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$K[x_1, \ldots, x_n]$-algebras of multivariate polynomial heart-based rhotrix sets and their tensor products

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Abstract: This study constructs and investigates the m-dimensional multivariate polynomial heart-based rhotrix ("*mph*-rhotrix") algebras over an arbitrary commutative and unitary polynomial ring in n indeterminates, $K[x_1, \ldots, x_n]$, with entries in the same ring, where $m = 2k+1$, $k \in \mathbb{N} := \{0, 1, 2, \ldots\}$, $n \in \mathbb{Z}^+$, denoted by **CRAPol**(K, m, n, x), which is commutative, and **NRAPol**(K, m, n, x), which is noncommutative, and their generalized forms, **gCRAPol**(K, m, n_1, n_2, x) and **gNRAPol**(K, m, n_1, n_2, x), respectively, which consider rhotrix entries in $K[x_1, \ldots, x_{n_1}]$ and are over an arbitrary commutative unitary polynomial ring $K[x_1, \ldots, x_{n_2}]$. This study is motivated by existing developments in rhotrix theory and the lack of formal investigation into rhotrix algebras and their tensor products. This paper first establishes the set $\mathcal{R}_m(K, n, x)$ of m-dimensional *mph*-rhotrices, proving that it forms an associative commutative unitary $K[x_1, \ldots, x_n]$-algebra. Eventually, the tensor products of the algebras were defined and examined. The results of this study enrich the theoretical landscape of rhotrices by expanding it on the realm of algebras. Furthermore, this study is theoretical in scope and does not cover practical applications, although the authors recommend a critical exploration of potential extensions and applications of the results found herein.

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1 Introduction

The study of algebraic structures has long been one of the foundations of modern mathematics, providing the language and tools through which mathematical systems can be understood, classified, and generalized. Over time, matrices have inspired several mathematical extensions.

In 2023, K. T. Atanassov published in [14] his perspective on the development of the "tertion," objects that serve as an "intermediate place between complex numbers and quaternions", with its name being analogous to the word "quaternion". Atanassov claimed in the same paper [14] that he conceived the idea of the tertion around 1971–1972 and had it published in 1994 via nine preprints [4–12]. Consequently, he and A. G. Shannon published [13] together in 1998 which became the basis of several other publications, including those concerning "rhotrices": the objects under investigation in this study. Shannon in their joint authorship [13] suggested referring to tertions as "matrix-tertions" and to include "matrix-noitrets", a mirrored object of tertions (the word "noitret" being a reversed form of the word "tertion").

In light of the development of tertion theory, the objects "*A*-tertion" and "*V*-tertion" were defined in [14]. The *A*-tertion is described in [14] as an "intermediate" between the 2-dimensional vector $\langle a, b\rangle$ and the (2×2)-dimensional matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (see [16, 22] for "vector" and "matrix"), as in (1):

$$\left/ \begin{array}{ccc} & a & \\ b & & c \end{array} \right\backslash, \tag{1}$$

where a, b, c are elements of a fixed set. Meanwhile, the *V*-tertion possesses a structure mirroring the *A*-tertion, as in (2):

$$\left\backslash \begin{array}{ccc} b & & c \\ & a & \end{array} \right/. \tag{2}$$

In addition, these two objects possess dimensions indicated by subscripts. Let

$$A_2 = \left\{ \left/ \begin{array}{ccc} & a & \\ b & & c \end{array} \right\backslash \;\middle|\; a, b, c \in \mathbb{R} \right\}, \tag{3}$$

where $\mathbb{R}$ is the set of real numbers. The subscript "2" of A_2 indicates the tertion dimension corresponding to the number of *A*-tertion rows. Analogously,

$$V_2 = \left\{ \left\backslash \begin{array}{ccc} b & & c \\ & a & \end{array} \right/ \;\middle|\; a, b, c \in \mathbb{R} \right\}, \tag{4}$$

contains *V*-tertions of dimension 2. Operations such as $+$, multiplication, and the "ninth operation" $\circ_9$ were also defined and discussed in [14] over *A*-tertions and *V*-tertions, as well as their corresponding properties and identities.

Meanwhile, representations of complex numbers, standard quaternions, and non-standard quaternions via A-tertions and V-tertions were also discussed in [14], where the representations for the complex numbers were achieved via juxtaposition to the elements of the A-tertions and V-tertions, while a novel object arises in the case of quaternions. This object in [14] is (5):

$$\left\langle \begin{array}{ccc} & a & \\ b & & c \\ & d & \end{array} \right\rangle, \tag{5}$$

and was aptly named "quaternion," with representations by A-tertions and V-tertions provided in [14], as in (6), (7), (8):

$$\left/ \begin{array}{ccc} & a & \\ b & & x \end{array} \right\backslash \ *_1 \ \left\backslash \begin{array}{ccc} x & & c \\ & d & \end{array} \right/ = \left\langle \begin{array}{ccc} & a & \\ b & & c \\ & d & \end{array} \right\rangle, \tag{6}$$

$$\left/ \begin{array}{ccc} & a & \\ b & & c \end{array} \right\backslash \ *_2 \ \left\backslash \begin{array}{ccc} b & & c \\ & d & \end{array} \right/ = \left\langle \begin{array}{ccc} & a & \\ b & & c \\ & d & \end{array} \right\rangle, \tag{7}$$

$$\left/ \begin{array}{ccc} & a & \\ b & & c \end{array} \right\backslash \ *_3 \ \left\backslash \begin{array}{ccc} d & & e \\ & f & \end{array} \right/ = \left\langle \begin{array}{ccc} & a(d+e) & \\ be & & cd \\ & (b+c)f & \end{array} \right\rangle. \tag{8}$$

On the other hand, the case for so-called "non-standard" quaternions brought by changing underlying conditions was tackled in [14] through the object defined in (5). Ultimately, [14] lists the following open problems:

(i) What other operations can be defined over tertions from $A_2, V_2, A_3, V_3, \ldots$?

(ii) What other generating operations of complex numbers, matrices, and quaternions can be defined, and which properties they will have?

(iii) What other objects can be represented by tertions?

(iv) What other constructions of tertions can be introduced and which operations over them can be defined?

This study does not directly answer these problems. Instead, it partially answers problems similar to (i) and (iv) that concern another object, the "rhotrix" introduced in [1] by A. O. Ajibade in 2003. Such problems similar to (i) and (iv) are listed through (v) and (vi), respectively:

(v) What other operations can be defined over rhotrices?

(vi) What other constructions of rhotrices can be introduced and which operations over them can be defined?

In 2003, Ajibade introduced in [1] the mathematical object called "rhotrix" that are "in some ways, between (2×2) and (3×3) matrices," an extension to the ideas revolving around matrix-tertions and matrix-noitrets. Its name comes from the rhomboidal nature of the arrangement of its entries [1]. The 3-dimensional form of the rhotrix was defined in [1] through the set in (9):

$$\mathbf{R} = \left\{ \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \;\middle|\; a, b, c, d, e \in \mathbb{R} \right\}. \tag{9}$$

By letting the rhotrix $R = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle$, the element $h(R)$, located at the perpendicular intersection of the two diagonals of R, is referred to as "the heart of R" [1]. The addition operation, denoted by $+$, was defined in [1] via (10):

$$R + Q = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle + \left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & a+f & \\ b+g & h(R)+h(Q) & d+j \\ & e+k & \end{array} \right\rangle \tag{10}$$

This was shown in [1] to form a commutative group with the identity $0 = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 0 \\ & 0 & \end{array} \right\rangle$. This current study extends this addition with the operation "rhomplus," denoted by the new symbol $\diamondplus$, formalized in Sec. 3.1.2.

On the other hand, scalar multiplication was defined in [1] through (11):

$$\alpha R = \alpha \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & \alpha a & \\ \alpha b & \alpha h(R) & \alpha d \\ & \alpha e & \end{array} \right\rangle, \tag{11}$$

where $\alpha \in \mathbb{R}$. This concept was extended into the operations "rhomdot-*r*" $\diamonddot_r$ and "rhomdot-*l*" $\diamonddot_l$ in Sec. 3.1.3.

Meanwhile, the multiplication $\circ$ in [1], also termed the "heart-oriented rhotrix multiplication" in literature [17, 18], was defined as in (12):

$$R \circ Q = \left\langle \begin{array}{ccc} & ah(Q) + fh(R) & \\ bh(Q) + gh(R) & h(R)h(Q) & dh(Q) + jh(R) \\ & eh(Q) + kh(R) & \end{array} \right\rangle. \tag{12}$$

Additionally, Ajibade claimed, yet without showing proof, that the set $\mathbf{R}$ of all rhotrices in (9) is a commutative algebra. The identity $I = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$ and the inverse $P^{-1} = -\frac{1}{(h(P))^2} \left\langle \begin{array}{ccc} & a & \\ b & -h(P) & d \\ & e & \end{array} \right\rangle$ of $P = \left\langle \begin{array}{ccc} & a & \\ b & h(P) & d \\ & e & \end{array} \right\rangle$, where $h(P) \neq 0$, for the set $\mathbf{R}$, were also shown in [1].

The dimension of the rhotrix in [1] was defined to be odd and that a rhotrix R_n of dimension n will have $\frac{1}{2}(n^2+1)$ entries. Additionally, E. A. Ezugwu *et al.* in [17, 18] extended [1] by generalizing the heart-oriented rhotrix multiplication. The generalized form of the rhotrix in [17, 18], through R_n in (13), makes use of the horizontal single-index method of arbitrary rhotrix representation as defined by Chinedu Peter [27]:

$$R_n = \left\langle \begin{array}{ccccccccc} & & & & a_1 & & & & \\ & & & a_2 & a_3 & a_4 & & & \\ & & a_5 & a_6 & a_7 & a_8 & a_9 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_{\frac{n^2-2n+5}{4}} & \cdots & \cdots & \cdots & a_{\frac{1}{4}(n^2+3)} & \cdots & \cdots & \cdots & a_{\frac{n^2+2n+1}{4}} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & a_{\frac{1}{2}(n^2+1)-8} & a_{\frac{1}{2}(n^2+1)-7} & a_{\frac{1}{2}(n^2+1)-6} & a_{\frac{1}{2}(n^2+1)-5} & a_{\frac{1}{2}(n^2+1)-4} & & \\ & & & a_{\frac{1}{2}(n^2+1)-3} & a_{\frac{1}{2}(n^2+1)-2} & a_{\frac{1}{2}(n^2+1)-1} & & & \\ & & & & a_{\frac{1}{2}(n^2+1)} & & & & \end{array} \right\rangle. \tag{13}$$

It follows that every entry of R_n can be generalized as a_i, with $i = 1, 2, \dots, \frac{n^2+1}{2}$; by this, the heart entry has index $\frac{\frac{1}{2}(n^2+1)+1}{2} = \frac{n^2+3}{4}$ [18]. Given another rhotrix Q_n defined in (14):

$$Q_n = \left\langle \begin{array}{ccccccccc} & & & & b_1 & & & & \\ & & & b_2 & b_3 & b_4 & & & \\ & & b_5 & b_6 & b_7 & b_8 & b_9 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ b_{\frac{n^2-2n+5}{4}} & \cdots & \cdots & \cdots & b_{\frac{1}{4}(n^2+3)} & \cdots & \cdots & \cdots & b_{\frac{n^2+2n+1}{4}} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & b_{\frac{1}{2}(n^2+1)-8} & b_{\frac{1}{2}(n^2+1)-7} & b_{\frac{1}{2}(n^2+1)-6} & b_{\frac{1}{2}(n^2+1)-5} & b_{\frac{1}{2}(n^2+1)-4} & & \\ & & & b_{\frac{1}{2}(n^2+1)-3} & b_{\frac{1}{2}(n^2+1)-2} & b_{\frac{1}{2}(n^2+1)-1} & & & \\ & & & & b_{\frac{1}{2}(n^2+1)} & & & & \end{array} \right\rangle, \tag{14}$$

the product $R_n \circ Q_n$ is the rhotrix in (15):

$$\left\langle \begin{array}{ccccccccc} & & & & r_1 & & & & \\ & & & r_2 & r_3 & r_4 & & & \\ & & r_5 & r_6 & r_7 & r_8 & r_9 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ r_{\frac{n^2-2n+5}{4}} & \cdots & \cdots & \cdots & r_{\frac{1}{4}(n^2+3)} & \cdots & \cdots & \cdots & r_{\frac{n^2+2n+1}{4}} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & r_{\frac{1}{2}(n^2+1)-8} & r_{\frac{1}{2}(n^2+1)-7} & r_{\frac{1}{2}(n^2+1)-6} & r_{\frac{1}{2}(n^2+1)-5} & r_{\frac{1}{2}(n^2+1)-4} & & \\ & & & r_{\frac{1}{2}(n^2+1)-3} & r_{\frac{1}{2}(n^2+1)-2} & r_{\frac{1}{2}(n^2+1)-1} & & & \\ & & & & r_{\frac{1}{2}(n^2+1)} & & & & \end{array} \right\rangle, \tag{15}$$

where for $i = 1, 2, \dots, \frac{n^2+3}{4} - 1, \frac{n^2+3}{4} + 1, \dots, \frac{n^2+1}{2}$– all the entries except the heart – $r_i = a_i b_{\frac{n^2+3}{4}} + b_i a_{\frac{n^2+3}{4}}$ and $r_{\frac{n^2+3}{4}} = a_{\frac{n^2+3}{4}} b_{\frac{n^2+3}{4}}$ [18]. This has become the basis for the definition of the "commutative rhomtimes", one of the results of this study (see Section 3.1.4).

On the other hand, B. Sani in a 2004 paper [28] introduced an alternative method for multiplying rhotrices, which is similar to that for matrices, termed the "row–column method". This idea was developed further by examining high-dimensional rhotrices through [29]. Consider the n-dimensional rhotrix in [29], shown in (16):

$$
R_n = \left\langle \begin{array}{ccccccc}
 & & & a_{11} & & & \\
 & & a_{21} & c_{11} & a_{12} & & \\
 & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & a_{tt-2} & c_{t-1t-2} & a_{t-1t-1} & c_{t-2t-1} & a_{t-2t} & \\
 & & a_{tt-1} & c_{t-1t-1} & a_{t-1t} & & \\
 & & & a_{tt} & & &
\end{array} \right\rangle, \tag{16}
$$

where $t = \frac{n+1}{2}$. The main rows and columns of R_n, according to [29], are expressed in (17):

$$
\begin{array}{ccccccccc}
a_{t1} & a_{t-11} & \cdot & a_{11} & & & & & \\
 & a_{t2} & a_{t-12} & \cdot & a_{12} & & & & \\
 & & a_{t3} & a_{t-13} & \cdot & a_{13} & & & \\
 & & & \cdot & \cdot & \cdot & \cdot & & \\
 & & & & \cdot & \cdot & \cdot & \cdot & \\
 & & & & & a_{tt} & a_{t-1t} & \cdot & a_{1t}
\end{array}, \tag{17}
$$

$$
\begin{array}{ccccccc}
 & & & & a_{11} & a_{12} & \cdot & a_{1t} \\
 & & & a_{21} & a_{22} & \cdot & a_{2t} & \\
 & & a_{31} & a_{32} & \cdot & a_{3t} & & \\
 & \cdot & \cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & \cdot & & & & \\
a_{t1} & a_{t2} & \cdot & a_{tt} & & & &
\end{array}. \tag{18}
$$

The other rows and columns of R_n are formed by the entries c_{lk}, where $l, k = 1, 2, \ldots, t-1$ [29]. Representing the n-dimensional rhotrix in (16) by $R_n = \langle a_{ij}, c_{lk} \rangle$, where $i, j = 1, 2, \ldots, t$ [29], the row–column method of multiplication proceeds as in (19):

$$
R_n \circ Q_n = \langle a_{i_1 j_1}, c_{l_1 k_1} \rangle \circ \langle b_{i_2 j_2}, d_{l_2 k_2} \rangle = \left\langle \sum_{i_2 j_1 = 1}^{t} (a_{i_1 j_1} b_{i_2 j_2}), \sum_{l_2 k_1 = 1}^{t-1} (c_{l_1 k_1} d_{l_2 k_2}) \right\rangle, \tag{19}
$$

see [29].

