omega(n) - 1).$$

From Theorems 2 and 3 we obtain as particular cases for $r = 1$ the following well-known inequalities:

$$\varphi(n)\psi(n) \le n^2 - 1,$$
$$\varphi(n) + \psi(n) \ge 2n.$$

Let

$$d(n) \;\; = \prod_{i=1}^{k}(\alpha_i + 1), \; d(1) = 1.$$

Theorem 4. *For each natural number $n \ge 2$*

$$\varphi^*(n) + d(n) \le n. \tag{9}$$

Proof. Let n be a prime number. Then from (7) we obtain

$$\varphi^*(n) + d(n) = n - 2 + 2 = n,$$

i.e., (9) is valid. When $n = p^a$ for a prime p and a natural number $a \ge 2$,

$$p^a - \varphi^*(p^a) - d(p^a) = p^a - p^{a-1}(p - 2) - a - 1 = 2p^{a-1} - a - 1 \ge 0.$$

When n is an even number, then (9) is obvious.

Let us assume that (9) is valid for the odd numbers less than or equal to some odd number n. Let $p \notin \underline{\text{set}}(n)$ and $p \ge 2$. Then

$$\begin{aligned} np - \varphi^*(np) - d(np) &= np - \varphi^*(n)(p-2) - 2d(n) \\ &\geq (\varphi^*(n) + d(n))p - \varphi^*(n)(p-2) - 2d(n) \\ &= d(n)p + 2\varphi^*(n) - 2d(n) > 0. \end{aligned}$$

Let $p \in \underline{\text{set}}(n)$. Then $n = mp^a$ for some natural number $a \geq 1$, odd number $m > 1$ such that $p \notin \underline{\text{set}}(m)$, and

$$\begin{aligned} np - \varphi^*(np) - d(np) &= mp^{a+1} - \varphi^*(m)p^a(p-2) - d(m)(a+2) \\ &\geq (\varphi^*(m) + d(m))p^{a+1} - \varphi^*(m)p^a(p-2) - d(m)(a+2) \\ &= d(m)p^{a+1} + 2\varphi^*(m)p^a - d(m)(a+2) \\ &> d(m)(p^{a+1} - a - 2) \\ &\geq 2^{a+1} - a - 2 > 0, \end{aligned}$$

with which (9) is proved. □

Theorem 5. *For each natural number n*

$$\psi^*(n) \geq n + d(n)\omega(n). \tag{10}$$

Proof. Let n be a prime number. Then from (8) we obtain

$$\psi^*(n) - n - d(n)\omega(n) = n + 2 - n - 2 = 0,$$

i.e., (10) is valid. If $n = p^a$ for a prime number p and a natural number $a \geq 2$, then

$$\psi^*(p^a) - p^a - d(p^a)\omega(p^a) = p^{a-1}(p+2) - p^a - a - 1 = 2p^{a-1} - a - 1 \geq 0.$$

Let us assume that (10) is valid for the natural numbers less than or equal to some n and let $p \notin \underline{\text{set}}(n)$. Then

$$\begin{aligned} \psi^*(np) - np - d(np)\omega(np) &= \psi^*(n)(p+2) - np - 2d(n)(\omega(n)+1) \\ &\geq (n + d(n)\omega(n))(p+2) - np - 2d(n)(\omega(n)+1) \\ &= np + d(n)\omega(n)p + 2n + 2d(n)\omega(n) - np - 2d(n)\omega(n) - 2d(n) \\ &= d(n)\omega(n)p + 2n - 2d(n) > 0. \end{aligned}$$

Let $p \in \underline{\text{set}}(n)$. Then $n = mp^a$ for some natural number $a \geq 1$, some natural number $m > 1$, such that $p \notin \underline{\text{set}}(m)$, and

$$\begin{aligned} \psi^*(np) - np - d(np)\omega(np) &= \psi^*(mp^{a+1}) - mp^{a+1} - d(mp^{a+1})\omega(mp^{a+1}) \\ &= \psi^*(m)p^a(p+2) - mp^{a+1} - d(m)(a+2)(\omega(m)+1) \\ &\geq (m + d(m)\omega(m))p^a(p+2) - mp^{a+1} - d(m)(a+2)(\omega(m)+1) \\ &= mp^a(p+2) + d(m)\omega(m)p^a(p+2) - mp^{a+1} \\ &\quad -d(m)\omega(m)(a+2) - d(m)(a+2) \\ &\geq 2mp^a - d(m)(a+2) \\ &\geq 2^{a+1}m - d(m)(a+2) > 0, \end{aligned}$$

with which (10) is proved. □

Obviously, for every two prime numbers p, q, such that $p > q$:

$$\frac{p-2}{p} > \frac{q-2}{q}$$

and hence for each natural number $a \geq 2$

$$\frac{\varphi^*(np^a)}{np^a} = \frac{\varphi^*(np)}{np} = \frac{\varphi^*(n)(p-2)}{np} > \frac{\varphi^*(n)(q-2)}{nq} = \frac{\varphi^*(nq)}{nq} = \frac{\varphi^*(nq^a)}{nq^a}.$$

