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A combinatorial study of the generalized Leonardo polynomials

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Abstract: Let $\mathcal{GL}_n(x) = \mathcal{GL}_n^{a,b,T}(x)$ denote the sequence satisfying the recursion $\mathcal{GL}_n(x) = x\mathcal{GL}_{n-1}(x) + \mathcal{GL}_{n-2}(x) + T$ for $n \geq 2$, with $\mathcal{GL}_0(x) = a$ and $\mathcal{GL}_1(x) = bx$, where a, b and T are arbitrary. The $\mathcal{GL}_n(x)$ are referred to as generalized Leonardo polynomials. In this paper, we provide a combinatorial interpretation for $\mathcal{GL}_n(x)$ as a weight function on a certain class of linear tilings wherein the initial tiles are colored according to their type. We use our interpretation to provide combinatorial explanations of some identities for $\mathcal{GL}_n(x)$ which were previously found by various algebraic methods. Further, we are able to establish by combinatorial arguments several new identities for $\mathcal{GL}_n(x)$, among them, formulas for various sums of products and a Cassini-like relation.

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1 Introduction

Let $\mathcal{L}_n$ denote the n-th *Leonardo* number (see A001595 in [17]) defined recursively by

$$\mathcal{L}_n = \mathcal{L}_{n-1} + \mathcal{L}_{n-2} + 1, \qquad n \geq 2, \tag{1.1}$$

with initial conditions $\mathcal{L}_0 = \mathcal{L}_1 = 1$. The Leonardo numbers were apparently first introduced by Dijkstra in conjunction with his smoothsort algorithm [7, 8], and have subsequently been studied (see, e.g., [2, 5, 6]). Various extensions of $\mathcal{L}_n$ have been put forth, among them, the generalized Leonardo numbers defined by the recurrence

$$\mathcal{L}_{k,n} = \mathcal{L}_{k,n-1} + \mathcal{L}_{k,n-2} + k, \qquad n \geq 2, \tag{1.2}$$

with $\mathcal{L}_{k,0} = \mathcal{L}_{k,1} = 1$. Here, k is often assumed to be a fixed positive integer, but may also be taken to be a variable. Note that $\mathcal{L}_{k,n}$ provides a common generalization of the Leonardo and Fibonacci numbers, reducing to the former when $k = 1$ and to the latter when $k = 0$. The $\mathcal{L}_{k,n}$ were introduced by Kuhapatanakul and Chobsorn [11], who found several properties of this sequence extending those of $\mathcal{L}_n$. For other generalizations and/or analogues of $\mathcal{L}_n$, we refer the reader to [10, 12, 14, 15, 18, 21].

A further extension of $\mathcal{L}_{k,n}$ is given by the *generalized Leonardo polynomial* [1], denoted by $\mathcal{GL}_n(x) = \mathcal{GL}_n^{a,b,T}(x)$, and satisfying the recurrence

$$\mathcal{GL}_n(x) = x\mathcal{GL}_{n-1}(x) + \mathcal{GL}_{n-2}(x) + T, \qquad n \geq 2, \tag{1.3}$$

with $\mathcal{GL}_0(x) = a$ and $\mathcal{GL}_1(x) = bx$. The values of $\mathcal{GL}_n(x)$ for $2 \leq n \leq 5$ are given by

$$\begin{aligned}
&\mathcal{GL}_2(x) = a + bx^2 + T, \qquad \mathcal{GL}_3(x) = ax + bx(1 + x^2) + T(1 + x),\\
&\mathcal{GL}_4(x) = a(1 + x^2) + bx^2(2 + x^2) + T(2 + x + x^2),\\
&\mathcal{GL}_5(x) = ax(2 + x^2) + bx(1 + 3x^2 + x^4) + T(2 + 3x + x^2 + x^3).
\end{aligned}$$

The $\mathcal{GL}_n(x)$ are seen to reduce to $\mathcal{L}_{k,n}$ when $a = b = x = 1$, with $T = k$, and to polynomial extensions of the Fibonacci and Lucas numbers, upon taking $T = 0$ with various choices for the parameters a, b and x. In particular, if $a = b = 1$ and $T = 0$, then $\mathcal{GL}_n(x)$ reduces to $F_{n+1}(x)$, where $F_n(x)$ is the n-th Fibonacci polynomial defined recursively by $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$ for $n \geq 2$, with $F_0(x) = 0$ and $F_1(x) = 1$. Moreover, taking $a = 1$, $b = 2x - 1$ and $T = x$ in $\mathcal{GL}_n(x)$ yields the single variable Leonardo polynomial generalization studied by Prasad and Kumari [13]. Still more general sequences that extend $\mathcal{GL}_n(x)$ have been considered in [19, 20], where T is replaced by a function of n (typically a polynomial in n) and/or the order of the recurrence is arbitrary.

The $\mathcal{GL}_n(x)$ were introduced by Abd-Elhameed *et al.* [1], who found several relations involving the sequence employing various algebraic methods. Among the properties established were explicit formulas for $\mathcal{GL}_n(x)$ in terms of binomial coefficients and Fibonacci/Lucas polynomials as well as expressions for the product of $\mathcal{GL}_n(x)$ with other sequences and some related definite integrals. In this paper, we shall introduce a combinatorial framework in terms of colored tilings for $\mathcal{GL}_n(x)$ by which one may understand several of the results from [1] as well as derive additional formulas. See [16] for a comparable approach to $\mathcal{L}_{k,n}$ and also [3], where a combinatorial interpretation is developed for the Leonardo p-numbers introduced by Tan and Leung [21].

The organization of this paper is as follows. In the next section, we describe a combinatorial interpretation for $\mathcal{GL}_n(x)$ in terms of tilings. In the third section, we use this interpretation to give combinatorial proofs of some recent identities for $\mathcal{GL}_n(x)$ established in [1] by other methods. In the final section, we establish several new relations involving $\mathcal{GL}_n(x)$ by combinatorial arguments, which generalize identities for the Fibonacci and Leonardo numbers. Our arguments seemingly provide a more elementary, self-contained approach to those used to establish the results in [1], which were intricate in some cases and drew upon advanced concepts such as the Zeilberger algorithm and the Chebyshev inversion principle. Furthermore, our model allows for one to readily extend to $\mathcal{GL}_n(x)$ identities that are known for some of its special cases.

2 Combinatorial interpretation for $\mathcal{GL}_n^{a,b,T}(x)$

To provide a combinatorial interpretation for $\mathcal{GL}_n^{a,b,T}(x)$, we extend one given for the Leonardo numbers in [16]. A (linear) *tiling* is a covering of the numbers $1,\ldots,n$, written in a row, by rectangular pieces, called *tiles*, such that no number is covered by more than one tile and tiles are indistinguishable from one another outside of their length. A tile covering one number or two consecutive numbers is known as a *square* or *domino* and is denoted by s or d, respectively. The length of a tiling is the sum of the lengths of all of its tiles. Let $\mathcal{F}_n$ denote the set of tilings of length n consisting of squares and dominoes; see, e.g., [4, Chapter 1]. Note $|\mathcal{F}_n| = f_{n+1}$ for all $n \geq 0$, where f_n denotes the n-th Fibonacci number defined recursively by $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$, with $f_0 = 0$ and $f_1 = 1$. For example, if $n = 4$, then $\mathcal{F}_4 = \{d^2, ds^2, sds, s^2d, s^4\}$ and $|\mathcal{F}_4| = 5 = f_5$, where t^i denotes a sequence of i consecutive occurrences of a tile t within a tiling.

