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A bound for repeated binomial coefficients having large prime power factors

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Abstract: If $t > 1$ is an integer, we let $N(t)$ denote the number of ways of expressing t as a binomial coefficient. We illustrate a method of bounding $N(t)$ by utilizing properties of Pascal's Triangle under modular arithmetic. We use this method to show that $N(t)$ is bounded for t with proportionately "large" prime power factors, and we describe how it may be possible to derive stronger bounds.
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1 Introduction

If $t > 1$ is an integer, we let $N(t)$ denote the number of ways of expressing t as a binomial coefficient. That is, $N(t)$ is the number of solutions to $t = \binom{n}{m}$ with $n, m \in \mathbb{Z}_{\geq 0}$. In 1971, David

B. Singmaster proved that $N(t) = O(\log(t))$ [8], and further conjectured that $N(t) = O(1)$, which is known as "Singmaster's Conjecture".

In 1974, Abbott *et al.* showed that $N(t) = O(\log(t)/\log\log(t))$, which can be further improved to $N(t) = O(\log(t)^{2/3+\epsilon})$ for all $\epsilon > 0$ assuming Cramér's conjecture on gaps between primes [1]. Additionally, an explicit upper bound on $N(t)$ in terms of the number of prime factors of t is proven. The current best known unconditional upper bound is $N(t) = O\left(\frac{\log(t)\log\log\log(t)}{(\log\log(t))^3}\right)$ proven by Kane in 2007 [4]. More recently, in 2022, Matomäki *et al.* have shown that Singmaster's Conjecture holds true for a particular range of values of $t = \binom{n}{m}$; specifically, for any fixed $\epsilon > 0$, the conjecture holds when $\exp\left(\log^{2/3+\epsilon} n\right) \leq m \leq n - \exp\left(\log^{2/3+\epsilon} n\right)$ [6]. It is also worth noting that it has been shown that $N(t) = 6$ infinitely often [9], and the only known t for which $N(t) > 6$ is $t = 3003$ with $N(3003) = 8$. Other more recent work has looked at various natural generalizations of Singmaster's conjecture [2, 3].

Our main new contributions to this study are Theorem 5 and its more concise corollary, Corollary 5.1, which we state below. These results informally show that, in order for a number t to appear many times in Pascal's Triangle, its prime power factors must all be proportionally small.

Corollary 5.1. *For a fixed positive integer r, if $t > 1$ is sufficiently large, p^q is a prime power factor of t, and $t^{1/r} \leq p^q$, then $N(t) \leq 2r$.*

We show this by proving bounds on two arithmetic functions which independently track both the size of entries in Pascal's Triangle and their residue modulo t. While we believe our results are original and interesting on their own, their main significance is that they demonstrate the utility of our techniques as a novel approach to the study of Singmaster's conjecture. We elaborate on this and briefly discuss possible extensions of our work in Section 5.

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2 Pascal's Square

We conceptualize binomial coefficients as entries in a slightly altered version of Pascal's Triangle, which we refer to as *Pascal's Square* (see Figure 1). We here emphasize that Pascal's Square is simply a rotated version of Pascal's Triangle, which we use purely for notational and conceptual convenience. All results derived about Pascal's Square hold for Pascal's Triangle after a simple change of coordinates. Explicitly, the entry at row $r \geq 0$, column $c \geq 0$, of Pascal's Square, denoted $S(r, c)$, is given by

$$S(r, c) = \binom{r+c}{c}.$$

Furthermore, it can easily be seen that

$$\binom{r+c}{c} = \frac{(r+c)!}{c!r!} = \binom{c+r}{r} = S(c, r)$$

implying $S(r, c) = S(c, r)$, so Pascal's Square is symmetric about its main diagonal.

$$\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
1 & 2 & 3 & 4 & 5 & 6 & \cdots \\
1 & 3 & 6 & 10 & 15 & 21 & \cdots \\
1 & 4 & 10 & 20 & 35 & 56 & \cdots \\
1 & 5 & 15 & 35 & 70 & 126 & \cdots \\
1 & 6 & 21 & 56 & 126 & 252 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Figure 1. Pascal's Square

Alternatively, Pascal's Square can be defined recursively. For $r, c > 0$ we have

$$S(r, c) = S(r-1, c) + S(r, c-1), \tag{1}$$

with

$$S(0, c) = S(r, 0) = 1. \tag{2}$$

Singmaster's function $N(t)$ can equivalently be defined as the number of occurrences of t in Pascal's Square. For example, from Figure 1 we can see that $N(6) = 3$.

