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Fibonacci numbers along residue classes and convolutions

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Abstract: The sequence F_{dn+h} and its convolutions have (for $h = 0$) been studied in a recent paper at the arxiv. The instance with general h is more involved and uses Chebyshev polynomials.
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1 A review of the Chebyshev polynomials of the second kind

These polynomials might be defined and studied in many ways [1], but we consider $U_n(x)$ via the generating function:

$$\frac{1}{1-2tx+t^2} = \sum_{n\geq 0} U_n(x)t^n.$$

The denominator $1 - 2tx + t^2$ leads to the recursion $U_n(x) - 2xU_{n-1}(x) + U_{n-2}(x) = 0$, with $U_0(x) = 1$ and $U_1(x) = 2x$. From this one can obtain via induction the explicit formula

$$\begin{cases} U_{2n}(x) = \sum_k x^{2k}(-1)^{n-k}\binom{n+k}{2k}2^{2k}, \\ U_{2n+1}(x) = \sum_k x^{2k+1}(-1)^{n-k}\binom{n+k+1}{2k+1}2^{2k+1}. \end{cases}$$

Since

$$\frac{\partial}{\partial x}\frac{1}{1-2tx+t^2} = \frac{2t}{(1-2tx+t^2)^2} \quad \text{and} \quad \frac{\partial}{\partial x}\frac{1}{(1-2tx+t^2)^s} = \frac{2st}{(1-2tx+t^2)^{s+1}},$$

we may set

$$\frac{1}{(1-2tx+t^2)^{s+1}} = \sum_{n\ge 0} U_n^{(s)}(x)t^n$$

and find again explicit formulæ

$$\begin{cases} U_{2n}^{(s)}(x) = \sum_k x^{2k}(-1)^{n-k}\binom{n+k+s}{2k+s}\binom{2k+s}{s}2^{2k}, \\ U_{2n+1}^{(s)}(x) = \sum_k x^{2k+1}(-1)^{n-k}\binom{n+k+s+1}{2k+s+1}\binom{2k+s+1}{s}2^{2k+1}. \end{cases}$$

This may be derived via induction on the parameter s.

A small variation is via the introduction of a $\pm$ sign; let $\epsilon \in \{-1,+1\}$ and

$$\frac{1}{(1-2tx-\epsilon t^2)^{s+1}} = \sum_{n\ge 0} \overline{U}_n^{(s)}(x)t^n,$$

then

$$\begin{cases} \overline{U}_{2n}^{(s)}(x) = \sum_k x^{2k}\epsilon^{n-k}\binom{n+k+s}{2k+s}\binom{2k+s}{s}2^{2k}, \\ \overline{U}_{2n+1}^{(s)}(x) = \sum_k x^{2k+1}\epsilon^{n-k}\binom{n+k+s+1}{2k+s+1}\binom{2k+s+1}{s}2^{2k+1}. \end{cases}$$

2 The d-section of Fibonacci numbers

The recent paper [3] considers for Fibonacci numbers $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$

$$\sum_{n\ge 0} F_{2n}z^n = \frac{z}{1-3z+z^2},$$

and, more generally,

$$\sum_{n\ge 0} F_{dn}z^n.$$

Here, we are interested in

$$\sum_{n\ge 0} F_{dn+h}z^n$$

which makes the analysis more complete and also more challenging.

A pioneering paper on sub-sections of the Fibonacci sequence is [2].

Proposition 1.

$$\sum_{n\ge 0} F_{nd+h}z^n = \frac{F_h + (-1)^h zF_{d-h}}{1 - zL_d + (-1)^d z^2}.$$

Proof. We will use the Binet formulæ

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},\ L_n = \alpha^n + \beta^n \quad \text{with} \quad \alpha = \frac{1+\sqrt{5}}{2},\ \beta = \frac{1-\sqrt{5}}{2}.$$

