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On (k, t)-Fibonacci–François numbers

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Abstract: In this article, we present a study of a new member of the family of k-Fibonacci numbers, which we call the (k, t)-Fibonacci–François sequence $\{w_n^{(k,t)}\}_{n\geq 0}$. This sequence is defined by the same k-Fibonacci recurrence relation with the initial terms $w_0^{(k,t)} = k - 2$ and $w_1^{(k,t)} = k^2 - k + 2 + t$. We describe the structure of this family of sequences by providing explicit formulas and establishing several related algebraic identities. In addition, we derive a Binet-type formula, and extend several classical identities, including those of Tagiuri–Vajda and d'Ocagne, as well as some expressions for the negative indices. Furthermore, we investigate fundamental properties of this family, obtaining limit identities for the ratios of successive terms, as well as summation formulas for the partial sums of the (k, t)-Fibonacci–François sequence.

Keywords: François sequence, Generalized François sequence, Generalized k-Fibonacci sequence, Generating function, Tagiuri–Vajda identity.
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1 Introduction

In [4], the François sequence is defined as the sequence $\{\mathcal{F}_n\}_{n\geq 0}$ given by

$$\mathcal{F}_{n+1} = \mathcal{F}_n + \mathcal{F}_{n-1} + 1$$

with initial values $\mathcal{F}_0 = 2$ and $\mathcal{F}_1 = 1$, appearing as the sequence $A022318$ in the OEIS [14]. In [1], the generalized François sequence $\{\mathcal{F}_n^{(t)}\}_{n\geq 0}$ is presented by

$$\mathcal{F}_{n+1}^{(t)} = \mathcal{F}_n^{(t)} + \mathcal{F}_{n-1}^{(t)} + t$$

with initial terms $\mathcal{F}_{t,0} = 2$, $\mathcal{F}_{t,1} = 1$, and an arbitrary real number t.

The definition below generalizes the two parameters of the François numbers.

Definition 1.1. *For any integer $n \geq 0$ and any real numbers k, t, the (k,t)*-François sequence *$\{\mathcal{F}_n^{(k,t)}\}_{n\geq 0}$ is given by the non-homogeneous linear second-order recurrence*

$$\mathcal{F}_n^{(k,t)} = k\mathcal{F}_{n-1}^{(k,t)} + \mathcal{F}_{n-2}^{(k,t)} + t\,, \tag{1}$$

with initial terms $\mathcal{F}_0^{(k,t)} = 2$ and $\mathcal{F}_1^{(k,t)} = k$.

The first seven elements of the François, generalized François, and (k,t)-François numbers are presented in Table 1.

Table 1. François-type numbers (Source: Authors' own work.)

n	$\mathcal{F}_n$	$\mathcal{F}_n^{(t)}$	$\mathcal{F}_n^{(k,t)}$
0	2	2	2
1	1	1	k
2	4	$3+t$	k^2+2+t
3	6	$4+2t$	$k^3+3k+kt+t$
4	11	$7+4t$	$k^4+4k^2+k^2t+kt+2+2t$
5	18	$11+7t$	$k^5+5k^3+k^3t+k^2t+5k+3kt+2t$
6	30	$18+12t$	$k^6+6k^4+k^4t+k^3t+9k^2+4k^2t+3kt+2+3t$

Some special cases are:

- if $k=1$, $\mathcal{F}_n^{(1,t)}$ is equal to the generalized François numbers $\mathcal{F}_n^{(t)}$, see for instance [1].
- if $t=0$, $\mathcal{F}_n^{(k,0)}$ is the same as the k-Lucas numbers $L_{k,n}$, see [6].

- if $k = 1$ and $t = 0$, $\mathcal{F}_n^{(1,0)}$ is equivalent to the classical Lucas numbers L_n, see [5], among others.
- if $k = 1$ and $t = 1$, $\mathcal{F}_n^{(1,1)}$ is equal to the François numbers $\mathcal{F}_n$, consult [4].

As a consequence of the Equation (1), when we subtract $\mathcal{F}_n^{(k,t)}$ from $\mathcal{F}_{n+1}^{(k,t)}$, we obtain a third order homogeneous recurrence relation.

Proposition 1.1. *The (k,t)-François sequence satisfies the third order recurrence*

$$\mathcal{F}_n^{(k,t)} = (1+k)\mathcal{F}_{n-1}^{(k,t)} + (1-k)\mathcal{F}_{n-2}^{(k,t)} - \mathcal{F}_{n-3}^{(k,t)}, \quad n \geq 3, \tag{2}$$

with initial terms $\mathcal{F}_0^{(k,t)} = 2$, $\mathcal{F}_1^{(k,t)} = k$, and $\mathcal{F}_2^{(k,t)} = k^2 + 2 + t$.

The characteristic equation associated with (2) is

$$r^3 - (1+k)r^2 - (1-k)r + 1 = 0\,, \tag{3}$$

whose real roots are

$$\alpha = \frac{k + \sqrt{k^2+4}}{2}, \quad \beta = \frac{k - \sqrt{k^2+4}}{2}, \quad \gamma = 1\,.$$

Note that the characteristic equation can be written as

$$(r^2 - kr - 1)(r-1) = 0\,. \tag{4}$$

Equation (4) allows us to connect the François type sequence and the Fibonacci type sequence, then we will present the (k,t)-Fibonacci–François sequence in the next section. So, the aim of this paper is to define the (k,t)-Fibonacci–François sequence as an extension of the François and Fibonacci sequences and to examine some of their properties. In the study of sequences of numbers, we are interested in finding new sequences (whether integer or not), especially if there is a connection to other sequences, and in providing classical results about them. These results consist of their recurrence relations, properties, identities, and the generating function. For this purpose, there are several classical methods in the literature. The methodology employed here aligns with standard approaches to studying recurrence-based sequences. The Binet formula derived for the Fibonacci–François sequence is a powerful tool for discovering new identities.