A *Leonardo* tiling (see [16]) is one consisting of squares, dominoes and a special tile, denoted by p, which satisfies the following: (i) p is a tile of variable length m for any $m \geq 2$, (ii) p need not occur, but if it does, must occur as the first tile. When the length of a special tile p is known, it will often be denoted by a subscript. Let $\mathcal{K}_n$ denote the set of Leonardo tilings of length n; note $|\mathcal{K}_n| = \mathcal{L}_n$ for all $n \geq 0$. For example, if $n = 4$, then

$$\mathcal{K}_4 = \{d^2,\, ds^2,\, sds,\, s^2d,\, s^4,\, p_2d,\, p_2s^2,\, p_3s,\, p_4\}$$

and $|\mathcal{K}_4| = 9 = \mathcal{L}_4$.

We now consider the following extension of the set $\mathcal{K}_n$.

Definition 2.1. *Let a, b and T be positive integers. By a generalized Leonardo tiling, we mean a colored member of $\mathcal{K}_n$ wherein a starting square, domino or special tile comes in one of a, b or T colors, respectively, with the remaining tiles each not receiving a color. Let $\mathcal{T}_n = \mathcal{T}_n(a,b,T)$ denote the set of all generalized Leonardo tilings of length n.*

From the prior example above, it is seen that $\mathcal{T}_4$ has cardinality $2a + 3b + 4T$ for all a, b and T. We define a weighted sum on $\mathcal{T}_n$ as follows.

Definition 2.2. *Let the weight of a member $\pi \in \mathcal{T}_n$ be given by $x^{\mu(\pi)}$, where $\mu(\pi)$ denotes the number of squares in π. Let $w_n = w_n^{a,b,T}(x)$ for $n \geq 1$ denote the sum of the weights of all the members of $\mathcal{T}_n$, with $w_0 = a$*

Extending the prior example where $n = 4$, we have $w_4 = a(1+x^2)+bx^2(2+x^2)+T(2+x+x^2)$.

Notice that w_4 works out to $\mathcal{GL}_4^{a,b,T}(x)$ and this is not a coincidence.

Proposition 2.1. *We have* $w_n^{a,b,T}(x) = \mathcal{GL}_n^{a,b,T}(x)$ *for all* $n \geq 0$.

Proof. When $n = 0$ or 1, there is equality as both sequences are given by a and bx, respectively. To complete the proof, we show that w_n satisfies recurrence (1.3) for all $n \geq 2$. Consider the final piece within a member of $\mathcal{T}_n$. If this piece is a square or domino, then one gets respective contributions towards the weight of xw_{n-1} and w_{n-2}. If the final piece is a special tile, then it must be the only piece, and hence there are T possibilities in this case, each having weight $x^0 = 1$. Combining the cases above implies $w_n = xw_{n-1} + w_{n-2} + T$ for $n \geq 2$, as desired. □

We will make use of this combinatorial interpretation for $\mathcal{GL}_n^{a,b,T}(x)$ in the subsequent proofs. Note that the sum of the weights of all the members of $\mathcal{F}_n$ is given by $F_{n+1}(x)$, where the weight of $\pi \in \mathcal{F}_n$ is again defined as $x^{\mu(\pi)}$; see, e.g., [4, Section 9.4]. We now recall a couple of further sequences. The Lucas number ℓ_n for $n \geq 2$ is given recursively by $\ell_n = \ell_{n-1} + \ell_{n-2}$ for $n \geq 2$, with $\ell_0 = 2$ and $\ell_1 = 1$. Note ℓ_n enumerates the marked square-and-domino tilings of length n where a starting domino may be marked, the set of which will be denoted by $\mathcal{P}_n$. The n-th Lucas polynomial $L_n(x)$ satisfies the recurrence $L_n(x) = xL_{n-1}(x) + L_{n-2}(x)$ for $n \geq 2$, with $L_0(x) = 2$ and $L_1(x) = x$. Note $L_n(1) = \ell_n$ and $L_n(x) = F_{n+1}(x) + F_{n-1}(x)$ for all $n \geq 0$, where $F_{-1}(x) = 1$. Finally, we have that $L_n(x)$ gives the sum of the weights of the members of $\mathcal{P}_n$, where the weight is defined as before.

3 Combinatorial proofs of prior identities

In this section, we provide combinatorial explanations of some of the identities shown in [1]. The following result occurs as Theorems 2 and 3 in [1] and expresses $\mathcal{GL}_n(x)$ in terms of Fibonacci and Lucas polynomials.

Theorem 3.1. *If* $n \geq 1$*, then*

$$\mathcal{GL}_n^{a,b,T}(x) = (a-b)F_{n-1}(x) + bF_{n+1}(x) + T\sum_{p=1}^{n-1} F_p(x) \tag{3.1}$$

and

$$\begin{aligned}\mathcal{GL}_n^{a,b,T}(x) = bL_n(x) &+ T\sum_{p=0}^{\lfloor (n-3)/4 \rfloor} c_{n-4p-3}L_{n-4p-3}(x)\\ &+ \frac{1}{2}\sum_{p=1}^{\lfloor n/2 \rfloor} c_{n-2p}\left(T + (-1)^{p-1}(2a-4b+T)\right)L_{n-2p}(x),\end{aligned} \tag{3.2}$$

where $c_m = \begin{cases} 1, & \text{if } m > 0; \\ \frac{1}{2}, & \text{if } m = 0. \end{cases}$

Proof. To show (3.1), consider cases on the first piece within $\pi \in \mathcal{T}_n$. There are aF_{n-1} and $bxF_n = b(F_{n+1} - F_{n-1})$ possibilities if the first piece of π is a domino or a square, respectively, where $F_n = F_n(x)$. If π starts with a special tile of length $n-p$ for some $0 \leq p \leq n-2$, then there are TF_{p+1} possibilities as the remaining tiles comprise a member of $\mathcal{F}_p$ with squares weighted by x. Considering all possible p accounts for the summation on the right side and completes the proof of (3.1).

To show (3.2), we compare the coefficients of a, b and T on both sides and make use of (3.1). Equating coefficients of b, we need to establish the identity

$$F_{n+1} - F_{n-1} = xF_n = L_n - 2 \sum_{p=1}^{\lfloor n/2 \rfloor} (-1)^{p-1} c_{n-2p} L_{n-2p}, \qquad n \geq 1, \tag{3.3}$$

where $L_n = L_n(x)$. By the fact $L_n = xF_n + 2F_{n-1}$, which may be realized by considering whether or not a member of $\mathcal{P}_n$ starts with a square, we have that (3.3) is equivalent to (3.5) below, which is given a combinatorial proof. Note that equating coefficients of a in (3.2) also leads to (3.5).