Then

$$\begin{aligned}
X &:= (1 - zL_d + (-1)^d z^2) \sum_{n\ge 0} F_{nd+h} z^n \\
&= (1 - z(\alpha^d + \beta^d) + (-1)^d z^2) \sum_{n\ge 0} \frac{\alpha^{nd+h} - \beta^{nd+h}}{\alpha - \beta} z^n \\
&= \frac{1 - z(\alpha^d + \beta^d) + (-1)^d z^2}{\alpha - \beta} \sum_{n\ge 0} (\alpha^{nd+h} - \beta^{nd+h}) z^n \\
&= \frac{1 - z(\alpha^d + \beta^d) + (-1)^d z^2}{\alpha - \beta} \left[\frac{\alpha^h}{1 - z\alpha^d} - \frac{\beta^h}{1 - z\beta^d} \right] \\
&= \frac{1 - z(\alpha^d + \beta^d) + (-1)^d z^2}{\alpha - \beta} \frac{\alpha^h (1 - z\beta^d) - \beta^h (1 - z\alpha^d)}{1 - z(\alpha^d + \beta^d) + (-1)^d z^2} \\
&= \frac{\alpha^h (1 - z\beta^d) - \beta^h (1 - z\alpha^d)}{\alpha - \beta} \\
&= F_h - z \frac{\alpha^h (\beta^d) - \beta^h (\alpha^d)}{\alpha - \beta} \\
&= F_h - z(-1)^h \frac{\beta^{d-h} - \alpha^{d-h}}{\alpha - \beta} \\
&= F_h + z(-1)^h F_{d-h}.
\end{aligned}$$

□

The s-fold convolution of the sequence $\langle F_{nd+h} \rangle_n$ is (in [3] for $h = 0$ only) defined via the generating function

$$\left(\sum_{n\ge 0} F_{nd+h} z^n \right)^{s+1}.$$

Indeed, for $h = 0$, we have

$$\left. \frac{F_h + (-1)^h z F_{d-h}}{1 - zL_d + (-1)^d z^2} \right|_{h=0} = \frac{zF_d}{1 - zL_d + (-1)^d z^2},$$

and the simpler numerator makes life a bit easier. We have to deal with

$$(F_h + (-1)^h z F_{d-h})^{s+1} = \sum_{j=0}^{s+1} \binom{s+1}{j} F_h^{s+1-j} (-1)^{jh} z^j F_{d-h}^j$$

and eventually with

$$\sum_{j=0}^{s+1} \binom{s+1}{j} F_h^{s+1-j} (-1)^{jh} z^j F_{d-h}^j \frac{1}{(1 - zL_d + (-1)^d z^2)^{s+1}}.$$

Fortunately, following the introduction, the expansion of

$$\frac{1}{(1 - zL_d + (-1)^d z^2)^{s+1}} = \sum_{n\ge 0} \overline{U}_n^{(s)} \left(\frac{L_d}{2}\right) z^n$$

is known, and $\epsilon = (-1)^{d-1}$.

Therefore we have the fully explicit s-fold convolution of the sequence $\langle F_{nd+h}\rangle_n$ via its generating function:

Theorem 1.

$$\Big(\frac{F_h+(-1)^h zF_{d-h}}{1-zL_d+(-1)^d z^2}\Big)^{s+1}=\sum_{j=0}^{s+1}\binom{s+1}{j}F_h^{s+1-j}(-1)^{jh}z^jF_{d-h}^j\sum_{n\ge 0}\overline{U}_n^{(s)}\big(\frac{L_d}{2}\big)z^n.$$

Expanded,

$$\Big(\frac{F_h+(-1)^h zF_{d-h}}{1-zL_d+(-1)^d z^2}\Big)^{s+1}=\sum_{n\ge 0}z^n\sum_{j=0}^{s+1}\binom{s+1}{j}F_h^{s+1-j}(-1)^{jh}F_{d-h}^j\overline{U}_{n-j}^{(s)}\big(\frac{L_d}{2}\big).$$

The special case $h=0$ leads to

$$\sum_{n\ge 0}z^nF_d^{s+1}\overline{U}_n^{(s)}\big(\frac{L_d}{2}\big).$$

Remark 1. *The d-section of Lucas numbers can also be considered:*

$$\sum_{n\ge 0}L_{nd+h}z^n=\frac{L_h+(-1)^{h-1}zL_{d-h}}{1-zL_d+(-1)^d z^2}.$$

The details are left to the interested reader; the denominator is the same, so only the numerator needs to be changed.

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