The structure of the article is as follows: in the next section, we recall the formal definition of the k-Fibonacci and (k,t)-Fibonacci–François sequences, and we offer key results for the development of this work. Section 3 is dedicated to the study of some identities satisfied by the k-Fibonacci sequence. Section 4 is devoted to results involving the (k,t)-Fibonacci–François sequence. In particular, we provide the general term (Binet-like formula), the characteristic, and the negative subscripts for the (k,t)-Fibonacci–François sequence. Section 5 presents several identities, namely Tagiuri–Vajda's identity and, as consequences, Catalan's identity, d'Ocagne's identity, and Cassini's identity. In Section 6, we derive the generating function, the exponential generating function, and the Poisson generating function associated with the (k,t)-Fibonacci–François sequence. In Section 7, we investigate the properties and identities associated with the partial sums of the terms involving the (k,t)-Fibonacci–François numbers; moreover, the limit of some quotients is presented. Finally, some perspectives for future research are discussed. Therefore, in this article, we establish a new sequence, the (k,t)-Fibonacci–François sequence, and our motivation is to provide new properties, identities, and relations with the classical François and Fibonacci sequences.

2 k-Fibonacci and (k,t)-Fibonacci–François sequence

The k-Fibonacci sequence $\{F_{k,n}\}_{n\geq 0}$ is defined as follows.

Definition 2.1 (According [9]). *The k-Fibonacci sequence $\{F_{k,n}\}_{n\geq 0}$ is given by the recurrence*

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \quad n \geq 1, \tag{5}$$

with initial terms $F_{k,0} = 0$ and $F_{k,1} = 1$.

The first five terms of this sequence are: 0, 1, k, k^2+1, k^3+2k.

If we consider the recurrence (5) with arbitrary initial terms a and b, we have a generalized k-Fibonacci sequence, according [12]. A wide variety of works in the literature lead to the k-Fibonacci-like sequence. See, for instance, [2, 6, 8, 9, 12].

Inspired by [13], we will consider the shift operator E; that is, for any real or complex sequence $\{x_n\}_{n\geq 0}$, we have

$$E(x_n) = x_{n+1}. \tag{6}$$

The following result is derived directly from Equations (4) and (6).

Proposition 2.1. *For any non-negative integers k and t, we have*

$$(E^2 - kE - I)(E - I)(\mathcal{F}_n^{(k,t)}) = 0\,,$$

where E and I denote the shift and identity operators, respectively.

In this work, we are interested in the following sequence:

Definition 2.2. *The (k,t)-Fibonacci–François sequence $\{w_n^{(k,t)}\}_{n\geq 0}$ is given by*

$$w_n^{(k,t)} = (E - I)(\mathcal{F}_n^{(k,t)})\,, \tag{7}$$

where E is the shift operator, I is the identity operator, and $\{\mathcal{F}_n^{(k,t)}\}_{n\geq 0}$ is the (k,t)-François sequence.

Note that $\{w_n^{(k,t)}\}_{n\geq 0}$ is a Horadam-like sequence (see [10]) and satisfies the k-Fibonacci recurrence (see [9]). As we describe in the next result.

Proposition 2.2. *For $n \geq 0$, the (k,t)-*Fibonacci–François *sequence $\{w_n^{(k,t)}\}_{n\geq 0}$ satisfies the recurrence relation*

$$w_{n+1}^{(k,t)} = kw_n^{(k,t)} + w_{n-1}^{(k,t)}\,, \quad n \geq 1, \tag{8}$$

with initial terms $w_0^{(k,t)} = k-2$ and $w_1^{(k,t)} = k^2 - k + 2 + t$.

Proof. Using Equations (1) and (7), we obtain, for all integers $n \geq 1$,

$$\begin{aligned} kw_n^{(k,t)} + w_{n-1}^{(k,t)} &= k(E-I)(\mathcal{F}_n^{(k,t)}) + (E-I)(\mathcal{F}_{n-1}^{(k,t)}) \\ &= (E-I)(k\mathcal{F}_n^{(k,t)} + \mathcal{F}_{n-1}^{(k,t)}) \\ &= (E-I)(\mathcal{F}_{n+1}^{(k,t)} - t) \\ &= (E-I)(\mathcal{F}_{n+1}^{(k,t)}) - (E-I)(t). \end{aligned}$$

Since $w_{n+1}^{(k,t)} = (E-I)(\mathcal{F}_{n+1}^{(k,t)})$ and $(E-I)(t) = 0$, the result follows. □

As an illustrative example, we have calculated a few terms from the (k,t)-Fibonacci–François sequence; see Table 2.