Finally, comparing the coefficients of T in (3.2), we need to show

$$\sum_{p=0}^{n-2} F_{p+1} = \sum_{r=0}^{\lfloor (n-2)/4 \rfloor} c_{n-4r-2} L_{n-4r-2} + \sum_{r=0}^{\lfloor (n-3)/4 \rfloor} c_{n-4r-3} L_{n-4r-3}, \qquad n \geq 1. \tag{3.4}$$

Upon letting $n = 4m + i$, where $m \geq 0$ and $0 \leq i \leq 3$, we have that (3.4) is equivalent to the following four identities in m:

$$\begin{aligned}
\sum_{p=0}^{4m-2} F_{p+1} &= \sum_{r=0}^{m-1} (L_{4m-4r-2} + L_{4m-4r-3}), \\
\sum_{p=0}^{4m-1} F_{p+1} &= \sum_{r=0}^{m-1} (L_{4m-4r-1} + L_{4m-4r-2}), \\
\sum_{p=0}^{4m} F_{p+1} &= 1 + \sum_{r=0}^{m-1} (L_{4m-4r} + L_{4m-4r-1}), \\
\sum_{p=0}^{4m+1} F_{p+1} &= 1 + x + \sum_{r=0}^{m-1} (L_{4m-4r+1} + L_{4m-4r}).
\end{aligned}$$

These identities each follow from applying $L_n + L_{n+1} = F_{n-1} + F_n + F_{n+1} + F_{n+2}$ for $n \geq 1$. This last fact follows from two applications of the relation $L_n = F_{n-1} + F_{n+1}$, which can be seen combinatorially by considering whether or not a member of $\mathcal{P}_n$ starts with a marked domino. This completes the proof of (3.4), and hence of (3.2). $\square$

Lemma 3.1. *If $n \geq 1$, then*

$$F_{n-1}(x) = \sum_{p=1}^{\lfloor n/2 \rfloor} (-1)^{p-1} c_{n-2p} L_{n-2p}(x), \qquad n \geq 1, \tag{3.5}$$

where $c_m = \begin{cases} 1, & \text{if } m > 0; \\ \frac{1}{2}, & \text{if } m = 0. \end{cases}$

Proof. First assume n is even and we consider further subcases mod 4. Suppose $n = 4m + 2$ for some $m \geq 0$. Then (3.5) in this case is equivalent to

$$F_{4m+1} - 1 = L_{4m} - L_{4m-2} + \cdots - L_2, \qquad m \geq 0. \tag{3.6}$$

Note that (3.6) is trivial if $m = 0$, so assume $m \geq 1$. Let $\mathcal{A} = \mathcal{A}_m$ denote the set of "marked" Lucas tilings of length $4m$ wherein a tiling ending in exactly k dominoes for some $0 \leq k \leq 2m$ may have its final i dominoes marked for some $0 \leq i \leq k$, with the exception that $i < k$ if $k = 2m$. Define the (signed) weight of $\lambda \in \mathcal{A}$ ending in exactly i marked d's by $(-1)^i x^{\mu(\lambda)}$, where $\mu(\lambda)$ denotes the number of squares in λ. Note that an initial domino within a member of $\mathcal{A}$ may be marked, like in members of $\mathcal{P}_n$, though this marking is to be distinct from the one used to indicate a string of terminal dominoes. In particular, the member of $\mathcal{A}$ ending in $2m - 1$ marked d's and starting with a marked d will have $i = 2m - 1$.

Then the right-hand side of (3.6) is seen to give the sum of the weights of all the members of $\mathcal{A}$. Let $\mathcal{A}^{(i)}$ denote the subset of $\mathcal{A}$ whose members end in i marked d's. Define an involution on $\mathcal{A}$ by either unmarking the leftmost marked d within members of $\mathcal{A}^{(i)}$ where i is odd or performing the reverse operation on members of $\mathcal{A}^{(i)}$ where i is even and the leftmost marked d is directly preceded by another d. The survivors of this involution consist of those members of $\cup_{j=0}^{m-1} \mathcal{A}^{(2j)}$ where the leftmost marked d is preceded by an s if $j > 0$ or that end in s if $j = 0$.

Note that each survivor has positive sign and the sum of their weights is given by $x(L_{4m-1} + L_{4m-5} + \cdots + L_3)$. Thus, to establish (3.6), we need to show

$$F_{4m+1} = 1 + x(L_{4m-1} + L_{4m-5} + \cdots + L_3), \qquad m \geq 1. \tag{3.7}$$

To do so, suppose $\lambda \in \mathcal{P}_{4i-1}$ for some $i \in [m]$. If λ does not start with a marked d, then we append sd^{2m-2i} to λ to obtain a member of $\mathcal{F}_{4m}$ ending in an even number of d's. If λ starts with a marked d, then we delete this d and append $sd^{2m-2i+1}$ to obtain a tiling ending in an odd number of d's. Allowing i to vary implies that all members of $\mathcal{F}_{4m}$, except for d^{2m}, arise in this manner, with the weight of these tilings being given by $x(L_{4m-1} + L_{4m-5} + \cdots + L_3)$. This implies (3.7) and completes the proof of (3.6).

Now suppose $n = 4m$ for some $m \geq 1$, and we need to show

$$F_{4m-1} + 1 = L_{4m-2} - L_{2m-4} + \cdots + L_2, \qquad m \geq 1. \tag{3.8}$$

We may assume $m \geq 2$ and let $\mathcal{B} = \mathcal{B}_m$ denote the set of "marked" Lucas tilings of length $4m-2$ in which a tiling may end in exactly i marked d's for some $0 \leq i \leq 2m - 2$. With the weights defined as before, we define a sign-changing involution on $\mathcal{B}$ comparable to the one defined on $\mathcal{A}$ above which pairs members of $\mathcal{B}$ where i is odd with those where i is even for $0 \leq i \leq 2m - 3$, setting aside all members of $\mathcal{B}$ for which $i = 2m - 2$. Considering the weight of the set of survivors, we then need to show

$$F_{4m-1} = x^2 + 1 + x(L_{4m-3} + L_{4m-7} + \cdots + L_5), \qquad m \geq 2. \tag{3.9}$$

Upon reasoning as before, there are $x(L_{4m-3} + \cdots + L_5)$ members of $\mathcal{F}_{4m-2}$ that end in exactly j d's for some $0 \leq j \leq 2m - 3$. To this, we add $x^2 + 1$, which accounts for the weight of the members of $\mathcal{F}_{4m-2}$ ending in $2m - 2$ or $2m - 1$ d's. This implies (3.9) and completes the proof of (3.8).

A comparable argument, which we quickly describe, applies if n is odd. Letting $n = 2m+1$, formula (3.5) in this case is equivalent to

$$F_{2m} = L_{2m-1} - L_{2m-3} + \cdots + (-1)^{m-1} L_1. \tag{3.10}$$

One may then define a sign-changing involution comparable to before based on the right-hand side of (3.10). Considering the weight of the set of survivors of this involution leads to the identity

$$F_{2m} = \frac{1-(-1)^m}{2} x + x \sum_{i=0}^{\lfloor (m-2)/2 \rfloor} L_{2m-2-4i}, \tag{3.11}$$

which can be shown by appropriately modifying the arguments given above for (3.7) and (3.9). This yields the odd case of (3.5) and completes the proof. □

Remark: For a different combinatorial proof of (3.10) in the case $x = 1$ that does not use an involution, see [4, Identity 55]. Note that (3.7) and (3.9) also follow from applying the relation $L_n = F_{n+1} + F_{n-1}$ to each Lucas summand and then recalling the fact $F_{2m+1} = 1 + x(F_{2m} + F_{2m-2} + \cdots + F_2)$. See [4, Identity 12] for the combinatorial proof of the $x = 1$ case of this formula for F_{2m+1}. In a similar manner, formula (3.11) can be obtained using $F_{2m} = x(F_{2m-1} + F_{2m-3} + \cdots + F_1)$; see [4, Identity 2] for the $x = 1$ case.

Let $z_m = z(z+1)\cdots(z+m-1)$ if $m \geq 1$, with $z_0 = 1$. Further, we define $z_{-1} = \frac{1}{z-1}$ and $z_{-2} = \frac{1}{(z-1)(z-2)}$. The following explicit formula for $\mathcal{GL}_n^{a,b,T}(x)$ in terms of the products z_m corresponds to [1, Theorem 1].