3 Main definitions and preliminary results

We first define three arithmetic functions which will be used to study $N(t)$. In what follows, unless otherwise stated, we will use the convention that r and t are both nonnegative integers.

Definition 1. *Let $\beta_r(t)$ denote the column of the largest entry on row r of Pascal's Square which is less than or equal to t. Explicitly, $\beta_r(t)$ is the largest integer c such that $S(r, c) = \binom{r+c}{r} \leq t$.*

Definition 2. *Let $\alpha_r(t)$ denote the column of the first entry on row r of Pascal's Square which is congruent to 0 modulo t. Explicitly, $\alpha_r(t)$ is the smallest positive integer c such that $S(r, c) = \binom{r+c}{r} \equiv 0 \pmod{t}$.*

Definition 3. *Given a nonnegative integer n and a prime p,*

$$\nu_p(n) = \max\{v \in \mathbb{Z} : p^v \mid n\},$$

i.e., ν_p is the p-adic valuation.

We now establish two lemmas. Lemma 1 shows that $\alpha_r(t)$ is well defined. Lemma 2 will be used more extensively and establishes, as an immediate corollary, that $\beta_r(t)$ is well defined.

Lemma 1. *For each $t, r \geq 1$, we have $\alpha_r(t) \leq (t-1)r$.*

Proof. Observe that if $c = (t-1)r$, then we have

$$S(r,c) = \binom{c+r}{r} = \binom{tr}{r} = \frac{(rt)!}{r!(rt-r)!}$$
$$= \frac{(rt)(rt-1)!}{r(r-1)!(rt-r)!} = t\frac{(rt-1)!}{(r-1)!(rt-r)!} = t\binom{rt-1}{r-1}.$$

Hence, given that $rt \geq 1$, we see that $S(r,c) \equiv 0 \pmod{t}$, and the result follows. □

Lemma 2. *If $r \geq 1$ and $0 \leq c_0 < c_1$, then $S(r,c_0) < S(r,c_1)$. Likewise, if $c \geq 1$ and $0 \leq r_0 < r_1$, then $S(r_0,c) < S(r_1,c)$.*

Proof. Since $r > 0$, we have from (1) that $S(r,c_0+1) = S(r,c_0) + S(r-1,c_0+1)$ and since $S(r-1,c_0+1) > 0$, $S(r,c_0+1) > S(r,c_0)$. It follows by induction that if $c_1 > c_0$, then $S(r,c_1) > S(r,c_0)$. Likewise, since $S(r,c) = S(c,r)$ it follows that if $c \geq 1$ and $0 \leq r_0 < r_1$, then $S(r_0,c) < S(r_1,c)$. □

We next have our first significant result, Theorem 1, which implies that, in order to show an integer t cannot occur on row r of Pascal's Square, it suffices to prove $\beta_r(t) \neq \alpha_r(t)$, or equivalently, $\beta_r(t) < \alpha_r(t)$. Employing this strategy, we will prove general bounds on $\beta_r(t)$ and $\alpha_r(t)$, then use these bounds to establish our main result, Theorem 5.

Theorem 1. *If $r \geq 1$ and $t > 1$, then $\beta_r(t) \leq \alpha_r(t)$ and $\beta_r(t) = \alpha_r(t) := c$ if and only if $t = S(r,c) = \binom{r+c}{r}$.*

Proof. Let c be an integer such that $1 \leq c < \beta_r(t)$. It must be the case that $S(r,c) < S(r,\beta_r(t))$ by Lemma 2. Additionally, $S(r,\beta_r(t)) \leq t$ by definition, so $S(r,c) < t$. We know that $S(r,c) > 0$, so $0 < S(r,c) < t$ which implies that $S(r,c) \not\equiv 0 \pmod{t}$ and hence $\alpha_r(t) \neq c$. It thus follows that $\beta_r(t) \leq \alpha_r(t)$.

Now, suppose that $\alpha_r(t) = \beta_r(t) = c$, then we have that $0 < S(r,c) \leq t$ and $S(r,c) \equiv 0 \pmod{t}$ which implies that $S(r,c) = t$.