Table 2. The six first terms of the (k,t)-*Fibonacci–François* sequence.

n	$w_n^{(k,t)}$
0	$k-2$
1	$k^2-k+2+t$
2	$k^3-k^2+kt+3k-2$
3	$k^4-k^3+k^2t+4k^2-3k+2+t$
4	$k^5-k^4+k^3t+5k^3-4k^2+2kt+5k-2$
5	$k^6-k^5+k^4t+6k^4-5k^3+3k^2t+9k^2-5k+2+t$

Consider the following auxiliary sequence. For all integers $n \geq 0$, the sequence $\{P_n^{(k)}\}_{n\geq 0}$, with initial terms $P_0^{(k)} = k-2$ and $P_1^{(k)} = k^2-k+2$, is a k-Fibonacci-type sequence. As we can see, there are a few terms in Table 3.

Table 3. Decomposition of the first six terms of $w_n^{(k,t)}$ into $P_{k,n}$ and $F_{k,n}$.

n	$P_{k,n}$	$F_{k,n}$
0	$k-2$	0
1	k^2-k+2	1
2	k^3-k^2+3k-2	k
3	$k^4-k^3+4k^2-3k+2$	k^2+1
4	$k^5-k^4+5k^3-4k^2+5k-2$	k^3+2k
5	$k^6-k^5+6k^4-5k^3+9k^2-5k+2$	k^4+3k^2+1

Definition 2.3 ([12]). *The generalized k-Fibonacci sequence $\{g_{k,n}\}_{n\geq 0}$ satisfies the recurrence relation*

$$g_{k,n+1} = kg_{k,n} + g_{k,n-1}\,, \quad n \geq 1,$$

with arbitrary initial terms $g_{k,0} = a_k$ and $g_{k,1} = b_k$.

Proposition 2.3. *The (k,t)-Fibonacci–François sequence admits the decomposition*

$$w_n^{(k,t)} = P_{k,n} + tF_{k,n}$$

where $\{P_{k,n}\}_{n\geq 0}$ is a generalized k-Fibonacci sequence, with initial terms $P_{k,0} = k-2$, $P_{k,1} = k^2 - k + 2$, and $\{F_{k,n}\}_{n\geq 0}$ denotes the k-Fibonacci sequence.

Proof. The proof will be done using the principle of mathematical induction. Note that

$$\begin{aligned} P_{k,0} + tF_{k,0} &= k - 2 + t \cdot 0 = w_0^{(k,t)}\,, \\ P_{k,1} + tF_{k,1} &= k^2 - k + 2 + t \cdot 1 = w_1^{(k,t)}\,, \end{aligned}$$

where $\{P_n^{(k)}\}_{n\geq 0}$ is a k-Fibonacci-type sequence, with initial terms $P_0^{(k)} = k-2$, $P_1^{(k)} = k^2 - k + 2$.

Suppose that the result is true for every $n \leq m$ for some m. Then,

$$\begin{aligned} P_{k,m+1} + tF_{k,m+1} &= kP_{k,m} + P_{k,m-1} + t(kF_{k,m} + F_{k,m-1}) \\ &= k(P_{k,m} + tF_{k,m}) + P_{k,m-1} + tF_{k,m-1} = w_{m+1}^{(k,t)}\,, \end{aligned}$$

hence, the result is valid for every non-negative integer. □

3 *k*-Fibonacci-type sequence

As we know, the k-Fibonacci-type sequence is known to be associated with the characteristic equation:

$$r^2 = kr + 1\,, \tag{9}$$

whose two distinct roots are denoted by α and β, and are given by $\alpha = (k + \sqrt{k^2+4})/2$ and $\beta = (k - \sqrt{k^2+4})/2$.

Note that

$$\alpha + \beta = k, \;\; \alpha - \beta = \sqrt{k^2+4}, \;\; \alpha\beta = -1\,. \tag{10}$$

Lemma 3.1 ([3], Proposition 2)**.** *Let $\{F_{k,n}\}_{n\geq 0}$ be the k-Fibonacci sequence. Then*

$$F_{k,n} = \frac{\alpha^n - \beta^n}{\alpha - \beta}\,, \tag{11}$$

where α and β are the distinct roots of the Equation (9).

Lemma 3.2 ([6], Theorem 2.2)**.** *Let $\{L_{k,n}\}_{n\geq 0}$ be the k-Lucas sequence. Then*

$$L_{k,n} = \alpha^n + \beta^n\,, \tag{12}$$

where α and β are the distinct roots of the Equation (9).

Equations (11) and (12), respectively, present Binet's formula for the k-Fibonacci and k-Lucas sequences.

For the purpose of establishing identities involving the (k,t)-Fibonacci–François sequence, in this work we will consider the following identities for k-Fibonacci and k-Lucas numbers as a support.

Lemma 3.3 ([6], Theorem 2.4). *For $n \geq 1$ and any positive real number k, we have*

$$L_{k,n} = F_{k,n-1} + F_{k,n+1}\,, \tag{13}$$

where $\{F_{k,n}\}_{n\geq 0}$ and $\{L_{k,n}\}_{n\geq 0}$ are the, respectively, k-Fibonacci sequence and the k-Lucas sequence.

Lemma 3.4 ([2], Equation (26)). *For $n \geq 1$ and any positive real number k, we have*

$$(k^2+4)F_{k,n} = L_{k,n-1} + L_{k,n+1}\,, \tag{14}$$

where $\{F_{k,n}\}_{n\geq 0}$ and $\{L_{k,n}\}_{n\geq 0}$ are the k-Fibonacci sequence and the k-Lucas sequence, respectively.