The present paper hinges on (19) to define the "noncommutative rhomtimes" (see Section 3.1.5). Because this method links rhotrices with matrices, Sani introduced the idea of a "coupled matrix" and formalized the conversion of a rhotrix into this object through [30]. Moreover, [27] provides nomenclature on the rhotrix entry representation used in [28,29] by calling the embedded matrices (a_{ij}) and (c_{kl}) in the rhotrix $\langle a_{ij}, c_{kl} \rangle$ the "principal" and "complementary" matrices, respectively.

Seeing that the rhotrix only has odd dimensions, A. O. Isere in [20, 21] extended the rhotrix dimensions by formalizing even-dimensional rhotrices. These were also called 'heartless rhotrices" or *hl*-rhotrices [20, 21] due to their lack of the heart entry. A two-dimensional rhotrix A (which is an *hl*-rhotrix) is defined in [20] to be the rhotrix $\left\langle \begin{matrix} & a & \\ b & & d \\ & e & \end{matrix} \right\rangle$. Due to this extension, the name "heart-based rhotrix" or "*h*-rhotrix", presented in [20,21] to refer to the odd-dimensional rhotrices according to Ajibade's definition, became relevant in specifying rhotrix dimensionality. With this, the current paper only investigates *h*-rhotrices; hence, extending the results of this study to *hl*-rhotrices remains open.

On another note, A. Mohammed and Y. Tella in [26] extended the rhotrix entries to sets aside from $\mathbb{R}$. Also, S. M. Tudunkaya in [31] introduced the "polynomial rhotrix", a rhotrix whose entries are polynomials in the usual algebraic sense. This current study defines "multivariate polynomial *h*-rhotrices" (see Sec. 3.1.1) the primary objects of study, an extension to the concepts published by various authors.

Meanwhile, several authors have extended the operations on rhotrices by characterizing certain algebraic structures concerning rhotrices, e.g., groups, rings, fields (see [2, 23–25, 32]). Despite these developments, the modular and algebraic aspects of rhotrix theory remain largely unexplored. Hence, this current study makes use of the usual algebraic concept of polynomial rings in not necessarily one indeterminate to characterize certain algebras. Since polynomial rings serve as a cornerstone of commutative algebra, their incorporation into rhotrix theory opens a promising avenue for generalization and abstraction within algebraic systems.

Ultimately, this study extends the literature on heart-based rhotrices by characterizing certain $K[x_1, \dots, x_n]$-algebras of multivariate polynomial *h*-rhotrix sets by way of the multiplication methods of Ajibade [1] and Sani [28]. The tensor products of these algebras (see [15]) were also characterized through novel concepts, as this area in rhotrix theory lacks formal inquiry. By establishing these results, this paper has contributed to extending rhotrix theory toward potential future advancements in mathematical theory and applications.

2 Preliminaries

2.1 Ring

Definition 2.1. [3] *A* unitary ring $(K, +, \cdot, 1)$*, or simply* K*, is a quadruple, where* K *is a set such that:*

- *(A) There is an operation* $+$ *on* K *which associates to each pair* $(\alpha, \beta) \in K \times K$ *its sum* $\alpha + \beta \in K$*, and makes the pair* $(K, +)$ *an abelian group, that is, it satisfies the following properties:*
  - *(A1) It is commutative, that is,* $\alpha + \beta = \beta + \alpha$*, for any* $\alpha, \beta \in K$*.*
  - *(A2) It is associative, that is,* $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$*, for any* $\alpha, \beta, \gamma \in K$*.*
  - *(A3) There exists an element* $0 \in K$ *such that* $\alpha + 0 = 0 + \alpha = \alpha$*, for any* $\alpha \in K$*.*
  - *(A4) To any* $\alpha \in K$ *is associated an element* $(-\alpha) \in K$ *such that* $\alpha + (-\alpha) = (-\alpha) + \alpha = 0$*.*

(B) There is an operation $\cdot$ on K which associates to each pair $(\alpha, \beta) \in K \times K$ its product $\alpha \cdot \beta \in K$, also denoted simply by $\alpha\beta$, and an element 1 of K such that the triple $(K, \cdot, 1)$ satisfies the following properties:

(B1) It is associative, that is, $\alpha(\beta\gamma) = (\alpha\beta)\gamma$ for any $\alpha, \beta, \gamma \in K$.

(B2) The element $1 \in K$, called "identity" (or "unity") or one, is such that $\alpha 1 = 1\alpha = \alpha$, for all $\alpha \in K$.

(C) Multiplication is left and right distributive over addition:

(C1) $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$, for any $\alpha, \beta, \gamma \in K$.

(C2) $(\beta + \gamma)\alpha = \beta\alpha + \gamma\alpha$, for any $\alpha, \beta, \gamma \in K$.

Remark 2.1. *For any $a, b \in R$, where R is a ring, the juxtaposition ba is equivalent to the product $b \cdot a$, where the operation $\cdot$ is the multiplication in R.*

Definition 2.2. [3] *A* commutative ring *K is a ring such that for any $\alpha, \beta \in K$, $\alpha\beta = \beta\alpha$.*

Definition 2.3. [19] *Let R be a ring. The set $R[x_1, \ldots, x_n]$ is a* ring of polynomials *in the $n \in \mathbb{N}$ indeterminates x_i with coefficients in the ring R, equipped with the usual addition and multiplication of polynomials.*

Remark 2.2. [19] *$R[x_1, \ldots, x_n]$ is commutative if and only if R is commutative.*

2.2 Algebra

Definition 2.4. [3] *For a commutative unitary ring K, a K-*algebra *A is a K-module together with an operation $A \times A \longrightarrow A$, called multiplication, which is right and left distributive over addition and compatible with the K-action on A, that is, for all $a, b \in A$ and $\alpha \in K$, $(ab)\alpha = a(b\alpha) = (a\alpha)b$.*

Definition 2.5. [3] *If A is a K-algebra with K a commutative unitary ring, then A is an* associative algebra *if the operation of multiplication is associative, that is, if $a(bc) = (ab)c$, for any $a, b, c \in A$.*

Definition 2.6. [3] *If A is a K-algebra with K a commutative unitary ring, then A is a* commutative algebra *if the operation of multiplication is commutative, that is, if $ab = ba$, for any $a, b \in A$.*

Definition 2.7. [3] *If A is a K-algebra with K a commutative unitary ring, then A is a* unitary algebra *if there exists $1_A \in A$ such that $a \cdot 1_A = 1_A \cdot a = a$, for any $a \in A$.*

Definition 2.8. [3] *If A is a K-algebra with K a commutative unitary ring, then the* opposite algebra *of A, denoted by A^{op}, is the same set with the same K-module structure, but the product of the elements a, b of the set is defined as $a \cdot b = ba$.*

Remark 2.3. [3] *$A = A^{op}$ if and only if A is commutative.*

2.3 Tensor products of algebras

Let B, C be two A-algebras. Since B and C are A-modules, their tensor product is $D = B \otimes_A C$, which is an A-module [15]. Consider the mapping $B \times C \times B \times C \longrightarrow D$ defined by $(b, c, b', c') \mapsto bb' \otimes cc'$. This is A-linear in each factor and therefore, induces an A-module homomorphism [15]

$$B \otimes C \otimes B \otimes C \longrightarrow D; \tag{20}$$

hence an A-module homomorphism $D \otimes D \longrightarrow D$ [15]. This corresponds to an A-bilinear mapping $\mu : D \times D \longrightarrow D$, such that

$$\mu(b \otimes c, b' \otimes c') = bb' \otimes cc'. \tag{21}$$

The multiplication on D is defined for all elements $b \otimes c, b' \otimes c' \in D$ as

$$(b \otimes c)(b' \otimes c') = bb' \otimes cc'. \tag{22}$$

Remark 2.4. [15] *The tensor product D is an A-algebra.*

2.4 Heart-based rhotrix

Definition 2.9. [1] *The* rhotrix *is defined as*

$$R = \left\{ \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle : a, b, c, d, e \in \mathbb{R} \right\}. \tag{23}$$

Based on how the rhotrix is presented in Equation (23), it can be understood that the definition provided in [1] is not for a rhotrix, but rather, for a "real rhotrix set" of dimension 3, as defined in [26].

Definition 2.10. [1] *Let R be a 3-dimensional rhotrix such that*

$$R = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle. \tag{24}$$

The element at the perpendicular intersection of the two diagonals of R in Equation (24)*, denoted by $h(R)$, is called the* heart *of R.*

Definition 2.11. [20, 21] *A **heart-based rhotrix** or **h-rhotrix** is a rhotrix that has a heart entry and is odd-dimensional.*

2.5 Rhotrix representation

The following is a generalization of the *h*-rhotrix based on [17, 18].

$$
\left\langle \begin{array}{ccccccccc}
 & & & & a_1 & & & & \\
 & & & a_2 & a_3 & a_4 & & & \\
 & & a_5 & a_6 & a_7 & a_8 & a_9 & & \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
a_{\frac{n^2-2n+5}{4}} & \cdots & \cdots & \cdots & a_{\frac{1}{4}(n^2+3)} & \cdots & \cdots & \cdots & a_{\frac{n^2+2n+1}{4}} \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
 & & a_{\frac{1}{2}(n^2+1)-8} & a_{\frac{1}{2}(n^2+1)-7} & a_{\frac{1}{2}(n^2+1)-6} & a_{\frac{1}{2}(n^2+1)-5} & a_{\frac{1}{2}(n^2+1)-4} & & \\
 & & & a_{\frac{1}{2}(n^2+1)-3} & a_{\frac{1}{2}(n^2+1)-2} & a_{\frac{1}{2}(n^2+1)-1} & & & \\
 & & & & a_{\frac{1}{2}(n^2+1)} & & & &
\end{array} \right\rangle \tag{25}
$$

where each entry of (25) can be generalized as a_i, $i = 1, 2, \ldots, \frac{1}{2}(n^2+1) = \frac{n^2+1}{2}$ and the heart entry has the index $\frac{1}{4}(n^2+3) = \frac{n^2+3}{4}$. This type of indexing, which is by a single positive integer i, is called the *horizontal single-index method of rhotrix representation* in [27], which is just one way of representing the entries of a rhotrix.

2.6 Rhotrix sets

The notion of *rhotrix sets* was introduced in [26], where some of these are $\hat{R}_n(\mathbb{N})$, $\hat{R}_n(\mathbb{Z})$, $\hat{R}_n(\mathbb{Q})$, $\hat{R}_n(\mathbb{R})$, and $\hat{R}_n(\mathbb{C})$—sets containing all n-dimensional *h*-rhotrices whose entries all belong to $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, respectively.

2.7 Polynomial rhotrix

Definition 2.12. [31] *A* polynomial rhotrix *is a rhotrix whose entries are polynomials.*

The notation $R[f]$ was utilized in [31] to denote the n-dimensional heart-based polynomial rhotrix, as in the following:

$$
R[f] = \left\langle \begin{array}{ccccccc}
 & & & f_1(x) & & & \\
 & & f_2(x) & f_3(x) & f_4(x) & & \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \\
f_\alpha(x) & \cdots & \cdots & f_\beta(x) & \cdots & \cdots & f_\pi(x) \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \\
 & & f_{t-3}(x) & f_{t-2}(x) & f_{t-1}(x) & & \\
 & & & f_t(x) & & &
\end{array} \right\rangle, \tag{26}
$$

where $t = \frac{n^2+1}{2}$, $\alpha = \frac{t+1}{2} - \frac{n}{2}$, $\beta = \frac{t+1}{2}$, $\pi = \frac{t+1}{2} + \frac{n}{2}$, with $\frac{n}{2}$ being the integer value of the quotient of n when divided by 2.

3 Results and discussion

3.1 Algebras

This section tackles the construction of **CRAPol**(K, m, n, x) and **gCRAPol**(K, m, n_1, n_2, x).

3.1.1 The set $\mathcal{R}_m(K, n, x)$

Remark 3.1. *Any $h \in K[x_1, \ldots, x_n]$ has the form*

$$h := \sum_{i=0}^{p} \left(a_i \prod_{j=1}^{n} x_j^{\alpha_{i,j}} \right), \tag{27}$$

where $\prod_{j=1}^{n} x_j^{\alpha_{i,j}} = x_1^{\alpha_{i,1}} x_2^{\alpha_{i,2}} \cdots x_n^{\alpha_{i,n}}$, $a_i \in K$, $p \in \mathbb{Z}^+$, and $\alpha_{i,j} \in \mathbb{N} := \{0, 1, 2, \ldots\}$ is the exponent of x_j.

Lemma 3.1. *If K is a commutative unitary ring, then the ring $K[x_1, \ldots, x_n]$ of polynomials in n indeterminates x_j, $j = 1, 2, \ldots, n$, is commutative and unitary.*

Proof. By Definition 2.3, $K[x_1, \ldots, x_n]$ is a commutative ring. To prove that $K[x_1, \ldots, x_n]$ is unitary, let $f \in K[x_1, \ldots, x_n]$ such that, via Remark 3.1, $f = \sum_{i=0}^{p} \left(a_i \prod_{j=1}^{n} x_j^{\alpha_{i,j}} \right)$, where $a_i \in K$, $p \in \mathbb{Z}^+$, $\alpha_{i,j} \in \mathbb{N}$ is the exponent of x_j. Consider the unity $1_K \in K$. Thus, by Definition 2.1(B2),

$$f \cdot 1_K = \sum_{i=0}^{p} \left(a_i \prod_{j=1}^{n} x_j^{\alpha_{i,j}} \right) \cdot 1_K = \sum_{i=0}^{p} \left(a_i 1_K \prod_{j=1}^{n} x_j^{\alpha_{i,j}} \right) = \sum_{i=0}^{p} \left(a_i \prod_{j=1}^{n} x_j^{\alpha_{i,j}} \right) = f, \tag{28}$$

and

$$1_K \cdot f = 1_K \cdot \sum_{i=0}^{p} \left(a_i \prod_{j=1}^{n} x_j^{\alpha_{i,j}} \right) = \sum_{i=0}^{p} \left(1 a_i \prod_{j=1}^{n} x_j^{\alpha_{i,j}} \right) = \sum_{i=0}^{p} \left(a_i \prod_{j=1}^{n} x_j^{\alpha_{i,j}} \right) = f. \tag{29}$$

Hence, $K[x_1, \ldots, x_n]$ is unitary. $\square$

Definition 3.1. *For a commutative unitary ring $K[x_1, \ldots, x_n]$ of polynomials in n indeterminates x_j, $j = 1, 2, \ldots, n$, the set $\mathcal{R}_m(K, n, x)$, or equivalently, $\mathcal{R}_m(K[x_1, \ldots, x_n])$, contains all the h-rhotrices of odd dimension $m = 2k + 1$, $k \in \mathbb{N}$, with rhotrix entries in $K[x_1, \ldots, x_n]$.*

Definition 3.2. *Any h-rhotrix in $\mathcal{R}_m(K, n, x)$ is called a* multivariate polynomial heart-based rhotrix *or mph*-rhotrix.