Conversely, suppose that $\beta_r(t) \neq \alpha_r(t)$, then it follows that $\beta_r(t) < \alpha_r(t)$. Now, let c be any integer for which $S(r,c) \equiv 0 \pmod{t}$. Clearly, if t occurs anywhere on row r of Pascal's Square then it must be equal to $S(r,c)$ for some such c, as $t \equiv 0 \pmod{t}$. Since, by definition, $\alpha_r(t)$ is the smallest value of a for which $S(r,a) \equiv 0 \pmod{t}$, it must be the case that $c \geq \alpha_r(t) > \beta_r(t)$. As $c > \beta_r(t)$, Lemma 2 implies that $S(r,c) > S(r,\beta_r(t))$, so $S(r,c) > t$ by the definition of $\beta_r(t)$, and hence $S(r,c) \neq t$. Therefore, if $\beta_r(t) \neq \alpha_r(t)$, then t does not occur anywhere on row r of Pascal's Square. □

For the following results, recall "big theta" notation: If $f, g : X \to \mathbb{R}$ are functions with $X \subseteq \mathbb{R}$, then we say $f(x) = \Theta(g(x))$ if there exist constants $k_1, k_2 > 0$ and $x_0 \in \mathbb{R}$ such that for all $x > x_0$ we have $k_1 g(x) \leq f(x) \leq k_2 g(x)$. With this definition, we first prove the elementary Lemma 3.

Lemma 3. *If $f : [0,\infty) \to \mathbb{R}^+$ is an increasing function and $f(x) = \Theta(x^d)$ for some fixed positive d, then $f^{-1}(x) = \Theta(x^{1/d})$.*

Proof. By definition, there exist constants k_1, k_2 such that $k_1 x^d \leq f(x) \leq k_2 x^d$ for sufficiently large x. Since f is increasing and positive, so is its inverse and we can conclude $f^{-1}(k_1 x^d) \leq x \leq f^{-1}(k_2 x^d)$. Making the substitution $y = k_1 x^d$ or $x = (1/k_1)^{1/d} y^{1/d}$, the lower bound implies $f^{-1}(y) \leq (1/k_1)^{1/d} y^{1/d}$. Similarly, with the substitution $z = k_2 x^d$ the upper bound implies $f^{-1}(z) \geq (1/k_2)^{1/d} z^{1/d}$. Thus, for all sufficiently large x, $(1/k_2)^{1/d} x^{1/d} \leq f^{-1}(x) \leq (1/k_1)^{1/d} x^{1/d}$ or $f^{-1}(x) = \Theta(x^{1/d})$. □

We now establish our bound on $\beta_r(t)$.

Theorem 2. *If $r \geq 1$ and $t > 1$, then $\beta_r(t) = \Theta(t^{1/r})$.*

Proof. For any fixed $r \geq 1$, let

$$f(c) = S(r, c) = \binom{r+c}{c} = \frac{(r+c)!}{r!c!} = \frac{1}{r!} \prod_{i=1}^{r} (c+i).$$

Since f is a polynomial, its domain can be extended to $\mathbb{R}^+$, on which it is continuous, and Lemma 2 implies it is increasing and unbounded. Furthermore, if $0 \leq a < b$, then $\prod_{i=1}^{r}(a+i) < \prod_{i=1}^{r}(b+i)$ so $f(a) < f(b)$ and f is strictly increasing on $[0, \infty)$.

Since f is continuous and unbounded above on $[0, \infty)$, $f(0) = 1$, and $t > 1$, there must exist $z \in [0, \infty)$ such that $f(z) = t$. Furthermore, since f is strictly increasing, it is invertible, its inverse is strictly increasing, and $z = f^{-1}(t)$.