Theorem 3.2. *If* $n \geq 1$, *then*

$$\begin{aligned} \mathcal{GL}_n^{a,b,T}(x) = &\sum_{r=0}^{\lfloor n/2 \rfloor} \frac{((n-2r+1)(ar+b(n-2r)) + r(n-r)T)\,(n-2r+2)_{r-2}}{r!} x^{n-2r} \\ &+ T \sum_{r=1}^{\lfloor (n-1)/2 \rfloor} \frac{(n-2r+1)_{r-1}}{(r-1)!} x^{n-2r-1}. \end{aligned} \tag{3.12}$$

Proof. By the definition of the product z_m, one may rewrite (3.12) equivalently as

$$\begin{aligned} \mathcal{GL}_n^{a,b,T}(x) &= a \sum_{r=1}^{\lfloor n/2 \rfloor} \frac{(n-2r+1)_{r-1}}{(r-1)!} x^{n-2r} + b \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{(n-2r)_r}{r!} x^{n-2r} \\ &\quad + T \sum_{r=1}^{\lfloor n/2 \rfloor} \frac{(n-2r+2)_{r-1}}{(r-1)!} x^{n-2r} + T \sum_{r=1}^{\lfloor (n-1)/2 \rfloor} \frac{(n-2r+1)_{r-1}}{(r-1)!} x^{n-2r-1} \\ &= a \sum_{r=1}^{\lfloor n/2 \rfloor} \binom{n-r-1}{r-1} x^{n-2r} + b \sum_{r=0}^{\lfloor n/2 \rfloor} \binom{n-r-1}{r} x^{n-2r} \\ &\quad + T \sum_{r=1}^{\lfloor n/2 \rfloor} \binom{n-r}{r-1} x^{n-2r} + T \sum_{r=1}^{\lfloor (n-1)/2 \rfloor} \binom{n-r-1}{r-1} x^{n-2r-1}. \end{aligned} \tag{3.13}$$

We now provide a combinatorial explanation of (3.13). Let $\mathcal{T}_n = \mathcal{T}_n^{(a)} \cup \mathcal{T}_n^{(b)} \cup \mathcal{T}_n^{(T)}$, where $\mathcal{T}_n^{(a)}$, $\mathcal{T}_n^{(b)}$ and $\mathcal{T}_n^{(T)}$ denote the subsets of $\mathcal{T}_n$ whose members start with a domino, square or special tile, respectively. Hence, the coefficient of a, b or T in the expression for $\mathcal{GL}_n^{a,b,T}(x)$ gives the sum of the weights of the members of $\mathcal{T}_n^{(a)}$, $\mathcal{T}_n^{(b)}$ or $\mathcal{T}_n^{(T)}$, respectively, without regard to the color assigned to the initial tile. First, suppose $\pi \in \mathcal{T}_n^{(a)}$ and let r denote the total number of d's in π. Then π contains $n-2r$ squares, and thus there are $\binom{n-2r+r-1}{r-1} = \binom{n-r-1}{r-1}$ possibilities, as the first piece must be a d. Each such π has weight x^{n-2r} and thus considering all possible $1 \leq r \leq \lfloor n/2 \rfloor$ accounts for the first summation in (3.13). If $\pi \in \mathcal{T}_n^{(b)}$ and r again denotes the number of d's in π, then there are $\binom{n-2r-1+r}{r} = \binom{n-r-1}{r}$ possibilities, as π must start with an s in this case. Considering all $0 \leq r \leq \lfloor n/2 \rfloor$ then accounts for the second summation in (3.13).

Now suppose $\pi \in \mathcal{T}_n^{(T)}$ and contains exactly i dominoes and j squares. Then π starting with a special tile implies $0 \leq j \leq n-2$ and $0 \leq i \leq \lfloor \frac{n-j-2}{2} \rfloor$. There are $\binom{j+i}{i}$ such π, and considering all possible j and i implies that the weight w of the (uncolored) members of $\mathcal{T}_n^{(T)}$ is given by

$$w = \sum_{j=0}^{n-2} x^j \sum_{i=0}^{\lfloor \frac{n-j-2}{2} \rfloor} \binom{j+i}{i} = \sum_{j=0}^{n-2} \binom{j + \lfloor \frac{n-j}{2} \rfloor}{j+1} x^j = \sum_{j=2}^{n} \binom{n-j+\lfloor \frac{j}{2} \rfloor}{\lfloor \frac{j-2}{2} \rfloor} x^{n-j}, \tag{3.14}$$

where we have used in the second equality the parallel summation formula for binomial coefficients (see, e.g., [9, p. 174] and [4, p. 68] for a combinatorial proof). Letting $j = 2r$ or $j = 2r+1$ for some $r \geq 1$ in (3.14) implies

$$w = \sum_{r=1}^{\lfloor n/2 \rfloor} \binom{n-r}{r-1} x^{n-2r} + \sum_{r=1}^{\lfloor (n-1)/2 \rfloor} \binom{n-r-1}{r-1} x^{n-2r-1}.$$

Thus, the weight w is accounted for by the final two sums on the right-hand side of (3.13), which completes the proof of (3.12). $\square$

The final result in this section gives formulas for the $x = 1$ case of $\mathcal{GL}_n^{a,b,T}(x)$ in terms of Fibonacci and Lucas numbers and corresponds to [1, Corollaries 5 and 6].

Theorem 3.3. *Let $U_n = \mathcal{GL}_n^{a,b,T}(1)$. If $n \geq 1$, then*

$$y^{n+1} U_{n+1} = a + \sum_{i=0}^{n} y^i \left(y(A+T) f_{i+1} + y(-2a+b-T) f_{i-1} + (y-1) U_i \right) \tag{3.15}$$

and

$$y^{n+1} U_{n+1} = a + \sum_{i=0}^{n} y^i \left(\frac{y}{5}(-a+2b+T)\ell_{i+1} + \frac{y}{5}(4a-3b+T)\ell_{i-1} + (y-1) U_i \right). \tag{3.16}$$

Proof. By convention, we take $f_{-2} = -1$, $f_{-1} = 1$ and $\ell_{-1} = -1$. We first argue combinatorially

$$U_i = a f_{i-3} + b f_{i-2} + T f_{i-1} + U_{i-1}, \qquad i \geq 1. \tag{3.17}$$

Both sides of (3.17) are equal to b when $i = 1$ and $a+b+T$ when $i = 2$, so assume $\pi \in \mathcal{T}_i$, where $i \geq 3$. If π ends in s, then there are clearly U_{i-1} possibilities. Suppose now π consists of a single special tile or is expressible as $\pi = p\pi' d$, where π' is possibly empty and p denotes a special tile. Since p can range in length from 2 to $i-2$, it follows that there are $1 + f_1 + f_2 + \cdots + f_{i-3} = f_{i-1}$ possibilities for π without regard to the color of the first tile. Since each of these π belongs to $\mathcal{T}_i^{(T)}$, we then get a contribution of $T f_{i-1}$ from these π. Finally, suppose π ends in d and does not start with p. Then there are af_{i-3} possibilities if π starts with d and bf_{i-2} if π starts with s, which accounts for the first two terms on the right side of (3.17). Combining the prior cases completes the enumeration of $\pi \in \mathcal{T}_i$ and hence yields (3.17).