Lemma 3.5 ([9], Equation (6)). *For $n \geq 1$ and any integer number k, we have*

$$F_{k,n-1}F_{k,n+1} - (F_{k,n})^2 = (-1)^n\,, \tag{15}$$

where $\{F_{k,n}\}_{n\geq 0}$ is the k-Fibonacci sequence.

Proposition 3.1. *For all non-negative integers n, we have*

$$F_{k,-n} = (-1)^{n+1}F_{k,n}, \tag{16}$$

$$L_{k,-n} = (-1)^{n}L_{k,n}, \tag{17}$$

where $\{F_{k,n}\}_{n\geq 0}$ is the k-Fibonacci sequence and $\{L_{k,n}\}_{n\geq 0}$ is the k-Lucas sequence.

Proof. It follows directly from Equations (11) and (12). □

Proposition 3.2. *For non-negative integers n and m, and for any generalized k-Fibonacci sequence $\{g_{k,\,n}\}_{n\geq 0}$, we have*

$$g_{k,\,n+m} = F_{k,m-1}\cdot g_{k,\,n} + F_{k,m}\cdot g_{k,\,n+1}\,, \tag{18}$$

where $\{F_{k,n}\}_{n\geq 0}$ denotes the k-Fibonacci sequence.

Proof. See [15, Equation (8)]. □

By combining Equations (16) and (18), we directly get the following result.

Corollary 3.1. *Let n and m be integers, and let $\{g_{k,\,n}\}_{n\geq 0}$ be a generalized k-Fibonacci sequence with initial terms $g_{k,0} = a_k$ and $g_{k,1} = b_k$. Then*

$$g_{k,\,n-m} = (-1)^m F_{k,m+1}\cdot g_{k,\,n} + (-1)^{m+1}F_{k,m}\cdot g_{k,\,n+1}\,,$$
$$g_{k,\,-m} = (-1)^m F_{k,m+1}\cdot a_k + (-1)^{m+1}F_{k,m}\cdot b_k\,,$$

where $\{F_{k,n}\}_{n\geq 0}$ denotes the k-Fibonacci sequence.

To finalize this section, we present the following auxiliary results involving two k-Fibonacci-type sequences and also can be seen as a version of k-Fibonacci sequence of Equation (18) presented in [15].

Proposition 3.3. *For any generalized k-Fibonacci-type sequences $\{g_{k,n}\}_{n\geq 0}$ and $\{h_{k,n}\}_{n\geq 0}$ the following identity holds :*

$$g_{k,\,n+m}h_{k,\,n+l} - g_{k,\,n}h_{k,\,n+m+l} = (-1)^n(g_{k,m}h_{k,l} - g_{k,0}h_{k,\,m+l})\,, \tag{19}$$

where n, m and l are non-negative integers.

Proof. We define

$$I_{k,n} := g_{k,\,n+m}h_{k,\,n+l} - g_{k,\,n}h_{k,\,n+m+l}\,.$$

We can use Equation (18) to show that

$$I_{k,n+1} = -I_{k,n}.$$

Indeed,

$$\begin{aligned}
I_{k,n} &= g_{k,\,n+m}h_{k,\,n+l} - g_{k,\,n}h_{k,\,n+m+l} \\
&= (F_{k,m-1}g_{k,\,n} + F_{k,m}g_{k,\,n+1})h_{k,\,n+l} - g_{k,\,n}(F_{k,m-1}h_{k,\,n+l} + F_{k,m}h_{k,\,n+l+1}) \\
&= F_{k,m}(g_{k,n+1}h_{k,n+l} - g_{k,n}h_{k,n+l+1})
\end{aligned}$$

and

$$\begin{aligned}
I_{k,n+1} &= (F_{k,m}g_{k,n} + F_{k,m+1}g_{k,n+1})h_{k,n+l+1} - g_{k,n+1}(F_{k,m}h_{k,n+l} + F_{k,m+1}h_{k,n+1+l}) \\
&= F_{k,m}(g_{k,n}h_{k,n+1+l} - g_{k,n+1}h_{k,n+l}),
\end{aligned}$$

as desired. Hence,

$$I_{k,1} = (-1)I_{k,0},\ \ldots,\ I_{k,n} = (-1)^n I_{k,0},$$

and we get

$$g_{k,\,n+m}h_{k,\,n+l} - g_{k,\,n}h_{k,\,n+m+l} = (-1)^n(g_{k,m}h_{k,l} - g_{k,0}h_{k,\,m+l}).$$

This completes the proof. □

In the next section, we will explore some known and new properties of the (k,t)-Fibonacci–François sequence $\{w_n^{(k,t)}\}_{n\geq 0}$ in order to obtain several properties for (k,t)-François numbers.

4 Some properties of the (k, t)-Fibonacci–François sequence

In this section, we present the Binet formula for $\{w_n^{(k,t)}\}_{n\geq 0}$ and examine some applications. Furthermore, we will exhibit the relationship between the k-Fibonacci type numbers and the (k,t)-Fibonacci–François numbers.

4.1 Relationship between (k, t)-Fibonacci–François and k-Fibonacci type numbers

In this subsection, we will explore the relationship between the k-Fibonacci type sequences and any (k, t)-Fibonacci–François sequence for some integer $k \geq 1$.