Now, $\beta_r(t)$ is defined as the largest $c \in \mathbb{Z}$ such that $f(c) \leq t$, which implies that $\beta_r(t) = \lfloor z \rfloor = \lfloor f^{-1}(t) \rfloor$. Since f is a polynomial of degree r, we have that $f(x) = \Theta(x^r)$ so, by Lemma 3, $f^{-1}(x) = \Theta(x^{1/r})$. Finally, as $\beta_r(t) = \lfloor f^{-1}(t) \rfloor$ and we know $x - 1 \leq \lfloor x \rfloor \leq x$ for all x, so since f^{-1} is strictly increasing, there must exist constants k_1, k_2 such that $k_1 f^{-1}(t) \leq \beta_r(t) \leq k_2 f^{-1}(t)$, hence $\beta_r(t) = \Theta(t^{1/r})$. □

To establish a lower bound for $\alpha_r(t)$, we will use Kummer's Theorem [5] which provides a method of computing the p-adic valuation of binomial coefficients. This result, along with many others, was compiled by Meštrović in [7].

Kummer's Theorem. *Given integers $0 \leq m \leq n$ and a prime number p, $\nu_p(\binom{n+m}{n})$ is equal to the number of carries when n is added to m in base p.*

To clarify, a 'carry' refers to a digit which is transferred from one column to another under ordinary base-p addition. For example, if n and m are expressed in base p as $n = n_0 + n_1 p + \cdots + n_s p^s$ and $m = m_0 + m_1 p + \cdots + m_q p^q$, then, if $n_i + m_i \geq p$, an extra p will 'carry' from the i-th column to the $(i+1)$-st column when n and m are added in base p.

Theorem 3. *Let p be a prime and $q > 0$. Then $\alpha_r(p^q) \geq p^q - r$.*

Proof. Given that $\alpha_r(p^q) \geq 0$, if $r \geq p^q$, then clearly we have $\alpha_r(p^q) \geq p^q - r$.

Now suppose $r < p^q$. Let c be a positive integer such that $S(r, c) \equiv 0 \pmod{p^q}$, then $\nu_p(S(r, c)) = \nu_p(\binom{r+c}{r}) \geq q$. By Kummer's Theorem this is equivalent to the statement that there are at least q carries when c is added to r in base p. Since $r < p^q$, the base-p expansion of r has

at most q digits and $r = r_0 + r_1 p + r_2 p^2 + \cdots + r_{q-1} p^{q-1}$ with $0 \leq r_i < p$ for each i. In order for there to be q carries when c is added to r in base p, either c must have more than q digits in base p, which would imply $c \geq p^q$, or there would need to be a carry in every digit place when r is added to c. The latter implies that, if $c = c_0 + c_1 p + c_2 p^2 + \cdots + c_{q-1} p^{q-1}$, then the smallest possible values the digits of c could take on must satisfy $c_0 + r_0 \geq p$ and $c_i + r_i \geq p - 1$ for each $1 \leq i \leq q - 1$. Thus

$$\begin{aligned} c &\geq (p - r_0) + (p - r_1 - 1)p + (p - r_2 - 1)p^2 + \cdots + (p - r_{q-1} - 1)p^{q-1} \\ &= (1 + (p-1) + (p-1)p \cdots + (p-1)p^{q-1}) - (r_0 + r_1 p + \cdots + r_{q-1} p^{q-1}) \\ &= p^q - r. \end{aligned}$$

Since this is true of any c for which $S(r, c) \equiv 0 \pmod{p^q}$, it hence follows that $\alpha_r(p^q) \geq p^q - r$ as desired. □

In order to apply our results on $\beta_r(t)$ and $\alpha_r(t)$ to establish a bound on $N(t)$, we will take advantage of the fact that Pascal's Square is symmetric about its main diagonal.

Definition 4. *Let $\rho(t)$ be the largest integer r such that $S(r, r) = \binom{2r}{r} \leq t$.*

In other words, $\rho(t)$ is the last row for which the value of the main diagonal is less than or equal to t. Note that, by Lemma 2, Pascal's Square must increase along the diagonal; this means that $\rho(t)$ is well defined for all $t > 1$.

Lemma 4. *If $S(r, c) = t$ and $r > \rho(t)$, then $c \leq \rho(t)$.*

Proof. By definition, $S(\rho(t)+1, \rho(t)+1) > t$. If $c > \rho(t)$, then $c \geq \rho(t)+1$ and, by assumption, $r \geq \rho(t) + 1$, implying by Lemma 2 that $S(r, c) \geq S(\rho(t) + 1, \rho(t) + 1) > t$, which contradicts the assumption that $S(r, c) = t$. Thus, it follows that $c \leq \rho(t)$. □

Informally, Lemma 4 tells us that, while looking for occurrences of t in Pascal's Square, one only needs to consider rows 1 through $\rho(t)$, and all other instances of t will occur symmetrically from reflections across the main diagonal. That is, t cannot occur on a row below $\rho(t)$ and then reflect across the diagonal to another row below $\rho(t)$.