Thus, by (3.17) and the Fibonacci number recurrence, we have

$$\begin{aligned} U_i &= a(f_i - 2f_{i-2}) + bf_{i-2} + T(f_i - f_{i-2}) + U_{i-1} \\ &= (a+T)f_i + (-2a+b-T)f_{i-2} + U_{i-1}, \qquad i \geq 1. \end{aligned} \tag{3.18}$$

Multiplying both sides of (3.18) by y^i, and summing over $1 \leq i \leq n+1$, gives

$$\begin{aligned} \sum_{i=0}^{n+1} y^i U_i &= a + \sum_{i=1}^{n+1} y^i \left((a+T)f_i + (-2a+b-T)f_{i-2} + U_{i-1}\right) \\ &= a + \sum_{i=0}^{n} y^{i+1} \left((a+T)f_{i+1} + (-2a+b-T)f_{i-1} + U_i\right). \end{aligned}$$

Subtracting $\sum_{i=0}^{n} y^i U_i$ from both sides of the last equality, and combining a difference of sums on the right-hand side, now yields (3.15).

If $i \geq 0$, then we have

$$\begin{aligned} &(a+T)f_{i+1} + (-2a+b-T)f_{i-1} = a(f_{i+1} - 2f_{i-1}) + bf_{i-1} + T(f_{i+1} - f_{i-1}) \\ &= af_{i-2} + bf_{i-1} + Tf_i \\ &= \frac{a(4\ell_{i-1} - \ell_{i+1}) + b(2\ell_{i+1} - 3\ell_{i-1}) + T(\ell_{i+1} + \ell_{i-1})}{5} \\ &= \frac{(-a+2b+T)\ell_{i+1} + (4a-3b+T)\ell_{i-1}}{5}, \end{aligned} \tag{3.19}$$

where in the third equality, we have made use of the facts $4\ell_{i-1} - \ell_{i+1} = 5f_{i-2}$, $2\ell_{i+1} - 3\ell_{i-1} = 5f_{i-1}$ and $\ell_{i+1} + \ell_{i-1} = 5f_i$; for the third fact, see [4, Identity 34]. Note that each of these three relations may be afforded a combinatorial explanation using the tiling interpretations for f_n and ℓ_n. Substituting (3.19) into (3.15) yields (3.16) and completes the proof. $\square$

4 Some further identities for $\mathcal{GL}_n^{a,b,T}(x)$

We denote $\mathcal{GL}_n^{a,b,T}(x)$ here by v_n for $n \geq 0$, where x is assumed to be nonzero. In this section, we prove using our combinatorial model several new identities for v_n.

Proposition 4.1. *If $n \geq 0$, then*

$$v_{n+2} = bx^{n+2} + \sum_{i=0}^{n}(Tx^i + x^{n-i}v_i), \tag{4.1}$$

$$v_{2n+1} = bx + nT + x\sum_{i=1}^{n} v_{2i} \tag{4.2}$$

and

$$v_{2n} = a + nT + x\sum_{i=1}^{n} v_{2i-1}. \tag{4.3}$$

Proof. The three identities are seen to hold if $n = 0$, the first as $v_2 = a + bx^2 + T$, so we may assume $n \geq 1$. To show (4.1), we enumerate $\pi \in \mathcal{T}_{n+2}$ according to the position of the rightmost d, if it exists. First note that there are $x^{n-i}v_i$ possible π of the form $\pi = \pi' ds^{n-i}$, where $\pi' \in \mathcal{T}_i$ and $0 \leq i \leq n$, upon considering separately the $i = 0$ case. Summing over all i then accounts for the summation on the right side of (4.1). The bx^{n+2} term gives the weight of the all squares tiling s^{n+2}. The remaining members of $\mathcal{T}_{n+2}$ start with a special tile and contain no dominoes. Thus, they are of the form $\pi = p_i s^{n+2-i}$, where $2 \leq i \leq n+2$, and hence contribute $T\sum_{i=0}^{n} x^i$. Combining the prior cases on π gives (4.1).

To show (4.2), let $\pi \in \mathcal{T}_{2n+1}$ and we consider the position of the rightmost s, if it exists. If $\pi = \pi' sd^{n-i}$, where $1 \leq i \leq n$, then $\pi' \in \mathcal{T}_{2i}$ and there are $x\sum_{i=1}^{n} v_{2i}$ possibilities in all. The tiling sd^n has weight bx, whereas there are exactly n members of $\mathcal{T}_{2n+1}$ not containing an s, namely, those of the form $\pi = p_{2i+1}d^{n-i}$ for some $i \in [n]$, which are accounted for by the nT term. Combining the prior cases then yields (4.2), with a similar argument applying to (4.3). $\square$

We will henceforth denote the Fibonacci polynomial $F_n(x)$ simply by F_n. The v_n satisfy the following further recurrence relations.

Proposition 4.2. *If* $n, m \geq 1$, *then*

$$v_{n+m} = v_{m-1}F_n + v_m F_{n+1} + T\sum_{i=1}^{n} F_i. \tag{4.4}$$

If $n \geq \ell \geq 1$, *then*

$$v_{n+\ell} = T\frac{(1+x)^\ell - 1}{x} + \sum_{i=0}^{\ell} x^{\ell-i}\binom{\ell}{i} v_{n-i}. \tag{4.5}$$

Proof. Let $\pi \in \mathcal{T}_{n+m}$. First suppose $\pi = \alpha\beta$, where $\alpha \in \mathcal{T}_m$. Then there are F_{n+1} possibilities for β as $m > 0$, and hence the weight of such π is $v_m F_{n+1}$. If $\pi = \alpha d\beta$, where $\alpha \in \mathcal{T}_{m-1}$, then $|\beta| = n-1$ and we get a contribution of $v_{m-1}F_n$ towards the weight in this case. The remaining members of $\mathcal{T}_{n+m}$ have the form $\pi = p_i\pi'$, where $m+1 \leq i \leq n+m$. Then $0 \leq |\pi'| \leq n-1$, which implies that the weight of such π is given by $T\sum_{i=1}^{n} F_i$. Combining the previous cases for π yields (4.4).

Now let $\pi \in \mathcal{T}_{n+\ell}$, where $n \geq \ell \geq 1$. First suppose that the combined number of squares and dominoes in π is at least ℓ. Let i denote the number of dominoes amongst the final ℓ pieces of π. Then there are $x^{\ell-i}\binom{\ell}{i}$ possibilities for the final ℓ pieces, which have a total length of $\ell + i$. Thus, the remaining section of π comprises a member of $\mathcal{T}_{n-i}$ implying this section is nonempty if $i < n$. Hence, there are $x^{\ell-i}\binom{\ell}{i}v_{n-i}$ possibilities for π if $i < n$, with this formula also seen to hold if $i = n$ (which can occur only when $\ell = n$) as the tiling in question in this case is d^n. Considering all $0 \leq i \leq \ell$ then accounts for the summation on the right side of (4.5). On the other hand, if π does not contain at least ℓ squares and dominoes, and hence the total length of these pieces is at most $2\ell - 2$, then $n \geq \ell$ implies π must start with a special tile. Suppose π contains $k - i$ squares and i dominoes altogether for some $0 \leq i \leq k < \ell$. Then we have $\pi = p\pi'$, where p is a special tile of length $n + \ell - k - i$ (note $n + \ell - k - i \geq 2$) and π' contains i dominoes.