We can calculate directly that for every two natural numbers $a \geq 1, b \geq 1$:

$$\varphi^*(3^a) = 3^{a-1} < \frac{3^a}{\sqrt{3}},$$

$$\varphi^*(5^a) = 3.5^{a-1} > \frac{5^a}{\sqrt{5}},$$

$$\varphi^*(3^a5^b) = 3^a5^{b-1} < \frac{3^a5^b}{\sqrt{15}},$$

$$\varphi^*(3^a7) = 3^{a-1}5 > \frac{3^a7}{\sqrt{21}},$$

Theorem 6. *For each odd number* $n \geq 17$ *and* $n \neq 3^a, 3^a5^b$ *for every two natural numbers* $a \geq 1$, $b \geq 1$*:*

$$\varphi^*(n) \geq \frac{n}{\sqrt{\underline{\mathrm{mult}}(n)}}. \tag{11}$$

Proof. First, we check that for the prime number $p \geq 7$ for each natural number a:

$$\varphi^*(3^ap) - \frac{3^ap}{\sqrt{3p}} = 3^{a-1}(p-2) - \frac{3^ap}{\sqrt{3p}} = \frac{3^{a-1}}{\sqrt{3p}}\left(\sqrt{3p}(p-2) - 3p\right) \geq 0,$$

because from $p \geq 7$ it follows that

$$\sqrt{3p}(p-2) - 3p \geq \sqrt{21}(p-2) - 3p \geq 4.5(p-2) - 3p = 1.5p - 9 > 0.$$

Second, we check that for the prime number $p \geq 7$ for every two natural numbers a and b:

$$\varphi^*(3^a5^bp) - \frac{3^a5^bp}{\sqrt{15p}} = 3^a5^{b-1}(p-2) - \frac{3^a5^bp}{\sqrt{15p}} = \frac{3^a5^{b-1}}{\sqrt{15p}}\left(\sqrt{15p}(p-2) - 5p\right) \geq 0,$$

because from $p \geq 7$ it follows that

$$\sqrt{15p}(p-2) - 5p \geq \sqrt{100}(p-2) - 5p = 10(p-2) - 5p = 5p - 20 > 0.$$

Let $n \geq 17$ be a prime number. Then from (11) we obtain

$$\varphi^*(n) - \frac{n}{\sqrt{n}} = n - 2 - \sqrt{n} > 0.$$

Let us assume that (11) is valid for the odd number $n \neq 3^a, 3^a5$, i.e., n contains at least one divisor different from 3 or 5, $n \geq 17$ and let $p \notin \underline{\mathrm{set}}(n)$. Then

$$\begin{aligned}
\varphi^*(np) - \frac{np}{\sqrt{\underline{\mathrm{mult}}(np)}} &= \varphi^*(n)(p-2) - \frac{np}{\sqrt{\underline{\mathrm{mult}}(n)p}} \\
&\geq \frac{n}{\sqrt{\underline{\mathrm{mult}}(n)}}(p-2) - \frac{np}{\sqrt{\underline{\mathrm{mult}}(n)p}} \\
&= \frac{n}{\sqrt{\underline{\mathrm{mult}}(n)}}\left(p - 2 - \sqrt{p}\right) > 0,
\end{aligned}$$

which is valid for each $p \geq 5$.