Lastly, the following three technical results establish restrictions on the growth rate of $\alpha_r(t)$ and $\beta_r(t)$, for use in the proof of Theorem 5.

Lemma 5. *If $1 \leq r \leq c$, then $S(r, c+1) \leq S(r+1, c)$.*

Proof. As $c \geq r$ we clearly have $(c+1)!/c! = c+1 \geq r+1 = (r+1)!/r!$, and thus $r!(c+1)! \geq (r+1)!c!$. Hence it follows that

$$\frac{r+c+1}{r!(c+1)!} \leq \frac{r+c+1}{(r+1)!c!}.$$

Therefore, $S(r, c+1) \leq S(r+1, c)$. □

Theorem 4. *If $r \geq 1$, then $\beta_{r+1}(t) \leq \beta_r(t)$ with $\beta_{r+1}(t) < \beta_r(t)$ when $1 \leq r < \rho(t)$.*

Proof. Suppose for contradiction that $\beta_{r+1}(t) > \beta_r(t)$, which implies $\beta_{r+1}(t) \geq \beta_r(t) + 1$. By definition, we know that $S(r, \beta_r(t) + 1) > t$ and that $S(r + 1, \beta_{r+1}(t)) \leq t$. Lemma 2 implies $S(r, \beta_r(t) + 1) < S(r + 1, \beta_r(t) + 1)$. Furthermore, if $\beta_{r+1}(t) \geq \beta_r(t) + 1$, then either $\beta_{r+1}(t) = \beta_r(t) + 1$, implying $S(r + 1, \beta_{r+1}(t)) = S(r + 1, \beta_r(t) + 1)$, or $\beta_{r+1}(t) > \beta_r(t) + 1$, implying, again by Lemma 2, that $S(r + 1, \beta_{r+1}(t)) > S(r + 1, \beta_r(t) + 1)$, hence, $S(r + 1, \beta_{r+1}(t)) \geq S(r+1, \beta_r(t)+1)$. Thus, we see that $t < S(r, \beta_r(t)+1) < S(r+1, \beta_r(t)_1) \leq S(r+t, \beta_{r+1}(t))$, which is a contradiction given that we know $S(r + 1, \beta_{r+1}(t)) \leq t$.

Now, suppose that $r < \rho(t)$. By the above argument, we already know that $\beta_{r+1}(t) \leq \beta_r(t)$, so we only need to rule out the possibility that $\beta_{r+1}(t) = \beta_r(t)$. By definition, we have $S(\rho(t), \rho(t)) \leq t$, and we know, by Lemma 2, that the entries along column $\rho(t)$ are increasing, so $S(r, \rho(t)) \leq S(\rho(t), \rho(t)) \leq t$ and hence $\beta_r(t) \geq \rho(t) > r$. Similarly, $S(r, \beta_r(t) + 1) > t$ and, given that $r < \beta_r(t)$, Lemma 5 implies that $S(r + 1, \beta_r(t)) \geq S(r, \beta_r(t) + 1) > t \geq S(r + 1, \beta_{r+1}(t))$ and hence $\beta_{r+1}(t) \neq \beta_r(t)$. □

Lemma 6. *If $t, n > 1$ and $r \geq 1$ with $n \mid t$, then $\alpha_r(t) \geq \alpha_r(n)$.*

Proof. If $0 \leq c < \alpha_r(n)$, then, by definition, $S(r, c) \not\equiv 0 \pmod{n}$. If $n \mid t$, then it follows that $S(r, c) \not\equiv 0 \pmod{t}$. Clearly, if $S(r, c) \equiv 0 \pmod{t}$, then $S(r, c) \equiv 0 \pmod{n}$ as $n \mid t$, and hence $\alpha_r(t) \geq \alpha_r(n)$. □

4 Main results

Finally, our main result, Theorem 5 and its corollary, Corollary 5.1, show that the number of occurrences as an entry in Pascal's Square of an integer t having large prime power factors is bounded.