Considering all possible k and i then yields a contribution towards the weight of

$$T\sum_{k=0}^{\ell-1}\sum_{i=0}^{k}x^{k-i}\binom{k}{i}=T\sum_{k=0}^{\ell-1}(1+x)^k=T\frac{(1+x)^\ell-1}{x}.$$

Combining this case with the prior yields (4.5) and completes the proof. $\square$

Remark: Note that (4.1), (4.2) and (4.3) reduce respectively to Identities 1, 2 and 12 in [4] when $a=b=x=1$ and $T=0$. Formula (4.4) when $m=1$ is seen to be equivalent to (3.1).

Theorem 4.1. *If* $m\geq 1$, *then*

$$v_{2m}^2-v_{2m+1}v_{2m-1}=a^2+abx^2-b^2x^2+\frac{T(2a+2T+ax-2bx+bx^2+(x-2)v_{2m}+2v_{2m-1})}{x} \tag{4.6}$$

and

$$v_{2m-1}^2-v_{2m}v_{2m-2}=-a^2-abx^2+b^2x^2-\frac{T(2a+ax-2bx+bx^2-(x-2)v_{2m-1}-2v_{2m-2})}{x}. \tag{4.7}$$

Proof. We prove only (4.6), with a suitable modification of the argument applying to (4.7). Let $\mathcal{X}=\mathcal{T}_{2m}\times\mathcal{T}_{2m}$ and $\mathcal{Y}=\mathcal{T}_{2m+1}\times\mathcal{T}_{2m-1}$, where $m\geq 1$. The tail-swapping procedure (see, e.g., [4, Chapter 1]) provides a one-to-one correspondence between the subsets of $\mathcal{X}$ and $\mathcal{Y}$ for which it is defined. Assume when tail-swapping is applied to members of $\mathcal{X}$ that α and β within $(\alpha,\beta)\in\mathcal{X}$ are arranged so that β lies below α and is translated to the right by one square length. Then γ and δ within $(\gamma,\delta)\in\mathcal{Y}$ are such that δ lies directly below the middle $2m-1$ positions of γ. Note that tail-swapping cannot be performed if there are no common faults or if it results in a special tile or a colored square or domino occurring as the second tile within some tiling. Let $\mathcal{X}'$ and $\mathcal{Y}'$ denote the subsets of $\mathcal{X}$ and $\mathcal{Y}$ for which tail-swapping is not defined.

We first enumerate the members of $\mathcal{X}'$. One may verify that $\mathcal{X}'$ consists of those $(\alpha,\beta)\in\mathcal{X}$ for which one of the following holds: (a) $\alpha=\beta=d^m$, (b) $\alpha=s^2d^{m-1}$, $\beta=d^m$, (c) $\alpha=p_{2i}d^{m-i}$, $\beta=\beta'd^{m-i+1}$ or (d) $\alpha=\alpha'd^{m-i}$, $\beta=p_{2i}d^{m-i}$, where $\alpha'\in\mathcal{T}_{2i}$, $\beta'\in\mathcal{T}_{2i-2}$ and $i\in[m]$. Note that there are v_{2i-2} possibilities for the section β' in part (c) if $i\geq 2$, with this formula holding for $i=1$ as well since the tiling β' is empty in this case and is directly followed by a d (and hence there are $a=v_0$ possibilities). Combining these cases then implies that the weight of the members of $\mathcal{X}'$ is given by

$$\begin{aligned}a^2+abx^2+T\sum_{i=1}^{m}(v_{2i-2}+v_{2i})&=a^2+abx^2+T(a-v_{2m})+2T\sum_{i=1}^{m}v_{2i}\\&=a^2+abx^2+T(a-v_{2m})+\frac{2T(v_{2m+1}-mT-bx)}{x},\end{aligned}$$

by (4.2).

Next, observe that members $(\gamma,\delta)\in\mathcal{Y}'$ must have one of the following forms: (a) $\gamma=sd^m$, $\delta=sd^{m-1}$, (b) $\gamma=p_{2i+1}d^{m-i}$, $\delta=\delta'd^{m-i}$, with $i\in[m]$, or (c) $\gamma=\gamma'd^{m-i}$, $\delta=p_{2i+1}d^{m-i-1}$, with $i\in[m-1]$, where $\gamma'\in\mathcal{T}_{2i+1}$ and $\delta'\in\mathcal{T}_{2i-1}$. This implies members of $\mathcal{Y}'$ have weight

$$b^2x^2 + T\sum_{i=1}^{m} v_{2i-1} + T\sum_{i=1}^{m-1} v_{2i+1} = b^2x^2 - bxT + 2T\sum_{i=1}^{m} v_{2i-1}$$
$$= b^2x^2 - bxT + \frac{2T(v_{2m} - mT - a)}{x},$$

by (4.3).

By tail-swapping, we have that the quantity $v_{2m}^2 - v_{2m+1}v_{2m-1}$ is given by the difference in the weights of the sets $\mathcal{X}'$ and $\mathcal{Y}'$. This implies

$$\begin{aligned}
v_{2m}^2 - v_{2m+1}v_{2m-1} &= a^2 + abx^2 + T(a - v_{2m}) + \frac{2T(v_{2m+1} - mT - bx)}{x} \\
&\quad - \left(b^2x^2 - bxT + \frac{2T(v_{2m} - mT - a)}{x}\right) \\
&= a^2 + abx^2 - b^2x^2 + T(a - 2b + bx + 2a/x) - \frac{T(v_{2m+1} - v_{2m-1} - T)}{x} \\
&\quad + \frac{2T(v_{2m+1} - v_{2m})}{x} \\
&= a^2 + abx^2 - b^2x^2 + T(a - 2b + bx + 2a/x) + \frac{T^2}{x} \\
&\quad + \frac{T(v_{2m+1} - 2v_{2m} + v_{2m-1})}{x}.
\end{aligned}$$

Substituting $v_{2m+1} = xv_{2m} + v_{2m-1} + T$ into the last expression, and simplifying, yields (4.6). $\square$

In the next two results, we let $\alpha_n = \mathcal{GL}_n^{a_1,b_1,T_1}(x)$ and $\beta_n = \mathcal{GL}_n^{a_2,b_2,T_2}(x)$.

Theorem 4.2. *If* $n \geq 1$, *then*

$$\alpha_{2n}\beta_{2n-1} = \frac{a_1b_2x^2 - (2n-1)T_1T_2 + T_1(\beta_{2n} - a_2) + T_2(\alpha_{2n-1} - b_1x)}{x} + x\sum_{i=1}^{2n-1} \alpha_i\beta_i \quad (4.8)$$

and

$$\alpha_{2n+1}\beta_{2n} = \frac{a_2b_1x^2 - 2nT_1T_2 + T_1(\beta_{2n+1} - b_2x) + T_2(\alpha_{2n} - a_1)}{x} + x\sum_{i=1}^{2n} \alpha_i\beta_i. \quad (4.9)$$

Proof. To show (4.8), first note that the left-hand side enumerates the set of ordered pairs (λ_1, λ_2), where $\lambda_1 \in \mathcal{T}_{2n}(a_1, b_1, T_1)$ and $\lambda_2 \in \mathcal{T}_{2n-1}(a_2, b_2, T_2)$ and the weight of an ordered pair is defined as the product of the weights of its components. Consider the largest $i \in [2n-1]$, if it exists, such that λ_1 and λ_2 may be decomposed as $\lambda_1 = \gamma_1\delta_1$ and $\lambda_2 = \gamma_2\delta_2$, where γ_1 and γ_2 are both of length i. Then $\delta_1 = sd^{n-\binom{i+1}{2}}, \delta_2 = d^{n-\binom{i+1}{2}}$ if i is odd and $\delta_1 = d^{n-\frac{i}{2}}, \delta_2 = sd^{n-\frac{i}{2}-1}$ if i is even, by the maximality of i. Thus, the ordered pairs for which such an i exists have weight $x\sum_{i=1}^{2n-1} \alpha_i\beta_i$, where the factor of x accounts for the initial s in either δ_1 or δ_2.