Example 3.1. *The sets of all 1-dimensional mph-rhotrices with entries in $\mathbb{R}[x]$, 3-dimensional mph-rhotrices with entries in $\mathbb{Z}[x_1, x_2]$, and 5-dimensional mph-rhotrices with entries in $\mathbb{Q}[x_1, x_2, x_3]$ are $\mathcal{R}_1\big(\mathbb{R}, 1, x\big)$, $\mathcal{R}_3\big(\mathbb{Z}, 2, x\big)$, $\mathcal{R}_5\big(\mathbb{Q}, 2, x\big)$, respectively.*

Example 3.2. *Let A be an mph-rhotrix such that*

$$A = \left\langle \begin{array}{ccc} & 2x_1^2 x_2^3 & \\ \frac{3}{2} x_1 x_2^2 x_3^3 + \sqrt{2} x_2 x_3^2 & 4x_1^3 x_2 + e x_2 + 5 & 6\pi x_3^2 + 3x_2^3 \\ & \pi x_1^2 x_2^3 & \end{array} \right\rangle \tag{30}$$

where $\pi = 3.141593\ldots$ and $e = 2.718282\ldots$. The mph-rhotrix A in Equation (30) is 3-dimensional and has entries in $\mathbb{R}[x_1, x_2, x_3]$. Hence, $A \in \mathcal{R}_3\big(\mathbb{R}, 3, x\big)$. The same can be represented via the *horizontal single-index method:*

$$A = \left\langle \begin{array}{ccc} & a_1 & \\ a_2 & a_3 & a_4 \\ & a_5 & \end{array} \right\rangle, \tag{31}$$

where $a_1 = 2x_1^2x_2^3$, $a_2 = \frac{3}{2}x_1x_2^2x_3^3 + \sqrt{2}x_2x_3^2$, $a_3 = 4x_1^3x_2 + ex_2 + 5$, $a_4 = 6\pi x_3^2 + 3x_2^3$, $a_5 = \pi x_1^2x_2^3$.

3.1.2 Rhomplus $\oplus$ (addition)

Definition 3.3. *The addition on $\mathcal{R}_m(K, n, x)$ is the binary operation* rhomplus $\oplus$ *on $\mathcal{R}_m(K, n, x)$ such that if $R, S, T \in \mathcal{R}_m(K, n, x)$, having entries r_i, s_i, t_i, respectively, $i = 1, 2, \ldots, \frac{m^2+1}{2}$, $R \oplus S = T$ if and only if $r_i + s_i = t_i$, for all i, with $+$ being the usual addition of polynomials in $K[x_1, \ldots, x_n]$.*

Lemma 3.2. The operation $\oplus$ is associative and commutative.

Proof. Let $A, B, C \in \mathcal{R}_m(K, n, x)$ with entries a_i, b_i, c_i, respectively, $i = 1, 2, \ldots, \frac{m^2+1}{2}$. By Definition 3.3, there exists $D, E, F, G \in \mathcal{R}_m(K, n, x)$ with entries d_i, e_i, respectively, such that $B \oplus C = D$, $A \oplus D = E$, $A \oplus B = F$, and $F \oplus C = G$, as $b_i + c_i = d_i$, $a_i + d_i = a_i + (b_i + c_i) = e_i$, $a_i + b_i = f_i$, and $f_i + c_i = (a_i + b_i) + c_i = g_i$, for all i. Note that by Definition 2.1(A2)

$$e_i = a_i + (b_i + c_i) = (a_i + b_i) + c_i = g_i, \tag{32}$$

implying that

$$E = A \oplus D = A \oplus (B \oplus C) = (A \oplus B) \oplus C = F \oplus C = G. \tag{33}$$

Thus, $\oplus$ is associative. Meanwhile, it follows from Definition 2.1(A1) $a_i + b_i = b_i + a_i$, for all i; hence, $A \oplus B = B \oplus A$. Therefore, $\oplus$ is commutative. $\square$

Example 3.3. *The set $\mathcal{R}_3(\mathbb{R}, 3, x)$ contains all the mph-rhotrices with entries in the commutative unitary ring $\mathbb{R}[x_1, x_2, x_3]$. It follows that the rhotrices*

$$A = \left\langle \begin{array}{ccc} & 2x_1x_2^2x_3^3 & \\ 5x_2x_3^2 & 6x_1^3x_2^2 & 3x_1^2x_3^2 \\ & 4x_2^4 & \end{array} \right\rangle, \quad B = \left\langle \begin{array}{ccc} & 3x_1x_2^2x_3^3 & \\ 2x_2x_3^2 & 4x_1^3x_2^2 & 7x_1^2x_3^2 \\ & 5x_2^4 & \end{array} \right\rangle,$$

$$\text{and} \quad C = \left\langle \begin{array}{ccc} & 8x_1x_2^2x_3^3 & \\ 9x_2x_3^2 & 7x_1^3x_2^2 & 6x_1^2x_3^2 \\ & 5x_2^4 & \end{array} \right\rangle$$

are in $\mathcal{R}_3(\mathbb{R}, 3, x)$. The addition of A and B via $\oplus$ proceeds as follows, in accordance with Definition 3.3:

$$A \oplus B = \left\langle \begin{array}{ccc} & 2x_1x_2^2x_3^3 & \\ 5x_2x_3^2 & 6x_1^3x_2^2 & 3x_1^2x_3^2 \\ & 4x_2^4 & \end{array} \right\rangle \oplus \left\langle \begin{array}{ccc} & 3x_1x_2^2x_3^3 & \\ 2x_2x_3^2 & 4x_1^3x_2^2 & 7x_1^2x_3^2 \\ & 5x_2^4 & \end{array} \right\rangle \tag{34}$$

$$= \left\langle \begin{matrix} & 2x_1x_2^2x_3^3 + 3x_1x_2^2x_3^3 & \\ 5x_2x_3^2 + 2x_2x_3^2 & 6x_1^3x_2^2 + 4x_1^3x_2^2 & 3x_1^2x_3^2 + 7x_1^2x_3^2 \\ & 4x_2^4 + 5x_2^4 & \end{matrix} \right\rangle \tag{35}$$

$$= \left\langle \begin{matrix} & 5x_1x_2^2x_3^3 & \\ 7x_2x_3^2 & 10x_1^3x_2^2 & 10x_1^2x_3^2 \\ & 9x_2^4 & \end{matrix} \right\rangle \tag{36}$$

$$= \left\langle \begin{matrix} & 3x_1x_2^2x_3^3 + 2x_1x_2^2x_3^3 & \\ 2x_2x_3^2 + 5x_2x_3^2 & 4x_1^3x_2^2 + 6x_1^3x_2^2 & 7x_1^2x_3^2 + 3x_1^2x_3^2 \\ & 5x_2^4 + 4x_2^4 & \end{matrix} \right\rangle \tag{37}$$

$$= \left\langle \begin{matrix} & 3x_1x_2^2x_3^3 & \\ 2x_2x_3^2 & 4x_1^3x_2^2 & 7x_1^2x_3^2 \\ & 5x_2^4 & \end{matrix} \right\rangle \diamondplus \left\langle \begin{matrix} & 2x_1x_2^2x_3^3 & \\ 5x_2x_3^2 & 6x_1^3x_2^2 & 3x_1^2x_3^2 \\ & 4x_2^4 & \end{matrix} \right\rangle \tag{38}$$

$$= B \diamondplus A. \tag{39}$$

Thus, the commutativity of $\diamondplus$, as in Lemma 3.2, is shown above. Now consider the following:

$$A \diamondplus (B \diamondplus C) = \left\langle \begin{matrix} & 2x_1x_2^2x_3^3 & \\ 5x_2x_3^2 & 6x_1^3x_2^2 & 3x_1^2x_3^2 \\ & 4x_2^4 & \end{matrix} \right\rangle \diamondplus \left(\left\langle \begin{matrix} & 3x_1x_2^2x_3^3 & \\ 2x_2x_3^2 & 4x_1^3x_2^2 & 7x_1^2x_3^2 \\ & 5x_2^4 & \end{matrix} \right\rangle \right. \tag{40}$$

$$\left. \diamondplus \left\langle \begin{matrix} & 8x_1x_2^2x_3^3 & \\ 9x_2x_3^2 & 7x_1^3x_2^2 & 6x_1^2x_3^2 \\ & 5x_2^4 & \end{matrix} \right\rangle \right) \tag{41}$$

$$= \left\langle \begin{matrix} & 2x_1x_2^2x_3^3 & \\ 5x_2x_3^2 & 6x_1^3x_2^2 & 3x_1^2x_3^2 \\ & 4x_2^4 & \end{matrix} \right\rangle \diamondplus \left\langle \begin{matrix} & 3x_1x_2^2x_3^3 + 8x_1x_2^2x_3^3 & \\ 2x_2x_3^2 + 9x_2x_3^2 & 4x_1^3x_2^2 + 7x_1^3x_2^2 & 7x_1^2x_3^2 + 6x_1^3x_2^2 \\ & 5x_2^4 + 5x_2^4 & \end{matrix} \right\rangle \tag{42}$$

$$= \left\langle \begin{matrix} & 2x_1x_2^2x_3^3 & \\ 5x_2x_3^2 & 6x_1^3x_2^2 & 3x_1^2x_3^2 \\ & 4x_2^4 & \end{matrix} \right\rangle \diamondplus \left\langle \begin{matrix} & 11x_1x_2^2x_3^3 & \\ 11x_2x_3^2 & 11x_1^3x_2^2 & 13x_1^3x_2^2 \\ & 10x_2^4 & \end{matrix} \right\rangle \tag{43}$$

$$= \left\langle \begin{matrix} & 2x_1x_2^2x_3^3 + 11x_1x_2^2x_3^3 & \\ 5x_2x_3^2 + 11x_2x_3^2 & 6x_1^3x_2^2 + 11x_1^3x_2^2 & 3x_1^2x_3^2 + 13x_1^2x_3^2 \\ & 4x_2^4 + 10x_2^4 & \end{matrix} \right\rangle \tag{44}$$

$$= \left\langle \begin{matrix} & 13x_1x_2^2x_3^3 & \\ 16x_2x_3^2 & 17x_1^3x_2^2 & 16x_1^2x_3^2 \\ & 14x_2^4 & \end{matrix} \right\rangle \tag{45}$$

$$= \left\langle \begin{matrix} & (2x_1x_2^2x_3^3 + 3x_1x_2^2x_3^3) + 8x_1x_2^2x_3^3 & \\ (5x_2x_3^2 + 2x_2x_3^2) + 9x_2x_3^2 & (6x_1^3x_2^2 + 4x_1^3x_2^2) + 7x_1^3x_2^2 & (3x_1^2x_3^2 + 7x_1^2x_3^2) + 6x_1^2x_3^2 \\ & (4x_2^4 + 5x_2^4) + 5x_2^4 & \end{matrix} \right\rangle \tag{46}$$

$$= \left\langle \begin{matrix} & 2x_1x_2^2x_3^3 + 3x_1x_2^2x_3^3 & \\ 5x_2x_3^2 + 2x_2x_3^2 & 6x_1^3x_2^2 + 4x_1^3x_2^2 & 3x_1^2x_3^2 + 7x_1^2x_3^2 \\ & 4x_2^4 + 5x_2^4 & \end{matrix} \right\rangle \diamondplus \left\langle \begin{matrix} & 8x_1x_2^2x_3^3 & \\ 9x_2x_3^2 & 7x_1^3x_2^2 & 6x_1^3x_2^2 \\ & 5x_2^4 & \end{matrix} \right\rangle \tag{47}$$

$$= \left(\left\langle \begin{matrix} & 2x_1x_2^2x_3^3 & \\ 5x_2x_3^2 & 6x_1^3x_2^2 & 3x_1^2x_3^2 \\ & 4x_2^4 & \end{matrix} \right\rangle \diamondplus \left\langle \begin{matrix} & 3x_1x_2^2x_3^3 & \\ 2x_2x_3^2 & 4x_1^3x_2^2 & 7x_1^2x_3^2 \\ & 5x_2^4 & \end{matrix} \right\rangle \right) \diamondplus \left\langle \begin{matrix} & 8x_1x_2^2x_3^3 & \\ 9x_2x_3^2 & 7x_1^3x_2^2 & 6x_1^3x_2^2 \\ & 5x_2^4 & \end{matrix} \right\rangle \tag{48}$$

$$= (A \diamondplus B) \diamondplus C. \tag{49}$$

The associativity of $\diamondplus$ (Lemma 3.2) is thus shown.

3.1.3 Rhomdot-*r* $\diamonddot_r$ and rhomdot-*l* $\diamonddot_l$ (scalar multiplication)

Lemma 3.3. *If $K[x_1], K[x_1, x_2], \dots, K[x_1, \dots, x_n]$ are commutative and unitary rings of polynomials with coefficients in K, $n \in \mathbb{Z}^+$, then $K[x_1] \subset K[x_1, x_2] \subset \cdots \subset K[x_1, \dots, x_n]$.*

Proof. Let $\mathcal{K} = \{K[x_1, \dots, x_i] \mid i \in \mathbb{Z}^+\}$. Consider $\mathcal{K}_j, \mathcal{K}_k \in \mathcal{K}$ such that $\mathcal{K}_j = K[x_1, \dots, x_j]$ and $\mathcal{K}_k = K[x_1, \dots, x_k]$, where $k \neq j$. Without loss of generality, suppose $j < k$. Note that any $f \in \mathcal{K}_j$ has indeterminates x_l, $l = 1, 2, \dots, j$, and any $g \in \mathcal{K}_k$ has indeterminates x_m, $m = 1, 2, \dots, j, \dots, k$. Hence, $f \in \mathcal{K}_k$. Since $j < k$, there exists $g_0 \in \mathcal{K}_k$ such that $g_0 \notin \mathcal{K}_j$. Thus, $\mathcal{K}_j \subset \mathcal{K}_k$. As j and k are arbitrary, $K[x_1] \subset K[x_1, x_2] \subset \cdots \subset K[x_1, \dots, x_n]$, for any $n \in \mathbb{Z}^+$. $\square$

Lemma 3.4. *For any $n_1, n_2 \in \mathbb{Z}^+$ such that $n_1 \geq n_2$, if $f \in K[x_1, \dots, x_{n_1}]$ and $g \in K[x_1, \dots, x_{n_2}]$, then $fg \in K[x_1, \dots, x_{n_1}]$.*

Proof. Let $f \in K[x_1, \dots, x_{n_1}]$ and $g \in K[x_1, \dots, x_{n_2}]$ where $n_1 \geq n_2$. If $n_1 = n_2$, then $fg \in K[x_1, \dots, x_{n_1}]$. If $n_1 > n_2$, then there exist $f_0 \in K[x_1, \dots, x_{n_1}]$ and $g_0 \in K[x_1, \dots, x_{n_2}] \subseteq K[x_1, \dots, x_{n_2}, \dots, x_{n_1}]$, such that by closure of multiplication and Lemma 3.3, $f_0g_0 \in K[x_1, \dots, x_{n_1}]$ but $f_0g_0 \notin K[x_1, \dots, x_{n_2}]$. Therefore, $fg \in K[x_1, \dots, x_{n_1}]$. $\square$

Definition 3.4. *For any $n_1, n_2 \in \mathbb{Z}^+$ such that $n_1 \geq n_2$, the right scalar multiplication on $\mathcal{R}_m(K, n_1, x)$ by a polynomial in $K[x_1, \dots, x_{n_2}]$ is the operation* rhomdot-*r* $\diamonddot_r : \mathcal{R}_m(K, n_1, x) \times K[x_1, \dots, x_{n_2}] \longrightarrow \mathcal{R}_m(K, n_1, x)$ *such that if $R, S \in \mathcal{R}_m(K, n_1, x)$ have entries r_i and s_i, respectively, $i = 1, 2, \dots, \frac{m^2+1}{2}$, and $h \in K[x_1, \dots, x_{n_2}]$, $R \diamonddot_r h = S$ if and only if $r_ih = s_i$, for all i, the operation being the usual multiplication of polynomials.*