Theorem 5. *Let $t > 1$ with p^q a prime power factor of t. If $\beta_r(t) < p^q - r$ for some $1 \leq r \leq \rho(t)$, then $N(t) \leq 2r - 2$.*

Proof. We will first show that for each $r \leq s \leq \rho(t)$, we have $\beta_s(t) < \alpha_s(t)$. We proceed by induction over the finite set $\{r, r+1, r+2, \ldots, \rho(t)\}$. We know, by assumption, that $r \leq \rho(t)$. For the base case, by Theorem 3, $\alpha_r(p^q) \geq p^q - r$, so it follows that $\alpha_r(t) \geq \alpha_r(p^q) \geq p^q - r > \beta_r(t)$ by Lemma 6, or $\alpha_r(t) > \beta_r(t)$.

Now, suppose that $r + k + 1 \leq \rho(t)$ and that $\beta_{r+k}(t) < p^q - r - k$ for some $k \geq 0$, which implies that $\beta_{r+k}(t) \leq p^q - r - k - 1$. Since $r + k + 1 \leq \rho(t)$, Theorem 4 implies that $\beta_{r+k+1}(t) < \beta_{r+k}(t) \leq p^q - r - k - 1$ or $\beta_{r+k+1}(t) < p^q - (r + k + 1)$. Again by Theorem 3, $\alpha_{r+k+1}(p^q) \geq p^q - (r+k+1)$, and so Lemma 6 implies $\alpha_{r+k+1}(t) \geq \alpha_{r+k+1}(p^q) \geq p^q - (r+k+1)$. Thus, $\beta_{r+k+1}(t) < p^q - (r + k + 1) \leq \alpha_{r+k+1}(t)$ or $\beta_{r+k+1}(t) < \alpha_{r+k+1}(t)$. It hence follows by induction that for any s satisfying $r \leq s \leq \rho(t)$ we have $\beta_s(t) < \alpha_s(t)$.

Thus, Theorem 1 implies that t cannot occur on rows r through $\rho(t)$ of Pascal's Square. That is, t can only occur as an entry of row s of Pascal's Square if $1 \leq s < r$, implying t can occur at most $r - 1$ times above the main diagonal. Every other occurrence of t will appear symmetrically below the main diagonal in Pascal's Square by Lemma 4 so $N(t) \leq 2(r - 1) = 2r - 2$. □

Corollary 5.1. *For a fixed positive integer r, if $t > 1$ is sufficiently large, p^q is a prime power factor of t, and $t^{1/r} \leq p^q$, then $N(t) \leq 2r$.*

Proof. By Theorem 2, $\beta_{r+1}(t) = \Theta(t^{1/(r+1)})$. Since $t^{1/r}$ grows asymptotically faster with t than $t^{1/(r+1)}$, there must exist t_0 such that if $t > t_0$, then $\beta_{r+1}(t) < t^{1/r} - r - 1 \leq p^q - (r+1)$. Likewise, since $\rho(t)$ is nondecreasing and unbounded, there must exist t_1 such that if $t > t_1$, $r + 1 \leq \rho(t)$. Thus, assuming $t > \max\{t_0, t_1\}$, we have $\beta_{r+1}(t) < p^q - (r + 1)$ and $1 \leq r + 1 \leq \rho(t)$, so, by Theorem 5, $N(t) \leq 2(r + 1) - 2 = 2r$. □

5 Conclusions and future work

Theorem 5 can be thought of as a consequence of the fairly simple lower bound on α for prime powers provided by Theorem 3. More generally, any lower bounds on $\alpha_r(t)$ stronger than the trivial lower bound of $\beta_r(t)$ given in Theorem 1 immediately yield restrictions on $N(t)$. The function $\beta_r(t)$ is generally well behaved, and much stronger upper bounds could be derived with a more careful analysis. The main difficulty with this approach comes with bounding $\alpha_r(t)$ below.

It is natural to consider $\alpha_r(t)$ as a function of r for fixed t. When t is a prime power, $\alpha_r(t)$ exhibits fairly regular, predictable behavior as r varies, but for all other values of t, $\alpha_r(t)$ exhibits complex behavior. Figure 2 illustrates a line plot of $\alpha_r(t)$ for $t = 25 = 5^2$ and $t = 20 = 2^25$ where the vertical axis is $\alpha_r(t)$ and the horizontal axis is r. One can clearly see the difference in the complexity of the behavior of $\alpha_r(t)$ in these two plots.