So assume no such $i \in [2n-1]$ as described above exists. If neither λ_1 nor λ_2 contains a special tile, then we must have $\lambda_1 = d^n$ and $\lambda_2 = sd^{n-1}$, which gives a_1b_2x. Otherwise, at least one of λ_1, λ_2 must start with a special tile. If both λ_1 and λ_2 start with a special tile, then the lengths of these pieces must be different due to the assumption that no i as above exists. First suppose that λ_1 starts with a special tile of length strictly greater than that of the one in λ_2, if it occurs, with λ_2 also allowed to start with s or d. Then the special tile of λ_1 must be of even length, for otherwise, there would be an index i for which both λ_1 and λ_2 have prefixes of length i. Then we must have $\lambda_1 = p_{2j}d^{n-j}$ for some $j \in [n]$, which in turn implies $\lambda_2 = \gamma d^{n-j}$, where $\gamma \in \mathcal{T}_{2j-1}(a_2, b_2, T_2)$. Such pairs (λ_1, λ_2) in this case have weight

$$T_1 \sum_{j=1}^{n} \beta_{2j-1} = \frac{T_1(\beta_{2n} - T_2 n - a_2)}{x},$$

by (4.3).

Now suppose λ_2 starts with a special tile of length strictly greater than that of the one in λ_1, where λ_1 may also start with s or d. By similar reasoning as before, we must have $\lambda_2 = p_{2j-1}d^{n-j}$, and hence $\lambda_1 = \gamma d^{n-j+1}$, where $1 < j \leq n$ and $\gamma \in \mathcal{T}_{2j-2}(a_1, b_1, T_1)$. Considering all possible j gives in this case a contribution towards the weight of

$$T_2 \sum_{j=2}^{n} \alpha_{2j-2} = \frac{T_2(\alpha_{2n-1} - T_1(n-1) - b_1 x)}{x},$$

by (4.2). Combining the prior cases on (λ_1, λ_2) yields

$$\alpha_{2n}\beta_{2n-1} = a_1 b_2 x + \frac{T_1(\beta_{2n} - T_2 n - a_2)}{x} + \frac{T_2(\alpha_{2n-1} - T_1(n-1) - b_1 x)}{x} + x \sum_{i=1}^{2n-1} \alpha_i \beta_1,$$

which may be rewritten as (4.8). A similar argument implies

$$\alpha_{2n+1}\beta_{2n} = a_2 b_1 x + T_1 \sum_{j=1}^{n} \beta_{2j} + T_2 \sum_{j=1}^{n} \alpha_{2j-1} + x \sum_{i=1}^{2n} \alpha_i \beta_i.$$

Applying (4.2) and (4.3) then yields (4.9) and completes the proof. □

Our next result expresses the quantity $\alpha_m\beta_n - \alpha_n\beta_m$ in terms of Fibonacci polynomials.

Theorem 4.3. *If $n \geq m \geq 1$, then*

$$\alpha_m\beta_n - \alpha_n\beta_m = (-1)^m(\gamma_1 x - \gamma_2/x + \gamma_3)F_{n-m} - (\gamma_2 F_{m-1} + \gamma_3 x F_m) \sum_{i=1}^{n-m} F_i + \frac{F_{n-m}}{x}(\gamma_2(F_{m-1} - F_{m-2}) + \gamma_3 x(F_m - F_{m-1})), \quad (4.10)$$

where $\gamma_1 = a_1 b_2 - a_2 b_1$, $\gamma_2 = a_2 T_1 - a_1 T_2$ and $\gamma_3 = b_2 T_1 - b_1 T_2$.

Proof. We will prove only the even case of (4.10) and leave the odd as an exercise for the reader. We may assume $n > m$, as both sides of (4.10) are seen to be zero when $n = m$. Let $\mathcal{A} = \mathcal{T}_m(a_1, b_1, T_1) \times \mathcal{T}_n(a_2, b_2, T_2)$ and $\mathcal{B} = \mathcal{T}_n(a_1, b_1, T_1) \times \mathcal{T}_m(a_2, b_2, T_2)$. We apply the tail-swapping procedure to $(\lambda_1, \lambda_2) \in \mathcal{A}$ or $\mathcal{B}$, where it is assumed that the tiling λ_1 lies directly above λ_2 and is not staggered. Let S_1 and S_2 denote respectively the subsets of $\mathcal{A}$ and $\mathcal{B}$ for which tail-swapping cannot be performed, i.e., there exist no common faults of λ_1 and λ_2 when they are arranged as described. Then we have $\alpha_m\beta_n - \alpha_n\beta_m = \mathrm{w}(S_1) - \mathrm{w}(S_2)$, where $\mathrm{w}(S)$ denotes the weight of a subset of $\mathcal{A}$ or $\mathcal{B}$.

Suppose $(\lambda_1, \lambda_2) \in \mathcal{A}$ belongs to S_1. If neither λ_1 nor λ_2 starts with a special tile, then m even implies $\lambda_1 = d^t$, with λ_2 starting sd^t, where $t = m/2$. Otherwise, at least one of λ_1 or λ_2 must start with a special tile, with these pieces being of different lengths in cases when both λ_1 and λ_2 contain one. If the longer special tile occurs in λ_1, then it must be of even length and followed by dominoes, for otherwise, a non-initial square would occur in λ_1, which would imply the existence of a common fault of λ_1 and λ_2 as $n > m$. On the other hand, if λ_2 contains the longer special tile, then it must have length in $[m+1, n]$ or be of odd length less than m. Putting the preceding observations together, we have that S_1 consists of those pairs (λ_1, λ_2) satisfying one of the following: (i) $\lambda_1 = d^t$, $\lambda_2 = sd^t\lambda'$, (ii) $\lambda_1 = p_{2i}d^{t-i}$, $\lambda_2 = \sigma d^{t-i+1}\lambda'$, (iii) $\lambda_1 = \sigma' d^{t-i}$, $\lambda_2 = p_{2i+1}d^{t-i}\lambda'$ or (iv) λ_1 arbitrary and $\lambda_2 = p_j\lambda^*$, where $\lambda' \in \mathcal{F}_{n-m-1}$, $\lambda^* \in \mathcal{F}_{n-j}$, $\sigma \in \mathcal{T}_{2i-1}(a_2, b_2, T_2)$, $\sigma' \in \mathcal{T}_{2i}(a_1, b_1, T_1)$ and $j \in [m+1, n]$, with $i \in [t]$ in (ii) and $i \in [t-1]$ in (iii).