Definition 3.5. *For any $n_1, n_2 \in \mathbb{Z}^+$ such that $n_1 \geq n_2$, the left scalar multiplication on $\mathcal{R}_m(K, n_1, x)$ by a polynomial in $K[x_1, \dots, x_{n_2}]$ is the operation* rhomdot-*l* $\diamonddot_l : K[x_1, \dots, x_{n_2}] \times \mathcal{R}_m(K, n_1, x) \longrightarrow \mathcal{R}_m(K, n_1, x)$ *such that if $R, S \in \mathcal{R}_m(K, n_1, x)$ have entries r_i and s_i, respectively, $i = 1, 2, \dots, \frac{m^2+1}{2}$, and $h \in K[x_1, \dots, x_{n_2}]$, $h \diamonddot_l R = S$ if and only if $hr_i = s_i$, for all i, the operation being the usual multiplication of polynomials.*

Theorem 3.1. *The operations $\diamondcdot_r$ and $\diamondcdot_l$ are equivalent, that is, for all $R \in \mathcal{R}_m(K, n_1, x)$ and every $f \in K[x_1, \ldots, x_{n_2}]$, where $n_1 \geq n_2$, $R \diamondcdot_r f = f \diamondcdot_l R$.*

Proof. Note that $r_i f = f r_i$ by Definition 2.2. Hence by Definitions 3.4 and 3.5, $R \diamondcdot_r f = f \diamondcdot_l R$. $\square$

Example 3.4. *Consider $D \in \mathcal{R}_3(\mathbb{R}, 2, x)$ and $f \in \mathbb{R}[x_1, x_2]$ such that*

$$D = \left\langle \begin{array}{ccc} & 2x_1 & \\ 3x_2^2 & 2x_1^2x_2^2 & 2x_1^2 \\ & 3x_2 & \end{array} \right\rangle$$

and $f = 7x_1^4x_2^3$. The right scalar multiplication (Definition 3.4) of D and f proceeds as follows:

$$D \diamondcdot_r f = \left\langle \begin{array}{ccc} & 2x_1 & \\ 3x_2^2 & 2x_1^2x_2^2 & 2x_1^2 \\ & 3x_2 & \end{array} \right\rangle \diamondcdot_r 7x_1^4x_2^3 \tag{50}$$

$$= \left\langle \begin{array}{ccc} & (2x_1)(7x_1^4x_2^3) & \\ (3x_2^2)(7x_1^4x_2^3) & (2x_1^2x_2^2)(7x_1^4x_2^3) & (2x_1^2)(7x_1^4x_2^3) \\ & (3x_2)(7x_1^4x_2^3) & \end{array} \right\rangle \tag{51}$$

$$= \left\langle \begin{array}{ccc} & 14x_1^5x_2^3 & \\ 21x_1^4x_2^5 & 14x_1^6x_2^5 & 14x_1^6x_2^3 \\ & 21x_1^4x_2^4 & \end{array} \right\rangle. \tag{52}$$

By Theorem 3.1, the left scalar multiplication of the f and D yields the same product as the above, that is,

$$f \diamondcdot_l D = D \diamondcdot_r f = \left\langle \begin{array}{ccc} & 14x_1^5x_2^3 & \\ 21x_1^4x_2^5 & 14x_1^6x_2^5 & 14x_1^6x_2^3 \\ & 21x_1^4x_2^4 & \end{array} \right\rangle. \tag{53}$$

3.1.4 Commutative rhomtimes $\diamondtimes_C$ (heart-based multiplication)

Definition 3.6. *The* heart-based multiplication *on $\mathcal{R}_m(K, n, x)$ is the binary operation* commutative rhomtimes $\diamondtimes_C$ *on $\mathcal{R}_m(K, n, x)$ such that if $A, B, C \in \mathcal{R}_m(K, n, x)$ with entries a_i, b_i, c_i, respectively, $i = 1, 2, \ldots, \frac{m^2+1}{2}$, $A \diamondtimes_C B = C$ if and only if*

$$c_i := \begin{cases} a_i b_i, & \textit{for } i = \frac{m^2+3}{4}, \\ a_i b_{\frac{m^2+3}{4}} + b_i a_{\frac{m^2+3}{4}}, & \textit{otherwise}. \end{cases}$$

Lemma 3.5. *The operation $\diamondtimes_C$ is associative and commutative.*

Proof. Let $A, B, C \in \mathcal{R}_m(K, n, x)$ with entries a_i, b_i, c_i, respectively, $i = 1, 2, \ldots, \frac{m^2+1}{2}$. By Definition 3.6, there exists $D, E \in \mathcal{R}_m(K, n, x)$ with entries d_i, e_i, respectively, such that $B \diamondtimes_C C = D$ and $A \diamondtimes_C D = E$. By the same definition,

$$d_i = \begin{cases} b_i c_i, & \text{for } i = \frac{m^2+3}{4}, \\ b_i c_{\frac{m^2+3}{4}} + c_i b_{\frac{m^2+3}{4}}, & \text{otherwise,} \end{cases} \tag{54}$$

$$e_i = \begin{cases} a_i d_i = a_i(b_i c_i), & \text{for } i = \frac{m^2+3}{4}, \\ a_i d_{\frac{m^2+3}{4}} + d_i a_{\frac{m^2+3}{4}} = a_i\left(b_{\frac{m^2+3}{4}} c_{\frac{m^2+3}{4}}\right) + \left(b_i c_{\frac{m^2+3}{4}} + c_i b_{\frac{m^2+3}{4}}\right) a_{\frac{m^2+3}{4}}, & \text{otherwise.} \end{cases} \tag{55}$$

By Definition 2.1(B1),

$$a_i(b_i c_i) = (a_i b_i) c_i \tag{56}$$

and

$$a_i \left(b_{\frac{m^2+3}{4}} c_{\frac{m^2+3}{4}}\right) = \left(a_i b_{\frac{m^2+3}{4}}\right) c_{\frac{m^2+3}{4}}. \tag{57}$$

By Definition 2.1(C2),

$$\left(b_i c_{\frac{m^2+3}{4}} + c_i b_{\frac{m^2+3}{4}}\right) a_{\frac{m^2+3}{4}} = b_i c_{\frac{m^2+3}{4}} a_{\frac{m^2+3}{4}} + c_i b_{\frac{m^2+3}{4}} a_{\frac{m^2+3}{4}}. \tag{58}$$

It follows from Definition 2.1(C2) that

$$\begin{aligned} \left(a_i b_{\frac{m^2+3}{4}}\right) c_{\frac{m^2+3}{4}} + b_i c_{\frac{m^2+3}{4}} a_{\frac{m^2+3}{4}} + c_i b_{\frac{m^2+3}{4}} a_{\frac{m^2+3}{4}} = {} & \left(a_i b_{\frac{m^2+3}{4}} + b_i a_{\frac{m^2+3}{4}}\right) c_{\frac{m^2+3}{4}} \\ & + c_i \left(b_{\frac{m^2+3}{4}} a_{\frac{m^2+3}{4}}\right). \end{aligned} \tag{59}$$

Thus by Equations (56), (57), and (59),

$$e_i = \begin{cases} (a_i b_i) c_i, & \text{for } i = \frac{m^2+3}{4}, \\ \left(a_i b_{\frac{m^2+3}{4}} + b_i a_{\frac{m^2+3}{4}}\right) c_{\frac{m^2+3}{4}} + c_i \left(b_{\frac{m^2+3}{4}} a_{\frac{m^2+3}{4}}\right), & \text{otherwise.} \end{cases} \tag{60}$$

If $F \in \mathcal{R}_m(K, n, x)$ with entries f_i such that $A \diamondplus_C B = F$ and

$$f_i = \begin{cases} a_i b_i, & \text{for } i = \frac{m^2+3}{4}, \\ a_i b_{\frac{m^2+3}{4}} + b_i a_{\frac{m^2+3}{4}}, & \text{otherwise,} \end{cases} \tag{61}$$

then

$$e_i = \begin{cases} f_i c_i, & \text{for } i = \frac{m^2+3}{4}, \\ f_i c_{\frac{m^2+3}{4}} + c_i f_{\frac{m^2+3}{4}}, & \text{otherwise.} \end{cases} \tag{62}$$

It follows then from Equations (61) and (62) that

$$E = A \diamondplus_C (B \diamondplus_C C) = (A \diamondplus_C B) \diamondplus_C C = F \diamondplus_C C. \tag{63}$$

Thus, $\diamondplus_C$ is associative.

Moreover, $a_i b_i = b_i a_i$ and $a_i b_{\frac{m^2+3}{4}} + b_i a_{\frac{m^2+3}{4}} = b_i a_{\frac{m^2+3}{4}} + a_i b_{\frac{m^2+3}{4}}$ by Definitions 2.2 and 2.1(A1), respectively. Hence, $A \diamondplus_C B = B \diamondplus_C A$, satisfying the commutativity of $\diamondplus_C$. $\square$

Example 3.5. *Consider $E, G, H \in \mathcal{R}_3(\mathbb{Z}, 3, x)$ such that*

$$E = \left\langle \begin{matrix} & 3x_1x_2^4x_3^2 & \\ 4x_2^3x_3^2 & 5x_1x_3^2 & 2x_2^5x_3 \\ & x_1^3x_2x_3^7 & \end{matrix} \right\rangle, \quad G = \left\langle \begin{matrix} & x_1x_2^4x_3^2 & \\ 4x_2^3x_3^2 & 2x_1x_3^2 & 5x_2^5x_3 \\ & 3x_1^3x_2x_3^7 & \end{matrix} \right\rangle,$$

$$\textit{and} \quad H = \left\langle \begin{matrix} & 3x_1x_2^4x_3^2 & \\ x_2^3x_3^2 & 2x_1x_3^2 & 4x_2^5x_3 \\ & 5x_1^3x_2x_3^7 & \end{matrix} \right\rangle.$$

Now,

$$E \diamondplus_C (G \diamondplus_C H) = \left\langle \begin{matrix} & 3x_1x_2^4x_3^2 & \\ 4x_2^3x_3^2 & 5x_1x_3^2 & 2x_2^5x_3 \\ & x_1^3x_2x_3^7 & \end{matrix} \right\rangle \diamondplus_C \left(\left\langle \begin{matrix} & x_1x_2^4x_3^2 & \\ 4x_2^3x_3^2 & 2x_1x_3^2 & 5x_2^5x_3 \\ & 3x_1^3x_2x_3^7 & \end{matrix} \right\rangle \right. \tag{64}$$

$$\left. \diamondplus_C \left\langle \begin{matrix} & 3x_1x_2^4x_3^2 & \\ x_2^3x_3^2 & 2x_1x_3^2 & 4x_2^5x_3 \\ & 5x_1^3x_2x_3^7 & \end{matrix} \right\rangle \right) \tag{65}$$

$$= \left\langle \begin{matrix} & 3x_1x_2^4x_3^2 & \\ 4x_2^3x_3^2 & 5x_1x_3^2 & 2x_2^5x_3 \\ & x_1^3x_2x_3^7 & \end{matrix} \right\rangle \diamondplus_C \left\langle \begin{matrix} & 8x_1^2x_2^4x_3^4 & \\ 10x_1x_2^3x_3^4 & 4x_1^2x_3^4 & 18x_1x_2^5x_3^3 \\ & 16x_1^4x_2x_3^9 & \end{matrix} \right\rangle \tag{66}$$

$$= \left\langle \begin{matrix} & 52x_1^3x_2^4x_3^6 & \\ 66x_1^2x_2^3x_3^6 & 20x_1^3x_3^6 & 98x_1^2x_2^5x_3^5 \\ & 84x_1^5x_2x_3^{11} & \end{matrix} \right\rangle. \tag{67}$$

Consider the following case with the grouping of operations changed:

$$(E \diamondplus_C G) \diamondplus_C H = \left(\left\langle \begin{matrix} & 3x_1x_2^4x_3^2 & \\ 4x_2^3x_3^2 & 5x_1x_3^2 & 2x_2^5x_3 \\ & x_1^3x_2x_3^7 & \end{matrix} \right\rangle \diamondplus_C \left\langle \begin{matrix} & x_1x_2^4x_3^2 & \\ 4x_2^3x_3^2 & 2x_1x_3^2 & 5x_2^5x_3 \\ & 3x_1^3x_2x_3^7 & \end{matrix} \right\rangle \right) \tag{68}$$

$$\diamondplus_C \left\langle \begin{matrix} & 3x_1x_2^4x_3^2 & \\ x_2^3x_3^2 & 2x_1x_3^2 & 4x_2^5x_3 \\ & 5x_1^3x_2x_3^7 & \end{matrix} \right\rangle \tag{69}$$

$$= \left\langle \begin{array}{ccc} & 11x_1^2x_2^4x_3^4 & \\ 28x_1x_2^3x_3^4 & 10x_1^2x_3^4 & 29x_1x_2^5x_3^3 \\ & 17x_1^4x_2x_3^9 & \end{array} \right\rangle \diamondtimes_C \left\langle \begin{array}{ccc} & 3x_1x_2^4x_3^2 & \\ x_2^3x_3^2 & 2x_1x_3^2 & 4x_2^5x_3 \\ & 5x_1^3x_2x_3^7 & \end{array} \right\rangle \tag{70}$$

$$= \left\langle \begin{array}{ccc} & 52x_1^3x_2^4x_3^6 & \\ 66x_1^2x_2^3x_3^6 & 20x_1^3x_3^6 & 98x_1^2x_2^5x_3^5 \\ & 84x_1^5x_2x_3^{11} & \end{array} \right\rangle = E \diamondtimes_C (G \diamondtimes_C H). \tag{71}$$

This shows the associativity of $\diamondtimes_C$ as in Lemma 3.5. Meanwhile, the commutativity of $\diamondtimes_C$ (Lemma 3.5) arises from the following:

$$E \diamondtimes_C G = \left\langle \begin{array}{ccc} & 3x_1x_2^4x_3^2 & \\ 4x_2^3x_3^2 & 5x_1x_3^2 & 2x_2^5x_3 \\ & x_1^3x_2x_3^7 & \end{array} \right\rangle \diamondtimes_C \left\langle \begin{array}{ccc} & x_1x_2^4x_3^2 & \\ 4x_2^3x_3^2 & 2x_1x_3^2 & 5x_2^5x_3 \\ & 3x_1^3x_2x_3^7 & \end{array} \right\rangle \tag{72}$$

$$= \left\langle \begin{array}{ccc} & 11x_1^2x_2^4x_3^4 & \\ 28x_1x_2^3x_3^4 & 10x_1^2x_3^4 & 29x_1x_2^5x_3^3 \\ & 17x_1^4x_2x_3^9 & \end{array} \right\rangle \tag{73}$$

$$= \left\langle \begin{array}{ccc} & x_1x_2^4x_3^2 & \\ 4x_2^3x_3^2 & 2x_1x_3^2 & 5x_2^5x_3 \\ & 3x_1^3x_2x_3^7 & \end{array} \right\rangle \diamondtimes_C \left\langle \begin{array}{ccc} & 3x_1x_2^4x_3^2 & \\ 4x_2^3x_3^2 & 5x_1x_3^2 & 2x_2^5x_3 \\ & x_1^3x_2x_3^7 & \end{array} \right\rangle \tag{74}$$

$$= G \diamondtimes_C E. \tag{75}$$

3.1.5 Noncommutative rhomtimes $\diamondtimes_N$ (row–column multiplication)

Definition 3.7. *The* row–column multiplication *on $\mathcal{R}_m(K, n, x)$ is the binary operation* noncommutative rhomtimes $\diamondtimes_N$ *on $\mathcal{R}_m(K, n, x)$ such that if $A, B, C \in \mathcal{R}_m(K, n, x)$ are expressed via Sani's notation in* [29], as in $A = \langle a_{ij}, d_{kl}\rangle$, $B = \langle b_{ij}, e_{kl}\rangle$, and $C = \langle c_{ij}, f_{kl}\rangle$, where a_{ij}, b_{ij}, and c_{ij} are the entries of the principal matrices, while d_{kl}, e_{kl}, and f_{kl} are the entries of the complementary matrices of A, B, and C, respectively, with $i, j = 1, 2, \ldots, \frac{m+1}{2}$, $k, l = 1, 2, \ldots, \frac{m-1}{2}$, $A \diamondtimes_N B = C$ if and only if

$$C = \langle c_{ij}, f_{kl}\rangle = \left\langle \sum_{p=1}^{\frac{m+1}{2}} a_{ip}b_{pj}, \sum_{p'=1}^{\frac{m-1}{2}} d_{kp'}e_{p'l} \right\rangle,$$

for all $i, j = 1, 2, \ldots, \frac{m+1}{2}$ and $k, l = 1, 2, \ldots, \frac{m-1}{2}$.