Proposition 4.1 (Binet-like Formula)**.** *Let $\{w_n^{(k,t)}\}_{n\geq 0}$ be the (k, t)-Fibonacci–François sequence. Then*

$$w_n^{(k,t)} = \frac{(w_1^{(k,t)} - w_0^{(k,t)}\beta)\alpha^n - (w_1^{(k,t)} - w_0^{(k,t)}\alpha)\beta^n}{\alpha - \beta}\,, \tag{20}$$

where α and β are the distinct roots of the Equation (9).

It follows from Proposition 4.1 that

Corollary 4.1. *Let $\{w_n^{(k,t)}\}_{n\geq 0}$ be the (k, t)-Fibonacci–François sequence. Then*

$$w_n^{(1,0)} = L_{n-1},$$

where α and β are the distinct roots of the characteristic equation $r^2 - r - 1 = 0$, and $\{L_n\}_{n\geq 0}$ is the Lucas sequence.

Proof. Take $k = 1$ and $t = 0$ in (20). Note that $w_0^{(1,0)} = 1-2 = -1$ and $w_1^{(1,0)} = 1-1+2+0 = 2$. Then,

$$\begin{aligned} w_n^{(1,0)} &= \frac{(2+\beta)\alpha^n - (2+\alpha)\beta^n}{\alpha - \beta} \\ &= \frac{1}{\alpha - \beta}[2(\alpha^n - \beta^n) - (\alpha^{n-1} - \beta^{n-1})] \\ &= 2F_n - F_{n-1} = F_n + F_n - F_{n-1} = F_n + F_{n-2} = L_{n-1}, \end{aligned}$$

where we use the classical identity, see [15, Equation (6)],

$$L_n = F_{n-1} + F_{n+1}.$$

□

It follows from Binet's formula that the following results hold:

Proposition 4.2. *Let $\{w_n^{(k,t)}\}_{n\geq 0}$ be the (k, t)-Fibonacci–François sequence. Then*

$$w_n^{(k,t)} = w_1^{(k,t)} F_{k,n} + w_0^{(k,t)} F_{k,n-1} \tag{21}$$

for all $n \geq 1$, where $\{F_{k,n}\}_{n\geq 0}$ is the k-Fibonacci sequence.

Proof. It is a consequence of Equations (20) and (11). □

Proposition 4.3. *Let $\{w_n^{(k,t)}\}_{n\geq 0}$ be the (k, t)-Fibonacci–François sequence. Then*

$$2w_n^{(k,t)} = \left(k^2 + 4 + 2t\right) F_{k,n} + (k-2)L_{k,n}\,, \tag{22}$$

for all $n \geq 0$, where $F_{k,n}$ and $L_{k,n}$ are the n-th k-Fibonacci and k-Lucas numbers, respectively.

Proof. We will use mathematical induction. Note that

$$\begin{aligned} 2w_0^{(k,t)} &= \left(k^2+4+2t\right)F_{k,0} + (k-2)L_{k,0} = 2(k-2); \\ 2w_1^{(k,t)} &= \left(k^2+4+2t\right)F_{k,1} + (k-2)L_{k,1} = k^2+4+2t+(k-2)k \\ &= 2k^2-2k+4+2t = 2(k^2-k+2+t)\,. \end{aligned}$$

Now suppose that the result is valid for any positive integer less than or equal to n, for some n. Then

$$\begin{aligned} &2w_{n+1}^{(k,t)} = 2kw_n^{(k,t)} + 2w_{n-1}^{(k,t)} \\ &= k\left(\left(k^2+4+2t\right)F_{k,n} + (k-2)L_{k,n}\right) + \left(k^2+4+2t\right)F_{k,n-1} + (k-2)L_{k,n-1} \\ &= \left(k^2+4+2t\right)F_{k,n+1} + (k-2)L_{k,n+1}\,. \end{aligned}$$

So, the result is valid for any non-negative integer n. □

Proposition 4.4. *Let $\{w_n^{(k,t)}\}_{n\geq 0}$ be the (k,t)-Fibonacci–François sequence. Then*

$$2(w_{n+1}^{(k,t)} + w_{n-1}^{(k,t)}) = (k^2+4)(k-2)F_{k,n} + \left(k^2+4+2t\right)L_{k,n}\,, \tag{23}$$

where $F_{k,n}$ and $L_{k,n}$ are the n-th k-Fibonacci and k-Lucas numbers, respectively.

Proof. It follows from Equations (13), (14), and (22). □

4.2 The characteristic of (k,t)-Fibonacci–François sequence

Let $\{w_n^{(k,t)}\}_{n\geq 0}$ be the (k,t)-Fibonacci–François sequence with initial values $w_0^{(k,t)} = w_0$ and $w_1^{(k,t)} = w_1$. Consider the constants $c = c(k,t) = w_1 - w_0\beta$ and $d = d(k,t) = w_1 - w_0\alpha$, where α and β represent the distinct roots of Equation (9).

According to [10, 11], the constant cd occurs in many of the formulas for Fibonacci-type numbers. It is called the *characteristic* of the Horadam sequence. In the following, we will calculate the characteristic of the (k,t)-Fibonacci–François sequence $\{w_n^{(k,t)}\}_{n\geq 0}$.

Note that

$$\begin{aligned} cd &= w_1^2 - w_0w_1(\alpha+\beta) + w_0^2\alpha\beta \\ &= w_1^2 - w_0w_1k - w_0^2\,, \end{aligned}$$

as $\alpha+\beta = k$ and $\alpha\beta = -1$, see Equation (10). Let $\mu_{k,t} = -cd$. Recall $w_0^{(k,t)} = k-2$ and $w_1^{(k,t)} = k^2-k+2+t$. So we have the following result.