Let $p \in \underline{\text{set}}(n)$. Then

$$\begin{aligned}\varphi^*(np) - \frac{np}{\sqrt{\underline{\text{mult}}(np)}} &= \varphi^*(n)p - \frac{np}{\sqrt{\underline{\text{mult}}(n)}} \\ &\geq \frac{n}{\sqrt{\underline{\text{mult}}(n)}}p - \frac{np}{\sqrt{\underline{\text{mult}}(n)}} = 0.\end{aligned}$$

which proves the theorem. □

Theorem 7. *For each natural number $n \geq 7$:*

$$\psi^*(n) \begin{cases} \leq \frac{(n^2-4\omega(n))\sqrt{n}}{n} & \text{if } n \text{ is an odd number} \\ < \frac{(n^2-4\omega(n))\sqrt{8n}}{n} & \text{if } n \text{ is an even number} \end{cases}. \tag{12}$$

Proof. First, we can see that

$$\begin{aligned}
&\psi^*(2) = 4 > 0 = \tfrac{(4-4)\sqrt{2}}{2}, \text{ because } \omega(2) = 1,\\
&\psi^*(3) = 5 > \tfrac{5\sqrt{3}}{3}, \text{ because } \omega(3) = 1,\\
&\psi^*(4) = 8 > \tfrac{12\sqrt{4}}{4}, \text{ because } \omega(4) = 1,\\
&\psi^*(5) = 7 < \tfrac{21\sqrt{5}}{5}, \text{ because } \omega(5) = 1,\\
&\psi^*(6) = 20 > \tfrac{28\sqrt{6}}{6}, \text{ because } \omega(6) = 2,\\
&\psi^*(7) = 9 < \tfrac{45\sqrt{7}}{7}, \text{ because } \omega(7) = 1,\\
&\psi^*(8) = 16 < \tfrac{60\sqrt{8}}{8}, \text{ because } \omega(8) = 1,\\
&\psi^*(9) = 15 < \tfrac{77}{3}, \text{ because } \omega(9) = 1,
\end{aligned}$$

Let $n \geq 7$ be a prime number. Then

$$\frac{(n^2 - 4\omega(n))\sqrt{n}}{n} - \psi^*(n) = \frac{(n^2 - 4)\sqrt{n}}{n} - (n + 2) = \frac{n + 2}{n}((n - 2)\sqrt{n} - 2) > 0.$$

Let us assume that (12) is valid for some odd number $n \geq 7$ and let $2 \neq p \notin \underline{\text{set}}n$. Then $p \geq 3$, $\sqrt{p} > 1.7$, $0.7p\sqrt{p} > 2$ and

$$\begin{aligned}
&\tfrac{(n^2p^2-4\omega(np))\sqrt{np}}{np} - \psi^*(np)\\
&= \tfrac{(n^2p^2-4(\omega(n)+1))\sqrt{np}}{np} - \psi^*(n)(p + 2)\\
&\geq \tfrac{(n^2p^2-4(\omega(n)+1))\sqrt{np}}{np} - \tfrac{(n^2-4\omega(n))\sqrt{n}}{n}(p + 2)\\
&= \tfrac{\sqrt{n}}{np}(n^2p^2\sqrt{p} - 4\omega(n)\sqrt{p} - 4\sqrt{p} - n^2p^2 + 4\omega(n)p^2 - 2n^2p + 8\omega(n)p)\\
&> \tfrac{\sqrt{n}}{np}(0.7n^2p^2\sqrt{p} + 4\omega(n)(p^2 - \sqrt{p}) - 4\sqrt{p} - 2n^2p + 8\omega(n)p)\\
&> \tfrac{\sqrt{n}}{np}(n^2p(0.7p\sqrt{p} - 2) + 4(2\omega(n)p - \sqrt{p}) > 0.
\end{aligned}$$

When $p \in \underline{\text{set}}(n)$, then

$$\begin{aligned}
&\tfrac{(n^2p^2-4\omega(np))\sqrt{np}}{np} - \psi^*(np)\\
&= \tfrac{(n^2p^2-4\omega(n))\sqrt{np}}{np} - \psi^*(n)p\\
&\geq \tfrac{(n^2p^2-4\omega(n))\sqrt{np}}{np} - \tfrac{(n^2-4\omega(n))\sqrt{n}}{n}p\\
&= \tfrac{\sqrt{n}}{np}(n^2p^2\sqrt{p} - 4\omega(n)\sqrt{p} - n^2p^2 + 4\omega(n)p^2) > 0.
\end{aligned}$$

Let $n = 2^a$, where from the condition $n \geq 7$ it follows that $a \geq 3$. Then

$$\psi^*(n) = 2^{a+1} < 8\frac{2^{2a}-4}{2^a} \leq \frac{(2^{2a}-4)\sqrt{2^{a+3}}}{2^a} = \frac{(n^2-4\omega(n))\sqrt{8n}}{n}.$$