Lemma 3.6. *The operation $\diamondtimes_N$ is associative and noncommutative.*

Proof. Let $A, B, C \in \mathcal{R}_m(K, n, x)$ expressed in coupled matrix notation as $A = \langle a_{ij}, d_{kl}\rangle$, $B = \langle b_{ij}, e_{kl}\rangle$, $C = \langle c_{ij}, f_{kl}\rangle$, where $i, j = 1, 2, \ldots, \frac{m+1}{2}$ and $k, l = 1, 2, \ldots, \frac{m-1}{2}$. Now,

$$A \diamondtimes_N (B \diamondtimes_N C) = \langle a_{ij}, d_{kl}\rangle \diamondtimes_N (\langle b_{ij}, e_{kl}\rangle \diamondtimes_N \langle c_{ij}, f_{kl}\rangle) \tag{76}$$

$$= \langle a_{ij}, d_{kl}\rangle \diamondtimes_N \left\langle \sum_{p=1}^{\frac{m+1}{2}} b_{ip}c_{pj}, \sum_{p'=1}^{\frac{m-1}{2}} e_{kp'}f_{p'l} \right\rangle \tag{77}$$

$$= \left\langle \sum_{q=1}^{\frac{m+1}{2}} \sum_{p=1}^{\frac{m+1}{2}} a_{iq}b_{qp}c_{pj}, \sum_{q'=1}^{\frac{m-1}{2}} \sum_{p'=1}^{\frac{m-1}{2}} d_{kq'}e_{q'p'}f_{p'l} \right\rangle \tag{78}$$

$$= \left\langle \sum_{q=1}^{\frac{m+1}{2}} a_{iq}b_{qp} \sum_{p=1}^{\frac{m+1}{2}} c_{pj}, \sum_{q'=1}^{\frac{m-1}{2}} d_{kq'}e_{q'p'} \sum_{p'=1}^{\frac{m-1}{2}} f_{p'l} \right\rangle \tag{79}$$

$$= \left\langle \sum_{q=1}^{\frac{m+1}{2}} a_{iq}b_{qp}, \sum_{q'=1}^{\frac{m-1}{2}} d_{kq'}e_{q'p'} \right\rangle \diamondtimes_N \langle c_{ij}, f_{kl}\rangle \tag{80}$$

$$= (\langle a_{ij}, d_{kl}\rangle \diamondtimes_N \langle b_{ij}, e_{kl}\rangle) \diamondtimes_N \langle c_{ij}, f_{kl}\rangle = (A \diamondtimes_N B) \diamondtimes_N C. \tag{81}$$

Consider

$$A \diamondtimes_N B = \left\langle \sum_{p=1}^{\frac{m+1}{2}} a_{ip}b_{pj}, \sum_{p'=1}^{\frac{m-1}{2}} d_{kp'}e_{p'l} \right\rangle \neq \left\langle \sum_{p=1}^{\frac{m+1}{2}} b_{ip}a_{pj}, \sum_{p'=1}^{\frac{m-1}{2}} e_{kp'}d_{p'l} \right\rangle = B \diamondtimes_N A. \tag{82}$$

Hence, $\diamondtimes_N$ is associative and noncommutative. $\square$

Example 3.6. *Consider the rhotrices $P, Q, R \in \mathcal{R}_3(\mathbb{Z}, 1, x)$ such that*

$$P = \left\langle \begin{array}{ccc} & 4x & \\ 5x & 2 & 3x \\ & 7x & \end{array} \right\rangle, Q = \left\langle \begin{array}{ccc} & 3x & \\ 5x & 6 & 2x \\ & 3x & \end{array} \right\rangle, \text{ and } R = \left\langle \begin{array}{ccc} & 7x & \\ 5x & 3 & 2x \\ & 2x & \end{array} \right\rangle.$$

Now,

$$P \diamondtimes_N (Q \diamondtimes_N R) = \left\langle \begin{array}{ccc} & 4x & \\ 5x & 2 & 3x \\ & 7x & \end{array} \right\rangle \diamondtimes_N \left(\left\langle \begin{array}{ccc} & 3x & \\ 5x & 6 & 2x \\ & 3x & \end{array} \right\rangle \diamondtimes_N \left\langle \begin{array}{ccc} & 7x & \\ 5x & 3 & 2x \\ & 2x & \end{array} \right\rangle \right) \tag{83}$$

$$= \left\langle \begin{array}{ccc} & 4x & \\ 5x & 2 & 3x \\ & 7x & \end{array} \right\rangle \diamondtimes_N \left\langle \begin{array}{ccc} & 31x^2 & \\ 50x^2 & 18 & 10x^2 \\ & 16x^2 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 274x^3 & \\ 505x^3 & 36 & 88x^3 \\ & 162x^3 & \end{array} \right\rangle. \tag{84}$$

Notice then that

$$(P \diamondtimes_N Q) \diamondtimes_N R = \left(\left\langle \begin{array}{ccc} & 4x & \\ 5x & 2 & 3x \\ & 7x & \end{array} \right\rangle \diamondtimes_N \left\langle \begin{array}{ccc} & 3x & \\ 5x & 6 & 2x \\ & 3x & \end{array} \right\rangle \right) \diamondtimes_N \left\langle \begin{array}{ccc} & 7x & \\ 5x & 3 & 2x \\ & 2x & \end{array} \right\rangle \tag{85}$$

$$= \left\langle \begin{array}{ccc} & 27x^2 & \\ 50x^2 & 12 & 17x^2 \\ & 31x^2 & \end{array} \right\rangle \diamondtimes_N \left\langle \begin{array}{ccc} & 7x & \\ 5x & 3 & 2x \\ & 2x & \end{array} \right\rangle \tag{86}$$

$$= \left\langle \begin{array}{ccc} & 274x^3 & \\ 505x^3 & 36 & 88x^3 \\ & 162x^3 & \end{array} \right\rangle = P \diamondtimes_N (Q \diamondtimes_N R). \tag{87}$$

The above shows the associativity of $\diamondtimes_N$. Note that

$$P \diamondtimes_N Q = \left\langle \begin{array}{ccc} & 4x & \\ 5x & 2 & 3x \\ & 7x & \end{array} \right\rangle \diamondtimes_N \left\langle \begin{array}{ccc} & 3x & \\ 5x & 6 & 2x \\ & 3x & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 27x^2 & \\ 50x^2 & 12 & 17x^2 \\ & 31x^2 & \end{array} \right\rangle \tag{88}$$

$$Q \diamondtimes_N P = \left\langle \begin{array}{ccc} & 3x & \\ 5x & 6 & 2x \\ & 3x & \end{array} \right\rangle \diamondtimes_N \left\langle \begin{array}{ccc} & 4x & \\ 5x & 2 & 3x \\ & 7x & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 22x^2 & \\ 35x^2 & 12 & 23x^2 \\ & 36x^2 & \end{array} \right\rangle \tag{89}$$

Thus, $P \diamondtimes_N Q \neq Q \diamondtimes_N P$, which shows the noncommutativity of $\diamondtimes_N$.

3.1.6 Algebras CRAPol(K, m, n, x) and gCRAPol(K, m, n_1, n_2, x)

Definition 3.8. *The* zero *mph*-rhotrix in $\mathcal{R}_m(K, n, x)$ *is denoted by* $\mathbf{0}_{\mathcal{R}m(K,n,x)}$ *and has all of its entries the additive identity* $0_{K[x_1,\ldots,x_n]} \in K[x_1, \ldots, x_n]$.

Definition 3.9. *The* unity *mph*-rhotrix in $\mathcal{R}_m(K, n, x)$ *is denoted by* $\mathbf{1}_{\mathcal{R}m(K,n,x)}$ *and has entries* p_i, $i = 1, 2, \ldots, \frac{m^2+1}{2}$, *such that*

$$p_i := \begin{cases} 1, & \text{for } i = \frac{m^2+3}{4}, \\ 0, & \text{otherwise.} \end{cases} \tag{90}$$

Theorem 3.2. $(\mathcal{R}_m(K, n, x), \diamondplus)$ *is an Abelian group.*

Proof. Let $\mathcal{R}_m(K, n, x)$ be the set of all m-dimensional *mph*-rhotrices with entries in the commutative unitary ring $K[x_1, \ldots, x_n]$. Let $A, B, C \in \mathcal{R}_m(K, n, x)$ with entries a_i, b_i, c_i, $i = 1, \ldots, \frac{m^2+1}{2}$. Since $a_i + b_i = b_i + a_i$, for all i, by Definition 2.1(A1) on $K[x_1, \ldots, x_n]$, $A \diamondplus B = B \diamondplus A$, satisfying Definition 2.1(A1). As $a_i + (b_i + c_i) = (a_i + b_i) + c_i$, for all i, by Definition 2.1(A2) on $K[x_1, \ldots, x_n]$, $A \diamondplus (B \diamondplus C) = (A \diamondplus B) \diamondplus C$ satisfying Definition 2.1(A2). Note that $\mathbf{0}_{\mathcal{R}m(K,n,x)} \in \mathcal{R}_m(K, n, x)$ and $A + \mathbf{0}_{\mathcal{R}m(K,n,x)} = \mathbf{0}_{\mathcal{R}m(K,n,x)} + A = A$, for any $A \in \mathcal{R}_m(K, n, x)$, thus satisfying Definition 2.1(A3). Now, any $-A \in \mathcal{R}_m(K, n, x)$ whose entries are $-a_i$ implies that $A \diamondplus (-A) = (-A) \diamondplus A = \mathbf{0}_{\mathcal{R}m(K,n,x)}$ as $a_i + (-a_i) = (-a_i) + a_i = 0$. Therefore, $(\mathcal{R}_m(K, n, x), \diamondplus)$ is an Abelian group. $\square$

Theorem 3.3. $(\mathcal{R}_m(K, n, x), \diamondplus, \diamondtimes_C, \mathbf{1}_{\mathcal{R}m(K,n,x)})$ *is a unitary commutative ring.*

Proof. Let $\mathcal{R}_m(K, n, x)$ be the set of all m-dimensional *mph*-rhotrices with entries in the commutative unitary ring $K[x_1, \ldots, x_n]$. Let $A, B, C \in \mathcal{R}_m(K, n, x)$ with entries a_i, b_i, c_i, $i = 1, \ldots, \frac{m^2+1}{2}$. Note that Theorem 3.2 satisfies Definition 2.1(A).

Now, Lemma 3.5 satisfies Definition 2.1(B1). Consider $\mathbf{1}_{\mathcal{R}m(K,n,x)} \in \mathcal{R}_m(K,n,x)$. Note that

$$A \diamondtimes_C \mathbf{1}_{\mathcal{R}m(K,n,x)} = A = \mathbf{1}_{\mathcal{R}m(K,n,x)} \diamondtimes_C A,$$

since

$$a_i = \begin{cases} a_i = a_i \cdot 1 = 1 \cdot a_i, & \text{for } i = \frac{m^2+3}{4}, \\ a_i = a_i \cdot 1 + 0 \cdot a_{\frac{m^2+3}{4}} = 0 \cdot a_{\frac{m^2+3}{4}} + a_i \cdot 1 & \text{otherwise.} \end{cases} \tag{91}$$

This satisfies Definition 2.1(B2).

For Definition 2.1(C1), note that for some $D, E \in \mathcal{R}_m(K,n,x)$,

$$A \diamondtimes_C (B \diamondplus C) = A \diamondtimes_C D = E, \tag{92}$$

where D has entries $d_i = b_i + c_i$, for all i, and E has entries

$$e_i = \begin{cases} a_{\frac{m^2+3}{4}} d_{\frac{m^2+3}{4}} = a_{\frac{m^2+3}{4}} \left(b_{\frac{m^2+3}{4}} + c_{\frac{m^2+3}{4}} \right) = a_{\frac{m^2+3}{4}} b_{\frac{m^2+3}{4}} + a_{\frac{m^2+3}{4}} c_{\frac{m^2+3}{4}}, & \text{for } i = \frac{m^2+3}{4}, \\ a_i d_{\frac{m^2+3}{4}} + d_i a_{\frac{m^2+3}{4}} = a_i \left(b_{\frac{m^2+3}{4}} + c_{\frac{m^2+3}{4}} \right) + (b_i + c_i) a_{\frac{m^2+3}{4}} & \\ \qquad = a_i b_{\frac{m^2+3}{4}} + a_i c_{\frac{m^2+3}{4}} + b_i a_{\frac{m^2+3}{4}} + c_i a_{\frac{m^2+3}{4}} & \\ \qquad = a_i b_{\frac{m^2+3}{4}} + b_i a_{\frac{m^2+3}{4}} + a_i c_{\frac{m^2+3}{4}} + c_i a_{\frac{m^2+3}{4}}, & \text{otherwise.} \end{cases} \tag{93}$$

Thus, $E = A \diamondtimes_C B \diamondplus A \diamondtimes_C C$ implying that $A \diamondtimes_C (B \diamondplus C) = A \diamondtimes_C B \diamondplus A \diamondtimes_C C$, satisfying Definition 2.1(C1). On the other hand, for some $F, G \in \mathcal{R}_m(K,n,x)$,

$$(B \diamondplus C) \diamondtimes_C A = F \diamondtimes_C A = G, \tag{94}$$

where F has entries $f_i = b_i + c_i$, for all i, and G has entries

$$g_i = \begin{cases} f_i a_i = (b_i + c_i) a_i = b_i a_i + c_i a_i, & \text{for } i = \frac{m^2+3}{4}, \\ f_i a_{\frac{m^2+3}{4}} + a_i f_{\frac{m^2+3}{4}} = (b_i + c_i) a_{\frac{m^2+3}{4}} + a_i \left(b_{\frac{m^2+3}{4}} + c_{\frac{m^2+3}{4}} \right) & \\ \qquad = b_i a_{\frac{m^2+3}{4}} + c_i a_{\frac{m^2+3}{4}} + a_i b_{\frac{m^2+3}{4}} + a_i c_{\frac{m^2+3}{4}} & \\ \qquad = b_i a_{\frac{m^2+3}{4}} + a_i b_{\frac{m^2+3}{4}} + c_i a_{\frac{m^2+3}{4}} + a_i c_{\frac{m^2+3}{4}}, & \text{otherwise.} \end{cases} \tag{95}$$

Hence, $G = B \diamondtimes_C A \diamondplus C \diamondtimes_C A$ implying that $(B \diamondplus C) \diamondtimes_C A = B \diamondtimes_C A \diamondplus C \diamondtimes_C A$, satisfying Definition 2.1(C2). Since $\diamondtimes_C$ is commutative, $(\mathcal{R}_m(K,n,x), \diamondplus, \diamondtimes_C)$ is indeed a unitary commutative ring. $\square$

Theorem 3.4. *With $\diamondplus$ as addition, $\diamondtimes_C$ as internal multiplication, and $\diamonddot_r$ as external multiplication,*

(i) The set $\mathcal{R}_m(K, n, x)$ of all m-dimensional mph-rhotrices, $m = 2k+1, k \in \mathbb{N}$, with entries in a commutative unitary ring $K[x_1, \ldots, x_n]$ of polynomials forms an associative commutative unitary $K[x_1, \ldots, x_n]$-algebra.