Proposition 4.5. *For all non-negative integers k and t, the characteristic $\mu_{k,t}$ of the (k,t)-Fibonacci–François sequence is given by*

$$\mu_{k,t} = -k^3 - k^2t - t^2 - 4k - 4t\,. \tag{24}$$

For example, for $k=1$ and $t=0$, the characteristic of the $(1,0)$-Fibonacci sequence $\{w_n^{(1,0)}\}$ is $\mu_{1,0} = -5$. Note that $\{w_n^{(1,0)}\}_{n\geq 0}$ is the shifted Lucas sequence, as we can see in Corollary 4.1.

4.3 Negative index for the (k,t)-Fibonacci–François sequence $\{w_n^{(k,t)}\}$

Using standard techniques for Fibonacci-type sequences, we establish a recurrence relation for negative indices and derive explicit formulas consistent with the properties of the sequence $\{w_n^{(k,t)}\}_{n\geq 0}$. To extend the (k,t)-Fibonacci–François sequence $\{w_n^{(k,t)}\}_{n\geq 0}$ to negative indices, we use the modified recurrence relation

$$w_{n-2}^{(k,t)} = w_n^{(k,t)} - kw_{n-1}^{(k,t)}.$$

This relation follows

$$w_{-1}^{(k,t)} = w_1^{(k,t)} - kw_0^{(k,t)} = -1(-w_1^{(k,t)} + kw_0^{(k,t)}) = -1(-F_{k,1}w_1^{(k,t)} + F_{k,2}w_0^{(k,t)}), \tag{25}$$

$$w_{-2}^{(k,t)} = w_0^{(k,t)} - kw_{-1}^{(k,t)} = w_0^{(k,t)} - kF_{k,1}w_1^{(k,t)} + kF_{k,2}w_0^{(k,t)} = 1(-w_1^{(k,t)}F_{k,2} + w_0^{(k,t)}F_{k,3}). \tag{26}$$

Proposition 4.6. *Let $\{w_n^{(k,t)}\}_{n\geq 0}$ be the (k,t)-Fibonacci–François sequence. For all integers $n \geq 1$, the n-th negative indices satisfy*

$$w_{-n}^{(k,t)} = (-1)^n \left(w_0^{(k,t)}F_{k,n+1} - w_1^{(k,t)}F_{k,n} \right),$$

where $\{F_{k,n}\}_{n\geq 0}$ is the k-Fibonacci sequence.

Proof. Let $W_{-n}^{(k,t)} = (-1)^n \left(w_0^{(k,t)}F_{k,n+1} - w_1^{(k,t)}F_{k,n} \right)$ for all integers $n \geq 0$. We want to show that $W_{-n}^{(k,t)}$ verifies the recurrence relation $W_{-n}^{(k,t)} = W_{-(n+2)}^{(k,t)} + kW_{-(n+1)}^{(k,t)}$. Indeed,

$$\begin{aligned}
&W_{-(n+2)}^{(k,t)} + kW_{-(n+1)}^{(k,t)} \\
&= (-1)^{n+2} \left(w_0^{(k,t)}F_{k,n+3} - w_1^{(k,t)}F_{k,n+2} \right) + (-1)^{n+1}k \left(w_0^{(k,t)}F_{k,n+2} - w_1^{(k,t)}F_{k,n+1} \right) \\
&= (-1)^n \left(w_0^{(k,t)}F_{k,n+1} - w_1^{(k,t)}F_{k,n} \right) = W_{-n}^{(k,t)}.
\end{aligned}$$

Moreover, by Equations (25) and (26), $W_{-1}^{(k,t)} = -(w_0^{(k,t)}F_{k,2} - w_1^{(k,t)}F_{k,1}) = w_{-1}^{(k,t)}$ and $W_{-2}^{(k,t)} = w_0^{(k,t)}F_{k,3} - w_1^{(k,t)}F_{k,2} = w_{-2}^{(k,t)}$. So, since $W_{-n}^{(k,t)}$ satisfies the recurrence that defines $w_{-n}^{(k,t)}$ with the same initial conditions, we conclude that $W_{-n}^{(k,t)} = w_{-n}^{(k,t)}$ for all $n \geq 1$. □

Another way of expressing $\{w_n^{(k,t)}\}_{n\geq 0}$ is given by:

Proposition 4.7. *Let $\{w_n^{(k,t)}\}_{n\geq 0}$ be the (k,t)-Fibonacci–François sequence. For each integer $n \geq 0$, the following identity holds:*

$$w_{-n}^{(k,t)} = (-1)^n \left(w_n^{(k,t)} - \left(k^2 + 4 + 2t\right) F_{k,n} \right), \tag{27}$$

where $\{F_{k,n}\}_{n\geq 0}$ denotes the k-Fibonacci sequence.