Combining the prior cases yields

$$\begin{aligned}
\mathrm{w}(S_1) &= a_1b_2xF_{n-m} + T_1F_{n-m}\sum_{i=1}^{t}\beta_{2i-1} + T_2F_{n-m}\sum_{i=1}^{t-1}\alpha_{2i} + T_2\alpha_m\sum_{j=m+1}^{n}F_{n-j+1}\\
&= a_1b_2xF_{n-m} + \frac{T_1F_{n-m}}{x}\left(\beta_m - \frac{mT_2}{2} - a_2\right) + \frac{T_2F_{n-m}}{x}\left(\alpha_{m-1} - \frac{(m-2)T_1}{2} - b_1x\right)\\
&\quad + T_2\alpha_m\sum_{i=1}^{n-m}F_i,
\end{aligned}$$

by (4.2) and (4.3). By similar reasoning, we have

$$\begin{aligned}
\mathrm{w}(S_2) &= a_2b_1xF_{n-m} + \frac{T_2F_{n-m}}{x}\left(\alpha_m - \frac{mT_1}{2} - a_1\right) + \frac{T_1F_{n-m}}{x}\left(\beta_{m-1} - \frac{(m-2)T_2}{2} - b_2x\right)\\
&\quad + T_1\beta_m\sum_{i=1}^{n-m}F_i.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
\alpha_m\beta_n - \alpha_n\beta_m &= \mathrm{w}(S_1) - \mathrm{w}(S_2)\\
&= \gamma_1xF_{n-m} + \frac{F_{n-m}}{x}(T_1\beta_m - T_2\alpha_m - \gamma_2) - \frac{F_{n-m}}{x}(T_1\beta_{m-1} - T_2\alpha_{m-1} - \gamma_3x)\\
&\quad - (T_1\beta_m - T_2\alpha_m)\sum_{i=1}^{n-m}F_i, \qquad (4.11)
\end{aligned}$$

where the γ_i for $1 \leq i \leq 3$ are as described above.

By (3.1), we have

$$\begin{aligned}
T_1\beta_m - T_2\alpha_m &= T_1\left(b_2xF_m + a_2F_{m-1} + T_2\sum_{i=1}^{m-1}F_i\right) - T_2\left(b_1xF_m + a_1F_{m-1} + T_1\sum_{i=1}^{m-1}F_i\right)\\
&= \gamma_2F_{m-1} + \gamma_3xF_m, \qquad m \geq 0. \qquad (4.12)
\end{aligned}$$

Substituting (4.12) in (4.11) implies

$$\begin{aligned}\alpha_m\beta_n - \alpha_n\beta_m &= (\gamma_1 x - \gamma_2/x + \gamma_3)F_{n-m} + \frac{F_{n-m}}{x}(\gamma_2 F_{m-1} + \gamma_3 x F_m)\\ &\quad - \frac{F_{n-m}}{x}(\gamma_2 F_{m-2} + \gamma_3 x F_{m-1}) - (\gamma_2 F_{m-1} + \gamma_3 x F_m)\sum_{i=1}^{n-m} F_i,\end{aligned}$$

which yields the even case of (4.10). □

Remark: Let G_n denote the sequence defined by $G_n = G_{n-1} + G_{n-2}$ for $n \geq 2$, with $G_0 = a$ and $G_1 = b$, where a and b are arbitrary. Note that $\mathcal{GL}_n^{a,b,T}(x)$ reduces to G_n when $x = 1$ and $T = 0$. Benjamin and Quinn [4, Chapter 3] refer to the G_n as *Gibonacci* numbers, which is shorthand for generalized Fibonacci numbers. Taking $x = 1$, $T = 0$ in (4.5) and (4.6)/(4.7) recovers respectively Identities 40 and 46 in [4], though expressed in a slightly different form. Further, we have that (4.8) and (4.10) when $x = 1$, $T_1 = T_2 = 0$ are equivalent to Identities 42 and 43 in [4].

Using (4.10), it is possible to derive some further combinatorial identities as follows. First note that by (3.1), we have

$$\begin{aligned}\alpha_m\beta_n &= \left(b_1 x F_m + a_1 F_{m-1} + T_1\sum_{j=1}^{m-1} F_j\right)\left(b_2 x F_n + a_2 F_{n-1} + T_2\sum_{j=1}^{n-1} F_j\right)\\ &= (b_1 x F_m + a_1 F_{m-1})(b_2 x F_n + a_2 F_{n-1}) + T_1(b_2 x F_n + a_2 F_{n-1})\sum_{j=1}^{m-1} F_j\\ &\quad + T_2(b_1 x F_m + a_1 F_{m-1})\sum_{j=1}^{n-1} F_j + T_1 T_2\sum_{j=1}^{m-1} F_j\sum_{j=1}^{n-1} F_j.\end{aligned}$$

Finding the analogous expression for $\alpha_n\beta_m$, and subtracting, we have

$$\begin{aligned}\alpha_m\beta_n - \alpha_n\beta_m &= (a_2 b_1 - a_1 b_2)x F_m F_{n-1} + (a_1 b_2 - a_2 b_1)x F_n F_{m-1}\\ &\quad + T_1(b_2 x F_n + a_2 F_{n-1})\sum_{j=1}^{m-1} F_j - T_1(b_2 x F_m + a_2 F_{m-1})\sum_{j=1}^{n-1} F_j\\ &\quad + T_2(b_1 x F_m + a_1 F_{m-1})\sum_{j=1}^{n-1} F_j - T_2(b_1 x F_n + a_1 F_{n-1})\sum_{j=1}^{m-1} F_j. \qquad (4.13)\end{aligned}$$

Equating the coefficients of $b_2 T_1$ in the expressions for $\alpha_m\beta_n - \alpha_n\beta_m$ in (4.10) and (4.13) in the case $n = m + 1$ yields

$$(1 - x)F_m - F_{m-1} + (-1)^m = x F_{m+1}\sum_{j=1}^{m-1} F_j - x F_m \sum_{j=1}^{m} F_j,$$

which may be rewritten as

$$F_m - F_{m+1} + (-1)^m = -x F_m^2 + x(F_{m+1} - F_m)\sum_{j=1}^{m-1} F_j.$$

Equating the comparable coefficients of a_1T_2 yields

$$\frac{(x-1)F_{m-1}+F_{m-2}+(-1)^m}{x} = F_{m-1}\sum_{j=1}^{m} F_j - F_m\sum_{j=1}^{m-1} F_j = F_mF_{m-1} + (F_{m-1}-F_m)\sum_{j=1}^{m-1} F_j.$$

Solving for $\sum_{j=1}^{m-1} F_j$ in the two relations above yields the following identity for the sum of Fibonacci polynomials.

Corollary 4.1. *If $m \geq 1$, then*

$$\sum_{j=1}^{m-1} F_j = \frac{xF_m^2 + F_m - F_{m+1} + (-1)^m}{x(F_{m+1}-F_m)} = \frac{xF_mF_{m-1} + F_{m-1} - F_m - (-1)^m}{x(F_m - F_{m-1})}. \tag{4.14}$$

Note that (4.14) may also be shown directly by a somewhat involved inductive argument. A more familiar formula for sums of Fibonacci polynomials involves a convolution with powers of x and is given by

$$\sum_{j=1}^{m-1} x^{m-1-j}F_j = F_{m+1} - x^m, \qquad m \geq 1. \tag{4.15}$$

Identity (4.15) may be obtained readily by induction using the recurrence for F_j or by extending using weights the combinatorial argument (see [4, Identity 1]) for the $x = 1$ case of (4.15), whereas (4.14) apparently cannot be obtained in this manner. Note that the $x = 1$ case of (4.14) is an equality for Fibonacci numbers which follows from the $x = 1$ case of (4.15) combined with Cassini's identity $f_{n+1}f_{n-1} - f_n^2 = (-1)^n$ for $n \geq 0$. Finally, by using either formula for $\sum_{j=1}^{m-1} F_j$ from Corollary 4.1 in (4.4) or (4.10), one obtains expressions for v_{n+m} and $\alpha_m\beta_n - \alpha_n\beta_m$ respectively in terms of Fibonacci polynomials that do not involve a summation.

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