Let $n = 2^a3^b$, where $a, b \geq 1$ and from the condition $n \geq 7$ it follows that $a + b \geq 3$. Then

$$\begin{aligned}
&\tfrac{(n^2-4\omega(n))\sqrt{8n}}{n} - \psi^*(n)\\
&= \tfrac{(2^{2a}3^{2b}-4\omega(2^a3^b))\sqrt{2^{a+3}3^b}}{2^a3^b} - \psi^*(2^a3^b)\\
&= \tfrac{(2^{2a}3^{2b}-8)\sqrt{2^{a+3}3^b}}{2^a3^b} - 2^{a+1}3^{b-1}5\\
&= \tfrac{1}{2^a3^b}(2^{2a}3^{2b}\sqrt{2^{a+3}3^b} - 8\sqrt{2^{a+3}3^b} - 2^{2a+1}3^{2b-1}5) > 0,
\end{aligned}$$

because $\sqrt{2^{a+3}3^b} > \sqrt{2^6} = 2^3$ and

$$\begin{aligned}
&2^{2a}3^{2b}\sqrt{2^{a+3}3^b} - 8\sqrt{2^{a+3}3^b} - 2^{2a+1}3^{2b-1}5\\
&> 2^{2a+3}3^{2b} - 8\sqrt{2^{a+3}3^b} - 2^{2a+1}3^{2b-1}5\\
&= 2^{2a+1}3^{2b-1}7 - 8\sqrt{2^{a+3}3^b} > 0.
\end{aligned}$$

Finally, when $n = 2^am$, where $m \geq 5$ is an odd number and $a \geq 1$ is a natural number, then from above it follows that $\sqrt{2^{a+3}} \geq 4, 2^{2a}m^2 > 4\omega(m)$, and

$$\psi^*(n) = 2^{a+1}\psi^*(m) \leq \frac{2^{a+1}(m^2-4\omega(m))\sqrt{m}}{m} < \frac{(2^{2a}m^2-4\omega(m))\sqrt{2^{a+3}m}}{2^am},$$

because from $a \geq 1$ it follows that

$$\begin{aligned}
&\tfrac{(2^{2a}m^2-4\omega(m))\sqrt{2^{a+3}m}}{2^am} - \tfrac{2^{a+1}(m^2-4\omega(m))\sqrt{m}}{m}\\
&= \tfrac{\sqrt{m}}{2^am}\Big((2^{2a}m^2 - 4\omega(m))\sqrt{2^{a+3}} - 2^{2a+1}(m^2 - 4\omega(m))\Big)\\
&\geq \tfrac{\sqrt{m}}{2^am}\left(2^{2a+2}m^2 - 16\omega(m) - 2^{2a+1}m^2 + 2^{2a+4}\omega(m)\right)\\
&> \tfrac{\sqrt{m}}{2^am}\left(2^{2a+1}m^2 + 48\omega(m)\right) > 0
\end{aligned}$$

which proves the theorem. □

Theorem 8. *For each natural number $n \geq 2$ and $r \geq 2$:*

$$nr^{d(n)} \geq 2\psi^{(r)}(n). \tag{13}$$

Proof. For $n = 2$ and $r \geq 2$ from (13) we obtain:

$$2r^{d(2)} - 2\psi^{(r)}(2) = 2r^2 - 2(2 + r) = 2r^2 - 2r - 4 \geq 0$$

and the equality is when $r = 2$.

When $n \geq 3$ is a prime number and $r \geq 2$, we obtain from (13):

$$nr^2 - 2(n + r) = n(r^2 - 2) - 2r \geq 3r^2 - 2r - 6 > 0.$$

If $n = p^a$ for a prime p and a natural $a \geq 2$, then

$$p^a r^{a+1} - 2\psi^{(r)}(p^a) = p^a r^{a+1} - 2p^{a-1}(p + r) = p^{a-1}(pr^{a+1} - 2p - 2r) \geq 0.$$

Let $p \notin \underline{\mathrm{set}}(n)$. Then from (13) we obtain:

$$\begin{aligned} npr^{d(np)} - 2\psi^{(r)}(np) &= npr^{2d(n)} - 2\psi^{(r)}(n)(p + r) \\ &\geq npr^{d(n)}r^2 - nr^{d(n)}(p + r) \\ &= nr^{d(n)}(pr^2 - p - r) \\ &= nr^{d(n)}((pr - 1)r - p) \\ &\geq nr^{d(n)}(3p - 2) > 0. \end{aligned}$$

Let $p \in \underline{\mathrm{set}}(n)$. Then $n = mp^a$ for some natural number $a \geq 1$, $m > 1, p \notin \underline{\mathrm{set}}(m)$, and

$$\begin{aligned} npr^{d(np)} - 2\psi^{(r)}(np) &= mp^{a+1}r^{d(m)(a+2)} - 2\psi^{(r)}(m)p^a(p + r) \\ &\geq mp^{a+1}r^{d(m)(a+2)} - mr^{d(m)}p^a(p + r) \\ &\geq mp^{a+1}r^{d(m)}r^2 - mr^{d(m)}p^a(p + r) \\ &= mp^a r^{d(m)}(pr^2 - p - r) > 0. \end{aligned}$$

i.e., the theorem is valid. □

In particular, $nr^{d(n)} \geq 2\psi^*(n)$.