(ii) If $n_1 \geq n_2$, the set $\mathcal{R}_m(K, n_1, x)$ of all m-dimensional mph-rhotrices, $m = 2k+1, k \in \mathbb{N}$, with entries in a commutative unitary ring $K[x_1, \ldots, x_{n_2}]$ of polynomials forms an associative commutative unitary $K[x_1, \ldots, x_{n_2}]$-algebra.

Proof. Let $\mathcal{R}_m(K, n_0, x)$ be the set of all m-dimensional *mph*-rhotrices with entries in the commutative unitary rings $K[x_1, \ldots, x_{n_0}]$, $n_0 \in \mathbb{Z}^+$. By Definition 2.1(C) on $(\mathcal{R}_m(K, n_0, x), \diamondplus, \diamondtimes_C)$, $\diamondtimes_C$ is right and left distributive over $\diamondplus$. Take $A, B \in \mathcal{R}_m(K, n_0, x)$ and $f \in K[x_1, \ldots, x_{n_3}]$, such that $n_0 \geq n_3$, $n_3 \in \mathbb{Z}^+$. By Definition 3.6, there exists $C \in \mathcal{R}_m(K, n_0, x)$ with entries c_i such that $A \diamondtimes_C B = C$ and

$$c_i = \begin{cases} a_i b_i, & \text{for } i = \frac{m^2+3}{4}, \\ a_i b_{\frac{m^2+3}{4}} + b_i a_{\frac{m^2+3}{4}}, & \text{otherwise.} \end{cases} \tag{96}$$

By Definition 3.4, there exists $D \in \mathcal{R}_m(K, n_0, x)$ with entries d_i such that $C \diamonddot_r f = D$ and $c_i f = d_i$ for all i. It follows from Equation (96) and Definitions 2.1 and 2.2 that

$$d_i = \begin{cases} (a_i b_i) f = (a_i f) b_i, & \text{for } i = \frac{m^2+3}{4}, \\ \left(a_i b_{\frac{m^2+3}{4}} + b_i a_{\frac{m^2+3}{4}} \right) f = (a_i f)\, b_{\frac{m^2+3}{4}} + b_i \left(a_{\frac{m^2+3}{4}} f \right), & \text{otherwise.} \end{cases} \tag{97}$$

Thus, $A \diamondtimes_C (B \diamonddot_r f) = (A \diamonddot_r f) \diamondtimes_C B$. Note that $\diamondtimes_C$ is associative and commutative on $\mathcal{R}_m(K, n_0, x)$ by Lemma 3.5, and the unity exists in $\mathcal{R}_m(K, n_0, x)$. Therefore, $\mathcal{R}_m(K, n_0, x)$ forms an associative commutative unitary $K[x_1, \ldots, x_{n_3}]$-algebra. Because n_0 and n_3 satisfy $n_0 \geq n_3$, setting $n = n_0 = n_3$ proves (i) while $n_1 = n_0$ and $n_2 = n_3$ proves (ii). $\square$

Remark 3.2. *The set $\mathcal{R}_m(K, n, x)$ as a $K[x_1, \ldots, x_n]$-algebra in Theorem 3.4(i) is the* commutative m-dimensional *mph*-rhotrix algebra on the ring of polynomials in n indeterminates x_i with coefficients in a commutative unitary ring K, *where $i = 1, 2, \ldots, n$, or* **CRAPol**(K, m, n, x). *Meanwhile, the set $\mathcal{R}_m(K, n_1, x)$ as a $K[x_1, \ldots, x_{n_2}]$-algebra in Theorem 3.4(ii) is the* generalized commutative m-dimensional *mph*-rhotrix algebra on the ring of polynomials in n_1 indeterminates x_i with coefficients in K, *where $i = 1, 2, \ldots, n_1$, or* **gCRAPol**(K, m, n_1, n_2, x).

Theorem 3.5. **CRAPol**$(K, m, n, x) =$ **CRAPol**$(K, m, n, x)^{\mathrm{op}}$ *and* **gCRAPol**$(K, m, n_1, n_2, x) =$ **gCRAPol**$(K, m, n_1, n_2, x)^{\mathrm{op}}$.

Proof. Since **CRAPol**(K, m, n, x) and **gCRAPol**(K, m, n_1, n_2, x) are commutative (Theorem 3.4),

$$\mathbf{CRAPol}(K, m, n, x) = \mathbf{CRAPol}(K, m, n, x)^{\mathrm{op}} \tag{98}$$

and

$$\mathbf{gCRAPol}(K,m,n_1,n_2,x) = \mathbf{gCRAPol}(K,m,n_1,n_2,x)^{\mathrm{op}} \tag{99}$$

by Remark 2.3. $\square$

Theorem 3.6. $\mathbf{gCRAPol}(K,m,n,n,x) = \mathbf{CRAPol}(K,m,n,x)$.

Proof. Note that if $n_1 = n_2 = n$, $\mathbf{gCRAPol}(K,m,n_1,n_2,x) = \mathbf{gCRAPol}(K,m,n,n,x)$. Hence, $\mathbf{gCRAPol}(K,m,n,n,x) = \mathbf{CRAPol}(K,m,n,x)$. $\square$

3.1.7 Algebras $\mathbf{NRAPol}(K,m,n,x)$ and $\mathbf{gNRAPol}(K,m,n_1,n_2,x)$

Definition 3.10. *The* identity *mph*-rhotrix *in* $\mathcal{R}_m(K,n,x)$ *is denoted by*

$$\mathbf{I}_{\mathcal{R}m(K,n,x)} = \left\langle I_{\frac{m+1}{2}\times\frac{m+1}{2}}, I_{\frac{m-1}{2}\times\frac{m-1}{2}} \right\rangle,$$

where $I_{\frac{m+1}{2}\times\frac{m+1}{2}}$ *and* $I_{\frac{m-1}{2}\times\frac{m-1}{2}}$ *are identity matrices with dimensions* $\frac{m+1}{2}\times\frac{m+1}{2}$ *and* $\frac{m-1}{2}\times\frac{m-1}{2}$*, respectively.*

Theorem 3.7. $\left(\mathcal{R}_m(K,n,x), \oplus, \otimes_N, \mathbf{I}_{\mathcal{R}m(K,n,x)}\right)$ *is a noncommutative unitary ring.*

Proof. Let $\mathcal{R}_m(K,n,x)$ be the set of all m-dimensional *mph*-rhotrices with entries in the commutative unitary ring $K[x_1,\ldots,x_n]$. Let $A,B,C \in \mathcal{R}_m(K,n,x)$ such that $A = \langle a_{ij}, z_{kl}\rangle$, $B = \langle b_{ij}, y_{kl}\rangle$, $C = \langle c_{ij}, x_{kl}\rangle$, $i,j = 1,\ldots,\frac{m+1}{2}$ and $k,l = 1,\ldots,\frac{m-1}{2}$. Note that Theorem 3.2 satisfies Definition 2.1(A).

It follows that Lemma 3.6 satisfies Definition 2.1(B1). Consider $\mathbf{I}_{\mathcal{R}m(K,n,x)} \in \mathcal{R}_m(K,n,x)$. Observe that

$$A \otimes_N \mathbf{I}_{\mathcal{R}m(K,n,x)} = \langle a_{ij}, d_{kl}\rangle \otimes_N \left\langle I_{\frac{m+1}{2}\times\frac{m+1}{2}}, I_{\frac{m-1}{2}\times\frac{m-1}{2}} \right\rangle = \langle a_{ij}, d_{kl}\rangle = A \tag{100}$$

and

$$\mathbf{I}_{\mathcal{R}m(K,n,x)} \otimes_N A = \left\langle I_{\frac{m+1}{2}\times\frac{m+1}{2}}, I_{\frac{m-1}{2}\times\frac{m-1}{2}} \right\rangle \otimes_N \langle a_{ij}, d_{kl}\rangle = \langle a_{ij}, d_{kl}\rangle = A. \tag{101}$$

Thus, Definition 2.1(B2) is satisfied.

Now for Definition 2.1(C1), note that for some $D,E \in \mathcal{R}_m(K,n,x)$,

$$A \otimes_N (B \oplus C) = A \otimes_N D = E, \tag{102}$$

where $D = \langle d_{ij}, w_{kl}\rangle = \langle b_{ij}+c_{ij}, y_{kl}+x_{kl}\rangle$ and

$$E = \left\langle \sum_{p=1}^{\frac{m+1}{2}} a_{ip}d_{pj}, \sum_{p'=1}^{\frac{m-1}{2}} z_{kp'}w_{p'l} \right\rangle = \left\langle \sum_{p=1}^{\frac{m+1}{2}} a_{ip}(b_{pj}+c_{pj}), \sum_{p'=1}^{\frac{m-1}{2}} z_{kp'}(y_{p'l}+x_{p'l}) \right\rangle \tag{103}$$

$$= \left\langle \sum_{p=1}^{\frac{m+1}{2}} (a_{ip}b_{pj} + a_{ip}c_{pj}), \sum_{p'=1}^{\frac{m-1}{2}} (z_{kp'}y_{p'l} + z_{kp'}x_{p'l}) \right\rangle \tag{104}$$

$$= \left\langle \sum_{p=1}^{\frac{m+1}{2}} a_{ip}b_{pj} + \sum_{p=1}^{\frac{m+1}{2}} a_{ip}c_{pj}, \sum_{p'=1}^{\frac{m-1}{2}} z_{kp'}y_{p'l} + \sum_{p'=1}^{\frac{m-1}{2}} z_{kp'}x_{p'l} \right\rangle \tag{105}$$

$$= \left\langle \sum_{p=1}^{\frac{m+1}{2}} a_{ip}b_{pj}, \sum_{p'=1}^{\frac{m-1}{2}} z_{kp'}y_{p'l} \right\rangle \oplus \left\langle \sum_{p=1}^{\frac{m+1}{2}} a_{ip}c_{pj}, \sum_{p'=1}^{\frac{m-1}{2}} z_{kp'}x_{p'l} \right\rangle = A \otimes_N B \oplus A \otimes_N C. \tag{106}$$

Thus, $A \otimes_N (B \oplus C) = A \otimes_N B \oplus A \otimes_N C$, satisfying Definition 2.1(C1). Meanwhile, for some $F, G \in \mathcal{R}_m(K, n, x)$,

$$(B \oplus C) \otimes_N A = F \otimes_N A = G, \tag{107}$$

where $F = \langle f_{ij}, u_{kl} \rangle = \langle b_{ij} + c_{ij}, y_{kl} + x_{kl} \rangle$ and

$$G = \left\langle \sum_{p=1}^{\frac{m+1}{2}} f_{ip}a_{pj}, \sum_{p'=1}^{\frac{m-1}{2}} u_{kp'}z_{p'l} \right\rangle = \left\langle \sum_{p=1}^{\frac{m+1}{2}} (b_{ip} + c_{ip})a_{pj}, \sum_{p'=1}^{\frac{m-1}{2}} (y_{kp'} + x_{kp'})z_{p'l} \right\rangle \tag{108}$$

$$= \left\langle \sum_{p=1}^{\frac{m+1}{2}} (b_{ip}a_{pj} + c_{ip}a_{pj}), \sum_{p'=1}^{\frac{m-1}{2}} (y_{kp'}z_{p'l} + x_{kp'}z_{p'l}) \right\rangle \tag{109}$$

$$= \left\langle \sum_{p=1}^{\frac{m+1}{2}} b_{ip}a_{pj}, \sum_{p'=1}^{\frac{m-1}{2}} y_{kp'}z_{p'l} \right\rangle \oplus \left\langle \sum_{p=1}^{\frac{m+1}{2}} c_{ip}a_{pj}, \sum_{p'=1}^{\frac{m-1}{2}} x_{kp'}z_{p'l} \right\rangle = B \otimes_N A \oplus C \otimes_N A. \tag{110}$$

Hence, $(B \oplus C) \otimes_N A = B \otimes_N A \oplus C \otimes_N A$, satisfying Definition 2.1(C2). Since $\otimes_N$ is noncommutative, $(\mathcal{R}_m(K, n, x), \oplus, \otimes_N)$ is indeed a noncommutative unitary ring. $\square$

Theorem 3.8. *With $\oplus$ as addition, $\otimes_N$ as internal multiplication, and $\diamond_r$ as external multiplication,*

(i) The set $\mathcal{R}_m(K, n, x)$ of all m-dimensional mph-rhotrices, $m = 2k + 1, k \in \mathbb{N}$, with entries in a commutative unitary ring $K[x_1, \ldots, x_n]$ of polynomials forms an associative noncommutative unitary $K[x_1, \ldots, x_n]$-algebra.

(ii) If $n_1 \geq n_2$, the set $\mathcal{R}_m(K, n_1, x)$ of all m-dimensional mph-rhotrices, $m = 2k + 1$, $k \in \mathbb{N}$, with entries in a commutative unitary ring $K[x_1, \ldots, x_{n_2}]$ of polynomials forms an associative noncommutative unitary $K[x_1, \ldots, x_{n_2}]$-algebra.