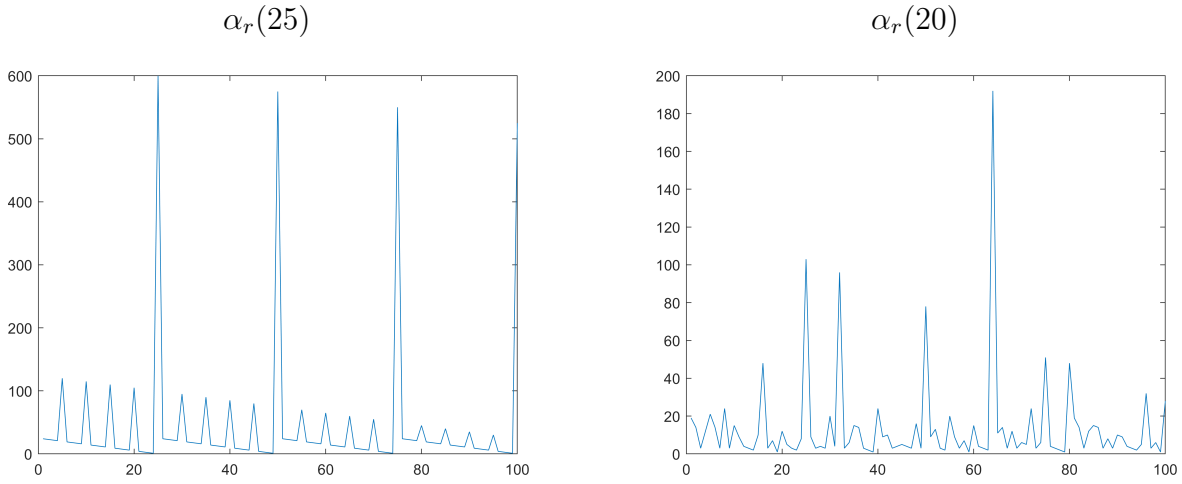

Figure 2. Line plot of $\alpha_r(t)$ as a function of r for $1 \leq r \leq 100$ with $t = 25$ and $t = 20$

One observation is that it is impossible to have $\alpha_r(t) = \beta_r(t)$ if $\alpha_r(t)$ increases with r. Theorem 6 shows that $\alpha_r(t) = \beta_r(t)$ can only occur when $\alpha_q(t)$ attains a minimum over the interval $1 \leq q \leq r$ at $q = r$. Thus, explicit lower bounds on $\alpha_r(t)$ are not completely necessary and it suffices to consider only when it is possible for $\alpha_r(t)$ to substantially decrease.

Theorem 6. *If $\beta_r(t) = \alpha_r(t)$ for some $t > 1$ and $r \geq 1$, then $\alpha_r(t) < \alpha_q(t)$ for all $1 \leq q < r$.*

Proof. Suppose $\alpha_r(t) \geq \alpha_q(t)$. By definition, $S(q, \alpha_q(t)) \equiv 0 \pmod{t}$ and we know that $S(q, \alpha_q(t)) \neq 0$, so $S(q, \alpha_q(t)) \geq t$. Thus, since $r > q$ and $\alpha_r(t) \geq \alpha_q(t)$, Lemma 2 implies that $S(r, \alpha_r(t)) > S(q, \alpha_q(t)) \geq t$, thus $S(r, \alpha_r(t)) \neq t$ and Theorem 1 implies $\alpha_r(t) \neq \beta_r(t)$. □

Due to the symmetry of Pascal's Square about the main diagonal, the behavior of $\alpha_r(t)$ only needs to be understood when $1 \leq r \leq \rho(t)$ in order to determine bounds on $N(t)$. Furthermore, if there exists a sequence of integers constituting a counterexample to Singmaster's conjecture that $N(t) = O(1)$, then the upper bounds on $N(t)$ given in [1] and by Theorem 5 imply that terms in any such sequence must have increasing numbers of prime factors and asymptotically small prime power factors; a proof of this conjecture, using these methods, therefore requires a better understanding of $\alpha_r(t)$ for t with large numbers of small prime power factors.

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