Obviously, for every natural number n and a prime number p,

$$\frac{\psi^{(r)}(np)}{np} > \frac{\psi^{(r)}(n)}{n}.$$

Theorem 9. *For every two natural numbers $n \geq 2$ and $r \geq 1$:*

$$\psi^{(r)}(n) \geq n + \frac{nr^{\omega(n)-1}}{\underline{\mathrm{mult}}(n)} \tag{14}$$

Proof. Let n be a prime number. Then $\omega(n) = 1$, $\underline{\mathrm{mult}}(n) = n$ and

$$\psi^{(r)}(n) = n + r = n + \frac{nr^{\omega(n)-1}}{\underline{\mathrm{mult}}(n)}.$$

Let us assume that (14) is valid and $p \notin \underline{\text{set}}(n)$. Then

$$\begin{aligned}
\psi^{(r)}(np) - np - \tfrac{npr^{\omega(np)-1}}{\underline{\text{mult}}(np)} &= \psi^{(r)}(n)(p+r) - np - \tfrac{npr^{\omega(n)}}{\underline{\text{mult}}(n)p} \\
&= \psi^{(r)}(n)(p+r) - np - \tfrac{nr^{\omega(n)}}{\underline{\text{mult}}(n)} \\
&\geq (n + \tfrac{nr^{\omega(n)-1}}{\underline{\text{mult}}(n)})(p+r) - np - \tfrac{nr^{\omega(n)}}{\underline{\text{mult}}(n)} \\
&= np + nr + \tfrac{nr^{\omega(n)-1}}{\underline{\text{mult}}(n)}p + \tfrac{nr^{\omega(n)-1}}{\underline{\text{mult}}(n)}r - np - \tfrac{nr^{\omega(n)}}{\underline{\text{mult}}(n)} \\
&= nr + \tfrac{nr^{\omega(n)-1}}{\underline{\text{mult}}(n)}p > 0.
\end{aligned}$$

When $p \in \underline{\text{set}}(n)$ we obtain

$$\begin{aligned}
\psi^{(r)}(np) - np - \tfrac{npr^{\omega(np)-1}}{\underline{\text{mult}}(np)} &= \psi^{(r)}(n)p - np - \tfrac{npr^{\omega(n)-1}}{\underline{\text{mult}}(n)} \\
&\geq (n + \tfrac{nr^{\omega(n)-1}}{\underline{\text{mult}}(n)})p - np - \tfrac{npr^{\omega(n)-1}}{\underline{\text{mult}}(n)} = 0,
\end{aligned}$$

i.e., the theorem is valid. □

Finally, let the well-known function σ be defined (see, e.g., [6–8, 10]) by:

$$\sigma(n) = \prod_{i=1}^{k} \frac{p^{\alpha_i+1} - 1}{p_i - 1},\ \sigma(1) = 1.$$

Theorem 10. *For each natural number $n \geq 2$:*

$$\psi^*(n) > \sigma(n). \tag{15}$$

Proof. Let $n \geq 2$ be an arbitrary natural number. From the definitions of functions ψ^* and σ we see that it is sufficient to prove that for each prime number $p \geq 2$ and for each natural number $a \geq 1$:

$$p^{a-1}(p+2) - \frac{p^{a+1}-1}{p-1} > 0.$$

Let

$$\begin{aligned}
p^{a-1}(p+2)(p-1) - (p^{a+1}-1) &= p^{a+1} + 2p^a - p^a - 2p^{a-1} - p^{a+1} + 1 \\
&= p^a - 2p^{a-1} + 1 > 0.
\end{aligned}$$

from where (15) follows directly. □

Corollary. For every two natural numbers $n, r \geq 2$, $\psi^{(r)}(n) > \sigma(n)$.

4 Conclusion

The author hopes that the new arithmetic function, in combination with other existing ones, will also reveal new interesting properties.

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