Proof. The proof is analogous to that of Theorem 3.4. Let $\mathcal{R}_m(K, n_0, x)$ be the set of all m-dimensional *mph*-rhotrices with entries in the commutative unitary rings $K[x_1, \ldots, x_{n_0}]$, $n_0 \in \mathbb{Z}^+$. By Definition 2.1(C) on $(\mathcal{R}_m(K, n_0, x), \oplus, \otimes_N)$, $\otimes_N$ is right and left distributive over $\oplus$. Take $A, B \in \mathcal{R}_m(K, n_0, x)$ and $f \in K[x_1, \ldots, x_{n_3}]$, such that $A = \langle a_{ij}, z_{kl} \rangle$, $B = \langle b_{ij}, y_{kl} \rangle$, and $n_0 \geq n_3, n_3 \in \mathbb{Z}^+$. By Definition 3.7, there exists $C \in \mathcal{R}_m(K, n_0, x)$ such that $A \diamondplus_N B = C$ and

$$C = \langle c_{ij}, x_{kl} \rangle = \left\langle \sum_{p=1}^{\frac{m+1}{2}} a_{ip} b_{pj}, \sum_{p'=1}^{\frac{m-1}{2}} z_{kp'} y_{p'l} \right\rangle. \tag{111}$$

By Definition 3.4, there exists $D \in \mathcal{R}_m(K, n_0, x)$ such that $C \diamondsuit_r f = D$. It follows from Equation (111) that

$$D = \langle d_{ij}, w_{kl} \rangle = \left\langle \left(\sum_{p=1}^{\frac{m+1}{2}} a_{ip} b_{pj} \right) f, \left(\sum_{p'=1}^{\frac{m-1}{2}} z_{kp'} y_{p'l} \right) f \right\rangle \tag{112}$$

$$= \left\langle \sum_{p=1}^{\frac{m+1}{2}} \left[a_{ip} \left(b_{pj} f \right) \right], \sum_{p'=1}^{\frac{m-1}{2}} \left[z_{kp'} \left(y_{p'l} f \right) \right] \right\rangle = \left\langle \sum_{p=1}^{\frac{m+1}{2}} \left[\left(a_{ip} f \right) b_{pj} \right], \sum_{p'=1}^{\frac{m-1}{2}} \left[\left(z_{kp'} f \right) y_{p'l} \right] \right\rangle. \tag{113}$$

Thus, $A \diamondplus_N (B \diamondsuit_r f) = (A \diamondsuit_r f) \diamondplus_N B$. Note that $\diamondplus_N$ is associative and noncommutative on $\mathcal{R}_m(K, n_0, x)$ by Lemma 3.6, and the unity exists in $\mathcal{R}_m(K, n_0, x)$ by $\mathbf{I}_{\mathcal{R}m(K,n,x)}$. Therefore, $\mathcal{R}_m(K, n_0, x)$ forms an associative noncommutative unitary $K[x_1, \ldots, x_{n_3}]$-algebra. Because n_0 and n_3 satisfy $n_0 \geq n_3$, setting $n = n_0 = n_3$ proves (i), while setting $n_1 = n_0$ and $n_2 = n_3$ proves (ii). □

Remark 3.3. *The set $\mathcal{R}_m(K, n, x)$ as a $K[x_1, \ldots, x_n]$-algebra in Theorem 3.8(i), which is by $\diamondplus_N$, is the* noncommutative m-dimensional *mph*-rhotrix algebra on the ring of polynomials in n indeterminates x_i with coefficients in a commutative unitary ring K, *where* $i = 1, 2, \ldots, n$, *or* **NRAPol**(K, m, n, x). *Meanwhile, the set $\mathcal{R}_m(K, n_1, x)$ as a $K[x_1, \ldots, x_{n_2}]$-algebra in Theorem 3.8(ii) is the* generalized noncommutative m-dimensional *mph*-rhotrix algebra on the ring of polynomials in n_1 indeterminates x_i with coefficients in K, *where* $i = 1, 2, \ldots, n_1$, *or* **gNRAPol**(K, m, n_1, n_2, x).

Theorem 3.9. **gNRAPol**$(K, m, n, n, x) =$ **NRAPol**(K, m, n, x).

Proof. Note that if $n_1 = n_2 = n$, **gNRAPol**$(K, m, n_1, n_2, x) =$ **gNRAPol**(K, m, n, n, x). Hence, **gNRAPol**$(K, m, n, n, x) =$ **NRAPol**(K, m, n, x). □

3.2 Tensor products

This section tackles the tensor products of the previously-defined algebras.

3.2.1 Rhomboidal array

Definition 3.11. *A* blank entry $\#$ *is an entry of a rectangular array that has no value, which can be represented by a blank space "   ".*

Definition 3.12. *Let $\mathcal{A}^{m \times m}(K, n, x)$ and $\mathcal{B}^{m \times m}(K, n, x)$ be sets of $m \times m$ rectangular arrays with entries in the ring $K[x_1, \ldots, x_n]$ of polynomials such that $m = 2k + 1, k \in \mathbb{N}$, $n \in \mathbb{Z}^+$, and*

K is a commutative unitary ring. Let the function $\rho : \mathcal{A}^{m\times m}(K,n,x) \longrightarrow \mathcal{B}^{m\times m}(K,n,x)$ be such that if $A \in \mathcal{A}^{m\times m}(K,n,x)$ and $B \in \mathcal{B}^{m\times m}(K,n,x)$, with entries a_{ij} and b_{ij}, respectively, where i corresponds to the row of a_{ij} and b_{ij}, and j to their column, satisfying $1 \le i,j \le m$,

$$\rho(A) = B \Longleftrightarrow \kappa(a_{ij}) = b_{ij}, \tag{114}$$

where

$$\kappa(a_{ij}) := \begin{cases} \#, & \text{for } 1 \le j \le \left|\frac{m+1}{2} - i\right| \text{ and } m - \left|\frac{m+1}{2} - i\right| + 1 \le j \le m,\ i \ne \frac{m+1}{2}, \\ a_{ij}, & \text{otherwise.} \end{cases} \tag{115}$$

Then, ρ is the direct rectangular-rhomboidal array transformation*, a transformation that preserves dimensions and non-blank entries.*

Definition 3.13. *Let $\rho : \mathcal{A}^{m\times m}(K,n,x) \longrightarrow \mathcal{B}^{m\times m}(K,n,x)$ be the direct rectangular-rhomboidal array transformation and $m = 2k+1, k \in \mathbb{N}$. An m-dimensional* rhomboidal array *$Q \in \mathcal{B}^{m\times m}(K,n,x)$ is defined as $Q := \rho(R')$, for some $R' \in \mathcal{A}^{m\times m}(K,n,x)$.*

Example 3.7. *Consider constructing a 7-dimensional rhomboidal array R' from a 7×7 rectangular array R:*

$$R = \begin{matrix} r_{11} & r_{12} & r_{13} & r_{14} & r_{15} & r_{16} & r_{17} \\ r_{21} & r_{22} & r_{23} & r_{24} & r_{25} & r_{26} & r_{27} \\ r_{31} & r_{32} & r_{33} & r_{34} & r_{35} & r_{36} & r_{37} \\ r_{41} & r_{42} & r_{43} & r_{44} & r_{45} & r_{46} & r_{47} \\ r_{51} & r_{52} & r_{53} & r_{54} & r_{55} & r_{56} & r_{57} \\ r_{61} & r_{62} & r_{63} & r_{64} & r_{65} & r_{66} & r_{67} \\ r_{71} & r_{72} & r_{73} & r_{74} & r_{75} & r_{76} & r_{77} \end{matrix}. \tag{116}$$

By Definition 3.13, the blank entries of $R' = \rho(R)$ are determined in Table 1.
It follows that R in (116) *becomes* (117)*,*

$$R' = \begin{matrix} & & & r_{14} & & & \\ & & r_{23} & r_{24} & r_{25} & & \\ & r_{32} & r_{33} & r_{34} & r_{35} & r_{36} & \\ r_{41} & r_{42} & r_{43} & r_{44} & r_{45} & r_{46} & r_{47} \\ & r_{52} & r_{53} & r_{54} & r_{55} & r_{56} & \\ & & r_{63} & r_{64} & r_{65} & & \\ & & & r_{74} & & & \end{matrix}. \tag{117}$$

Definition 3.14. *The set $\mathcal{B}^{m\times m}(K,n,x)$ in Definition 3.12 is the* set of all rhomboidal arrays of dimension $m = 2k+1, k \in \mathbb{N}$*, with entries in $K[x_1,\dots,x_n]$.*

Table 1. Blank entries of R'

Row	Blank Entries
At $\lambda = 1$ (first row)	$1 \leq \mu \leq \left\lvert \frac{7+1}{2} - 1 \right\rvert$ $\Longrightarrow 1 \leq \mu \leq 3$, entries r_{11}, r_{12}, r_{13} and $7 + 1 - \left\lvert \frac{7+1}{2} - 1 \right\rvert \leq \mu \leq 7$ $\Longrightarrow 5 \leq \mu \leq 7$, entries r_{15}, r_{16}, r_{17}
At $\lambda = 2$ (second row)	$1 \leq \mu \leq \left\lvert \frac{7+1}{2} - 2 \right\rvert$ $\Longrightarrow 1 \leq \mu \leq 2$, entries r_{21}, r_{22}, and $7 + 1 - \left\lvert \frac{7+1}{2} - 2 \right\rvert \leq \mu \leq 7$ $\Longrightarrow 6 \leq \mu \leq 7$, entries r_{26}, r_{27}
At $\lambda = 3$ (third row)	$1 \leq \mu \leq \left\lvert \frac{7+1}{2} - 3 \right\rvert$ $\Longrightarrow 1 \leq \mu \leq 1 \Longrightarrow \mu = 1$, entry r_{31}, and $7 + 1 - \left\lvert \frac{7+1}{2} - 3 \right\rvert \leq \mu \leq 7$ $\Longrightarrow 7 \leq \mu \leq 7 \Longrightarrow \mu = 7$, entry r_{37}
At $\lambda = 5$ (fifth row)	$1 \leq \mu \leq \left\lvert \frac{7+1}{2} - 5 \right\rvert$ $\Longrightarrow 1 \leq \mu \leq 1 \Longrightarrow \mu = 1$, entry r_{51}, and $7 + 1 - \left\lvert \frac{7+1}{2} - 5 \right\rvert \leq \mu \leq 7$ $\Longrightarrow 7 \leq \mu \leq 7 \Longrightarrow \mu = 7$, entry r_{57}
At $\lambda = 6$ (sixth row)	$1 \leq \mu \leq \left\lvert \frac{7+1}{2} - 6 \right\rvert$ $\Longrightarrow 1 \leq \mu \leq 2$, entries r_{61}, r_{62}, and $7 + 1 - \left\lvert \frac{7+1}{2} - 6 \right\rvert \leq \mu \leq 7$ $\Longrightarrow 6 \leq \mu \leq 7$, entries r_{66}, r_{67}
At $\lambda = 7$ (seventh row)	$1 \leq \mu \leq \left\lvert \frac{7+1}{2} - 7 \right\rvert$ $\Longrightarrow 1 \leq \mu \leq 3$, entries r_{71}, r_{72}, r_{73}, and $7 + 1 - \left\lvert \frac{7+1}{2} - 7 \right\rvert \leq \mu \leq 7$ $\Longrightarrow 5 \leq \mu \leq 7$, entries r_{75}, r_{76}, r_{77}

Definition 3.15. *Let* $\alpha : \mathcal{B}^{m\times m}(K, n, x) \longrightarrow \mathcal{R}_m(K, n, x)$ *be such that if* $Q \in \mathcal{B}^{m\times m}(K, n, x)$ *and* $R \in \mathcal{R}_m(K, n, x)$*,*

$$R = \alpha(Q) \Longleftrightarrow R = \langle\, Q \,\rangle, \tag{118}$$

where $\mathcal{B}^{m\times m}(K, n, x)$ *is the set of all rhomboidal arrays of dimension* $m = 2k + 1, k \in \mathbb{N}$*, with entries in* $K[x_1, \ldots, x_n]$ *and* $\mathcal{R}_m(K, n, x)$ *is the set of all* m*-dimensional mph-rhotrices,* $m = 2k + 1, k \in \mathbb{N}$*, with entries in* $K[x_1, \ldots, x_n]$*. The function* α *is the* direct rhomboidal array-*mph*-rhotrix transformation.

Definition 3.16. *An* m*-dimensional mph-*rhotrix $R \in \mathcal{R}_m(K, n, x)$*,* $m = 2k + 1, k \in \mathbb{N}$*, is an* m*-dimensional rhomboidal array enclosed by angled brackets,* $\langle \bullet \rangle$*, that is, there exists* $Q \in \mathcal{B}^{m\times m}(K, n, x)$ *such that* $R = \alpha\,(Q)$*, where* α *is the direct rhomboidal array-mph-rhotrix transformation.*

3.2.2 *Bh*-rhotrix, *uBh*-rhotrix, and rhomotimes $\diamondtimes$

Definition 3.17. *A q-dimensional* block heart-based rhotrix *or Bh*-rhotrix*, $q = 2j + 1, j \in \mathbb{N}$, is a q-dimensional h-rhotrix whose entries are all rhomboidal arrays.*

Definition 3.18. *A q-dimensional* uniform block heart-based rhotrix *or uBh*-rhotrix*, $m = 2k + 1$, $k \in \mathbb{N}$, is a q-dimensional Bh-rhotrix whose entries have equal dimensions.*

Definition 3.19. *The set $\mathcal{H}_{q,p}(K, n, x)$ contains all q-dimensional uBh-rhotrices, $m = 2k + 1$, $k \in \mathbb{N}$, whose entries are all rhomboidal arrays of dimension $p = 2j + 1$, $j \in \mathbb{N}$, in which the entries of these rhomboidal arrays belong to the commutative unitary ring $K[x_1, \ldots, x_n]$.*

Definition 3.20. *Let $\mathcal{R}_{m_1}(K, n_1, x)$ and $\mathcal{R}_{m_2}(K, n_2, x)$ be sets of m_1- and m_2-dimensional mph-rhotrices with entries in $K[x_1, \ldots, x_{n_1}]$ and $K[x_1, \ldots, x_{n_2}]$, respectively, such that $n_1 \geq n_2$. Then for some set $\mathcal{Q}^{m_1 \times m_2}(K, n', x)$, let*

$$\diamondtimes : \mathcal{R}_{m_1}(K, n_1, x) \times \mathcal{R}_{m_2}(K, n_2, x) \longrightarrow \mathcal{Q}^{m_1 \times m_2}(K, n', x), \tag{119}$$

be an operation such that if $R \in \mathcal{R}_{m_1}(K, n_1, x)$, $U \in \mathcal{R}_{m_2}(K, n_2, x)$, and $V \in \mathcal{Q}^{m_1 \times m_2}(K, n', x)$, with entries r_i, u_j, v_j, respectively, $i = 1, 2, \ldots, \frac{m_1^2+1}{2}$, $j = 1, 2, \ldots, \frac{m_2^2+1}{2}$, $R \diamondtimes U = V$ if and only if $v_j = \alpha^{-1}(R) \diamonddot_r u_j$, α^{-1} being the inverse of the direct rhomboidal array-mph-rhotrix transformation, for all j, such that $\diamonddot_r$ is carried over from the set of mph-rhotrices to the set of rhomboidal arrays with an analogous definition. The operation rhomotimes $\diamondtimes$ *shall yield the tensor product of mph-rhotrices in $\mathcal{R}_{m_1}(K, n_1, x)$ and $\mathcal{R}_{m_2}(K, n_2, x)$.*

Definition 3.21. *Each element of the set $\mathcal{Q}^{m_1 \times m_2}(K, n', x)$ is a* tensor quasi-*mph*-rhotrix *or Qmph*-rhotrix *which has dimensions $m_1 \times m_2$ and entries in $K[x_1, \ldots, x_{n'}]$, where $n' = \max\{n_1, n_2\}$.*

Example 3.8. *Consider $A \in \mathcal{Q}^{1\times1}(K, n', x), B \in \mathcal{Q}^{1\times3}(K, n', x), C \in \mathcal{Q}^{3\times1}(K, n', x), D \in \mathcal{Q}^{3\times3}(K, n', x)$, and $E \in \mathcal{Q}^{3\times5}(K, n', x)$, which are defined by*

$$A = \langle\ a_1\ \rangle, \tag{120}$$

$$B = \left\langle \begin{array}{ccc} & b_1 & \\ b_2 & b_3 & b_4 \\ & b_5 & \end{array} \right\rangle, \tag{121}$$

$$C = \left\langle \begin{array}{ccc} & c_1 & \\ c_2 & c_3 & c_4 \\ & c_5 & \end{array} \right\rangle, \tag{122}$$

$$
D = \left\langle \begin{array}{ccccccccc}
 & & & & d_1 & & & & \\
 & & & d_2 & d_3 & d_4 & & & \\
 & & & & d_5 & & & & \\
 & d_6 & & & d_7 & & & d_8 & \\
d_9 & d_{10} & d_{11} & d_{12} & d_{13} & d_{14} & d_{15} & d_{16} & d_{17} \\
 & d_{18} & & & d_{19} & & & d_{20} & \\
 & & & & d_{21} & & & & \\
 & & & d_{22} & d_{23} & d_{24} & & & \\
 & & & & d_{25} & & & &
\end{array} \right\rangle, \tag{123}
$$

$$
E = \left\langle \begin{array}{ccccccccccccccc}
 & & & & & & & e_1 & & & & & & & \\
 & & & & & & e_2 & e_3 & e_4 & & & & & & \\
 & & & & & e_5 & e_6 & e_7 & e_8 & e_9 & & & & & \\
 & & & & & & e_{10} & e_{11} & e_{12} & & & & & & \\
 & & & & & & & e_{13} & & & & & & & \\
 & & e_{14} & & & & & e_{15} & & & & & e_{16} & & \\
 & e_{17} & e_{18} & e_{19} & & & e_{20} & e_{21} & e_{22} & & & e_{23} & e_{24} & e_{25} & \\
e_{26} & e_{27} & e_{28} & e_{29} & e_{30} & e_{31} & e_{32} & e_{33} & e_{34} & e_{35} & e_{36} & e_{37} & e_{38} & e_{39} & e_{40} \\
 & e_{41} & e_{42} & e_{43} & & & e_{44} & e_{45} & e_{46} & & & e_{47} & e_{48} & e_{49} & \\
 & & e_{50} & & & & & e_{51} & & & & & e_{52} & & \\
 & & & & & & & e_{53} & & & & & & & \\
 & & & & & & e_{54} & e_{55} & e_{56} & & & & & & \\
 & & & & & e_{57} & e_{58} & e_{59} & e_{60} & e_{61} & & & & & \\
 & & & & & & e_{62} & e_{63} & e_{64} & & & & & & \\
 & & & & & & & e_{65} & & & & & & &
\end{array} \right\rangle. \tag{124}
$$

$A, B, C, D,$ and E are Qmph-rhotrices with dimensions 1×1, 1×3, 3×1, 3×3, and 3×5, respectively.