Proof. Combining Equations (22), (16) and (17) we have

$$\begin{aligned}
2w_{-n}^{(k,t)} &= \left(k^2 + 4 + 2t\right) F_{k,-n} + (k-2)L_{k,-n} \\
&= (-1)^{n+1} \left(k^2 + 4 + 2t\right) F_{k,n} + (-1)^n (k-2)L_{k,n} \\
&= (-1)^n \left[-2\left(k^2 + 4 + 2t\right) F_{k,n} + \left(k^2 + 4 + 2t\right) F_{k,n} + (k-2)L_{k,n}\right] \\
&= (-1)^n \left[-2\left(k^2 + 4 + 2t\right) F_{k,n} + 2w_n^{(k,t)}\right] \\
&= 2(-1)^n \left[w_n^{(k,t)} - \left(k^2 + 4 + 2t\right) F_{k,n}\right],
\end{aligned}$$

which verifies the result. □

5 Some classical identities

In this section, the classical Tagiuri–Vajda identity of the (k,t)-Fibonacci–François sequence is presented. From Tagiuri–Vajda's identity, we obtain the results establishing d'Ocagne's identity, Catalan's identity, and Cassini's identity for the sequence.

Lemma 5.1. *For non-negative integers n, m, and for the (k,t)-Fibonacci–François sequence $\{w_n^{(k,t)}\}_{n\geq 0}$, the following identity holds:*

$$4w_{n+m}^{(k,t)} = 2L_{k,m}w_n^{(k,t)} + F_{k,m}\left((k^2+4)(k-2)F_{k,n} + \left(k^2+4+2t\right)L_{k,n}\right), \tag{28}$$

where $\{L_{k,n}\}_{n\geq 0}$ is the k-Lucas sequence and $\{F_{k,n}\}_{n\geq 0}$ is the k-Fibonacci sequence.

Proof. Changing m by $-m$ in Equation (18), and by using Equations (16) and (13), we have

$$w_{n-m}^{(k,t)} = (-1)^m(F_{k,m+1}\cdot w_n^{(k,t)} - F_{k,m}\cdot w_{n+1}^{(k,t)}).$$

Then

$$\begin{aligned} w_{n+m}^{(k,t)} + (-1)^m w_{n-m}^{(k,t)} &= F_{k,m-1}w_n^{(k,t)} + F_{k,m}w_{n+1}^{(k,t)} + F_{k,m+1}w_n^{(k,t)} - F_{k,m}w_{n+1}^{(k,t)} \\ &= (F_{k,m-1} + F_{k,m+1})w_n^{(k,t)} \\ &= L_{k,m}\cdot w_n^{(k,t)}, \end{aligned}$$

and

$$\begin{aligned} w_{n+m}^{(k,t)} - (-1)^m w_{n-m}^{(k,t)} &= F_{k,m-1}w_n^{(k,t)} + F_{k,m}w_{n+1)}^{(k,t)} - F_{k,m+1}w_n^{(k,t)} + F_{k,m}w_{n+1}^{(k,t)} \\ &= (F_{k,m-1} - F_{k,m+1})w_n^{(k,t)} + 2F_{k,m}w_{n+1}^{(k,t)} \\ &= -kF_{k,m}w_n^{(k,t)} + 2F_{k,m}w_{n+1}^{(k,t)} \\ &= F_{k,m}(w_{n-1}^{(k,t)} + w_{n+1}^{(k,t)}). \end{aligned}$$

Now, by summing the last two equations multiplied by 2, we obtain

$$4w_{n+m}^{(k,t)} = 2L_{k,m}w_n^{(k,t)} + F_{k,m}2(w_{n-1}^{(k,t)} + w_{n+1}^{(k,t)}).$$

Using Equation (23), we obtain the result. □

In the following, we prove the Tagiuri–Vajda identity for the generalized k-Fibonacci sequence using Proposition 3.3.

Proposition 5.1 (Tagiuri–Vajda)**.** *Let $\{w_n^{(k,t)}\}_{n\geq 0}$ be the (k,t)-Fibonacci–François sequence and $\mu_{k,t}$ be its characteristic. Then, for all non-negative integers n, m, q, k, and t,*

$$w_{n+m}^{(k,t)}w_{n+q}^{(k,t)} - w_n^{(k,t)}w_{n+m+q}^{(k,t)} = (-1)^{n+1}\mu_{k,t}\cdot F_{k,m}F_{k,q}, \tag{29}$$

where $\{F_{k,n}\}_{n\geq 0}$ is the k-Fibonacci sequence, and $\mu_{k,t}$ is given in (24).

Proof. Taking $g_{k,i} = h_{k,i} = m_{k,t,i}$ in Equation (19), we have

$$\begin{aligned} 4\big(w_{n+m}^{(k,t)} w_{n+q}^{(k,t)} - w_n^{(k,t)} w_{n+m+q}^{(k,t)}\big) &= (-1)^n \left[4w_m^{(k,t)} w_q^{(k,t)} - w_0^{(k,t)} 4w_{m+q}^{(k,t)}\right] \\ &= (-1)^n \left[4w_m^{(k,t)} w_q^{(k,t)} - w_0^{(k,t)} 4w_{m+q}^{(k,t)}\right]. \end{aligned}$$