3.2.3 Tensor products of algebras

Definition 3.22. *The notation*

$$
\mathbf{T_C} \quad := \quad \mathbf{gCRAPol}(K, m, n_1, n_2, x) \otimes \mathbf{gCRAPol}(K, m, n_1', n_2, x), \tag{125}
$$

denotes the tensor product of algebras $\mathbf{gCRAPol}(K, m, n_1, n_2, x)$ and $\mathbf{gCRAPol}(K, m, n_1', n_2, x)$ *with multiplication defined for all $R_1 \Diamond R_2, S_1 \Diamond S_2 \in \mathbf{T_C}$ as*

$$
(R_1 \Diamond R_2)(S_1 \Diamond S_2) = (R_1 \Diamond_C S_1) \Diamond (R_2 \Diamond_C S_2). \tag{126}
$$

Theorem 3.10. *$\mathbf{T_C}$ is an associative commutative unitary $K[x_1, \ldots, x_{n_2}]$-algebra.*

Proof. By Remark 2.4, $\mathbf{T_C}$ is a $K[x_1, \ldots, x_{n_2}]$-algebra. Let $A_1 \Diamond A_1', A_2 \Diamond A_2', A_3 \Diamond A_3' \in \mathbf{T_C}$. Then,

$$
\begin{aligned}
(A_1 \mathbin{\Diamond} A'_1)[(A_2 \mathbin{\Diamond} A'_2)(A_3 \mathbin{\Diamond} A'_3)] &= (A_1 \mathbin{\Diamond} A'_1)[(A_2 \mathbin{\diamond}_C A_3) \mathbin{\Diamond} (A'_2 \mathbin{\diamond}_C A'_3)] \\
&= [A_1 \mathbin{\diamond}_C (A_2 \mathbin{\diamond}_C A_3)] \mathbin{\Diamond} [A'_1 \mathbin{\diamond}_C (A'_2 \mathbin{\diamond}_C A'_3)] \\
&= [(A_1 \mathbin{\diamond}_C A_2) \mathbin{\diamond}_C A_3] \mathbin{\Diamond} [(A'_1 \mathbin{\diamond}_C A'_2) \mathbin{\diamond}_C A'_3] \\
&\qquad \text{(by Lemma 3.5)} \\
&= [(A_1 \mathbin{\diamond}_C A_2) \mathbin{\Diamond} (A'_1 \mathbin{\diamond}_C A'_2)](A_3 \mathbin{\Diamond} A'_3) \\
&= [(A_1 \mathbin{\Diamond} A'_1)(A_2 \mathbin{\Diamond} A'_2)](A_3 \mathbin{\Diamond} A'_3).
\end{aligned}
$$

Thus, $\mathbf{T_C}$ is associative (Definition 2.5). Now,

$$
\begin{aligned}
(A_1 \mathbin{\Diamond} A'_1)(A_2 \mathbin{\Diamond} A'_2) &= (A_1 \mathbin{\diamond}_C A_2) \mathbin{\Diamond} (A'_1 \mathbin{\diamond}_C A'_2) \\
&= (A_2 \mathbin{\diamond}_C A_1) \mathbin{\Diamond} (A'_2 \mathbin{\diamond}_C A'_1) \qquad \text{(Lemma 3.5)} \\
&= (A_2 \mathbin{\Diamond} A'_2)(A_1 \mathbin{\Diamond} A'_1)
\end{aligned}
$$

Hence, $\mathbf{T_C}$ is also commutative (Definition 2.6). Moreover, by Definition 3.9,

$$
\mathbf{1}_{\mathcal{R}m(K,n,x)} \mathbin{\Diamond} \mathbf{1}_{\mathcal{R}m(K,n,x)} \in \mathbf{T_C}.
$$

Then,

$$
\begin{aligned}
(A_1 \mathbin{\Diamond} A'_1)(\mathbf{1}_{\mathcal{R}m(K,n,x)} \mathbin{\Diamond} \mathbf{1}_{\mathcal{R}m(K,n,x)}) &= (A_1 \mathbin{\diamond}_C \mathbf{1}_{\mathcal{R}m(K,n,x)}) \mathbin{\Diamond} (A'_1 \mathbin{\diamond}_C \mathbf{1}_{\mathcal{R}m(K,n,x)}) \\
&= A_1 \mathbin{\Diamond} A'_1 \qquad \text{(Definition 2.1(B2))} \\
&= (\mathbf{1}_{\mathcal{R}m(K,n,x)} \mathbin{\diamond}_C A_1) \mathbin{\Diamond} (\mathbf{1}_{\mathcal{R}m(K,n,x)} \mathbin{\diamond}_C A'_1) \\
&\qquad \text{(Definition 2.1(B2))} \\
&= (\mathbf{1}_{\mathcal{R}m(K,n,x)} \mathbin{\Diamond} \mathbf{1}_{\mathcal{R}m(K,n,x)})(A_1 \mathbin{\Diamond} A'_1).
\end{aligned}
$$

Thus, $\mathbf{T_C}$ is unitary. □

Definition 3.23. *The notation*

$$
\mathbf{T_{C0}} := \mathbf{CRAPol}(K,m,n,x) \otimes \mathbf{CRAPol}(K,m,n,x), \tag{127}
$$

denotes the tensor product of ***CRAPol***(K,m,n,x) *and itself* with multiplication defined for all $R_1 \mathbin{\Diamond} R_2, S_1 \mathbin{\Diamond} S_2 \in \mathbf{T_{C0}}$ as

$$
(R_1 \mathbin{\Diamond} R_2)(S_1 \mathbin{\Diamond} S_2) = (R_1 \mathbin{\diamond}_C S_1) \mathbin{\Diamond} (R_2 \mathbin{\diamond}_C S_2). \tag{128}
$$

Theorem 3.11. *If* $n_1 = n'_1 = n_2 = n$*, then* $\mathbf{T_C} = \mathbf{T_{C0}}$.

Proof. If $n_1 = n'_1 = n_2 = n$, then by Theorem 3.6,

$$
\begin{aligned}
\mathbf{T_C} &= \mathbf{gCRAPol}(K,m,n,n,x) \otimes \mathbf{gCRAPol}(K,m,n,n,x) \qquad (129) \\
&= \mathbf{CRAPol}(K,m,n,x) \otimes \mathbf{CRAPol}(K,m,n,x) = \mathbf{T_{C0}}. \qquad (130)
\end{aligned}
$$

This completes the proof. □

Corollary 3.1. $\mathbf{T_{C0}}$ *is an associative commutative unitary* $K[x_1, \dots, x_n]$*-algebra.*

Proof. By Theorem 3.11, $\mathbf{T_{C0}} = \mathbf{T_C}$ if $n_1 = n_1' = n_2 = n$. Hence, $\mathbf{T_{C0}}$ is an associative commutative unitary $K[x_1, \dots, x_n]$-algebra by Theorem 3.10. □

Corollary 3.2. $\mathbf{T_C} = \mathbf{T_C}^{op}$ *and* $\mathbf{T_{C0}} = \mathbf{T_{C0}}^{op}$*.*

Proof. Since $\mathbf{T_C}$ is commutative (Theorem 3.10) and $\mathbf{T_C} = \mathbf{T_{C0}}$ if $n_1 = n_1' = n_2 = n$ (Theorem 3.11), $\mathbf{T_C} = \mathbf{T_C}^{\mathrm{op}}$ and $\mathbf{T_{C0}} = \mathbf{T_{C0}}^{\mathrm{op}}$ by Remark 2.3. □

Definition 3.24. *The notation*

$$\mathbf{T_N} := \boldsymbol{gNRAPol}(K, m, n_1, n_2, x) \otimes \boldsymbol{gNRAPol}(K, m, n_1', n_2, x). \tag{131}$$

denotes the tensor product of algebras **gNRAPol**(K, m, n_1, n_2, x) and **gNRAPol**(K, m, n_1', n_2, x) with multiplication defined for all $R_1 \boxtimes R_2, S_1 \boxtimes S_2 \in \mathbf{T_N}$ as

$$(R_1 \boxtimes R_2)(S_1 \boxtimes S_2) = (R_1 \boxtimes_N S_1) \boxtimes (R_2 \boxtimes_N S_2). \tag{132}$$

Theorem 3.12. $\mathbf{T_N}$ *is an associative noncommutative unitary* $K[x_1, \dots, x_{n_2}]$*-algebra.*

Proof. By Remark 2.4, $\mathbf{T_N}$ is a $K[x_1, \dots, x_{n_2}]$-algebra. Let $A_1 \boxtimes A_1', A_2 \boxtimes A_2', A_3 \boxtimes A_3' \in \mathbf{T_N}$. Then,

$$\begin{aligned}
(A_1 \boxtimes A_1')[(A_2 \boxtimes A_2')(A_3 \boxtimes A_3')] &= (A_1 \boxtimes A_1')[(A_2 \boxtimes_N A_3) \boxtimes (A_2' \boxtimes_N A_3')] \\
&= [A_1 \boxtimes_N (A_2 \boxtimes_N A_3)] \boxtimes [A_1' \boxtimes_N (A_2' \boxtimes_N A_3')] \\
&= [(A_1 \boxtimes_N A_2) \boxtimes_N A_3] \boxtimes [(A_1' \boxtimes_N A_2') \boxtimes_N A_3'] \\
&\qquad \text{(by Lemma 3.6)} \\
&= [(A_1 \boxtimes_N A_2) \boxtimes (A_1' \boxtimes_N A_2')](A_3 \boxtimes A_3') \\
&= [(A_1 \boxtimes A_1')(A_2 \boxtimes A_2')](A_3 \boxtimes A_3').
\end{aligned}$$

Thus, $\mathbf{T_N}$ is associative (Definition 2.5). Now,

$$\begin{aligned}
(A_1 \boxtimes A_1')(A_2 \boxtimes A_2') &= (A_1 \boxtimes_N A_2) \boxtimes (A_1' \boxtimes_N A_2') \\
&\neq (A_2 \boxtimes_N A_1) \boxtimes (A_2' \boxtimes_N A_1') \qquad \text{(Lemma 3.6)} \\
&= (A_2 \boxtimes A_2')(A_1 \boxtimes A_1')
\end{aligned}$$

Hence, $\mathbf{T_N}$ is noncommutative (Definition 2.6). Moreover, by Definition 3.9,

$$\mathbf{I}_{\mathcal{R}m(K,n,x)} \boxtimes \mathbf{I}_{\mathcal{R}m(K,n,x)} \in \mathbf{T_N}.$$

Then,

$$\begin{aligned}(A_1 \diamondtimes A_1')(\mathbf{I}_{\mathcal{R}m(K,n,x)} \diamondtimes \mathbf{I}_{\mathcal{R}m(K,n,x)}) &= (A_1 \diamondtimes_N \mathbf{I}_{\mathcal{R}m(K,n,x)}) \diamondtimes (A_1' \diamondtimes_N \mathbf{I}_{\mathcal{R}m(K,n,x)}) \\ &= A_1 \diamondtimes A_1' \qquad \text{(Definition 2.1(B2))} \\ &= (\mathbf{1}_{\mathcal{R}m(K,n,x)} \diamondtimes_N A_1) \diamondtimes (\mathbf{1}_{\mathcal{R}m(K,n,x)} \diamondtimes_N A_1') \\ &\quad \text{(Definition 2.1(B2))} \\ &= (\mathbf{1}_{\mathcal{R}m(K,n,x)} \diamondtimes \mathbf{1}_{\mathcal{R}m(K,n,x)})(A_1 \diamondtimes A_1').\end{aligned}$$

Thus, $\mathbf{T_N}$ is unitary. □

Definition 3.25. *The notation*

$$\mathbf{T_{N0}} := \textbf{NRAPol}(K, m, n, x) \otimes \textbf{NRAPol}(K, m, n, x), \tag{133}$$

denotes the tensor product of **NRAPol**(K, m, n, x) and itself *with multiplication defined for all* $R_1 \diamondtimes R_2, S_1 \diamondtimes S_2 \in \mathbf{T_{N0}}$ *as*

$$(R_1 \diamondtimes R_2)(S_1 \diamondtimes S_2) = (R_1 \diamondtimes_N S_1) \diamondtimes (R_2 \diamondtimes_N S_2). \tag{134}$$

Theorem 3.13. *If* $n_1 = n_1' = n_2 = n$*, then* $\mathbf{T_N} = \mathbf{T_{N0}}$.

Proof. If $n_1 = n_1' = n_2 = n$, then by Theorem 3.9,

$$\begin{aligned}\mathbf{T_N} &= \textbf{gNRAPol}(K, m, n, n, x) \otimes \textbf{gNRAPol}(K, m, n, n, x) \qquad (135)\\ &= \textbf{NRAPol}(K, m, n, x) \otimes \textbf{NRAPol}(K, m, n, x) = \mathbf{T_{N0}}. \qquad (136)\end{aligned}$$

This completes the proof. □

Corollary 3.3. $\mathbf{T_{N0}}$ *is an associative noncommutative unitary* $K[x_1, \ldots, x_n]$*-algebra.*

Proof. By Theorem 3.13, $\mathbf{T_{N0}} = \mathbf{T_N}$ if $n_1 = n_1' = n_2 = n$. Hence, $\mathbf{T_{N0}}$ is an associative noncommutative unitary $K[x_1, \ldots, x_n]$-algebra by Theorem 3.12. □

4 Conclusion and recommendations

This study introduced novel concepts leading to the characterization of the newly-denoted algebras **CRAPol**(K, m, n, x), **NRAPol**(K, m, n, x), **gCRAPol**(K, m, n, x), and **gNRAPol**(K, m, n_1', n_2, x), and their tensor products. It has also investigated some of their key properties. With this, the authors recommend the examination of the substructures of the algebras presented, as well as extending the results to *hl*-rhotrices. Also, the authors suggest that the readers would critically explore the extensions and applications of the results found in this study.

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