Using the Equations (22) and (28), we have

$$\begin{aligned} &4w_m^{(k,t)} w_q^{(k,t)} - w_0^{(k,t)} 4w_{m+q}^{(k,t)} \\ &= 4w_m^{(k,t)} w_q^{(k,t)} - w_0^{(k,t)}\big(2L_{k,q} w_m^{(k,t)} + F_{k,q}\left((k^2+4)(k-2)F_{k,m} + \big(k^2+4+2t\big) L_{k,m}\right)\big) \\ &= 2w_m^{(k,t)}(2w_q^{(k,t)} - (k-2)L_{k,q}) - (k-2)F_{k,q}((k^2+4)(k-2)F_{k,m} + (k^2+4+2t)L_{k,m}) \\ &= 2w_m^{(k,t)}(k^2+4+2t)F_{k,q} - (k-2)F_{k,q}((k^2+4)(k-2)F_{k,m} + (k^2+4+2t)L_{k,m}) \\ &= F_{k,q}\left[(k^2+4+2t)2w_m^{(k,t)} - (k^2+4)(k-2)^2 F_{k,m} - (k-2)(k^2+4+2t)L_{k,m})\right] \\ &= F_{k,q}F_{k,m}[(k^2+4+2t)^2 - (k^2+4)(k-2)^2] \\ &= 4F_{k,q}F_{k,m}[k^3 + k^2 t + t^2 + 4k + 4t]. \end{aligned}$$

Since $-\mu_{k,t} = k^3 + k^2 t + t^2 + 4k + 4t$, we conclude the proof. □

As a consequence of the Tagiuri–Vajda identity, we have the following identities.

Proposition 5.2 (d'Ocagne's identity)**.** *Let h, n be non-negative integers with $h \geq n$, then*

$$w_h^{(k,t)} w_{n+1}^{(k,t)} - w_n^{(k,t)} w_{h+1}^{(k,t)} = (-1)^{n+1}\mu_{k,t}F_{k,h-n}\,, \tag{30}$$

where $\{w_n^{(k,t)}\}_{n\geq 0}$ is the (k,t)-Fibonacci–François sequence, $\{F_{k,n}\}_{n\geq 0}$ is the k-Fibonacci sequence, and $\mu_{k,t}$ is given in (24)*.*

Proof. Consider $m + n = h$ and $q = 1$ in Equation (29), then

$$\begin{aligned} w_h^{(k,t)} w_{n+1}^{(k,t)} - w_n^{(k,t)} w_{h+1}^{(k,t)} &= (-1)^{n+1}\mu_{k,t}F_{k,h-n}F_{k,1} \\ &= (-1)^{n+1}\mu_{k,t}F_{k,h-n}\,. \end{aligned}$$

Since $F_{k,1} = 1$, we have the result. □

Proposition 5.3 (Catalan's identity)**.** *Let n, m be non-negative integers $n \geq m$. Then*

$$(w_n^{(k,t)})^2 - w_{n-m}^{(k,t)} w_{n+m}^{(k,t)} = (-1)^{n+m+1}\mu_{k,t}(F_{k,m})^2\,, \tag{31}$$

where $\{w_n^{(k,t)}\}_{n\geq 0}$ is the (k,t)-Fibonacci–François sequence, $\{F_{k,n}\}_{n\geq 0}$ is the k-Fibonacci sequence, and $\mu_{k,t}$ is given in Equation (24)*.*

Proof. Taking $q = -m$ in Equation (29), we have

$$w_{n+m}^{(k,t)} w_{m-n}^{(k,t)} - (w_n^{(k,t)})^2 = (-1)^{n+1}\mu_{k,t}F_{k,m}F_{k,-m}\,.$$

The result follows using Equation (16). □

As a consequence of Catalan's identity, since $F_{k,1} = 1$ and by doing $m = 1$ in Equation (31), we have the following result.

Corollary 5.1 (Cassini-Simson's identity)**.** *For all non-negative integers n, we have*

$$(w_n^{(k,t)})^2 - w_{n-1}^{(k,t)} w_{n+1}^{(k,t)} = (-1)^n \mu_{k,t}\,, \tag{32}$$

where $\{w_n^{(k,t)}\}_{n\geq 0}$ is the (k,t)-Fibonacci–François sequence and $\mu_{k,t}$ is given in (24)*.*

A direct consequence is the Cassini-Simson identity for even subscripts is given above.

Corollary 5.2. *For all non-negative integers n, we have*

$$(w_{2n}^{(k,t)})^2 - w_{2n-1}^{(k,t)} w_{2n+1}^{(k,t)} = \mu_{k,t}\,,$$

where $\{w_n^{(k,t)}\}_{n\geq 0}$ is the (k,t)-Fibonacci–François sequence and $\mu_{k,t}$ is given in (24)*.*

This next result also follows directly from Catalan's identity.

Proposition 5.4 (Gelin-Cesàro's identity)**.** *The (k,t)-Fibonacci–François sequence $\{w_n^{(k,t)}\}_{n\geq 0}$ satisfies the following identity*

$$w_{n+2}^{(k,t)} w_{n+1}^{(k,t)} w_{n-1}^{(k,t)} w_{n-2}^{(k,t)} - (w_n^{(k,t)})^4 = (-1)^n \left(k^2 - 1\right) \mu_{k,t} (w_n^{(k,t)})^2 - k^2 \mu_{k,t}^2\,,$$

where $\mu_{k,t}$ is the characteristic of the (k,t)-Fibonacci–François sequence given in (24)*.*

Proof. Using Catalan's identity (31) with $m = 2$ we obtain

$$(w_n^{(k,t)})^2 - w_{n-2}^{(k,t)} w_{n+2}^{(k,t)} = (-1)^{n+1} \mu_{k,t} F_{k,2}^2\,,$$

and multiplying by Equation (32), we get

$$\begin{aligned}
&w_{n+2}^{(k,t)} w_{n+1}^{(k,t)} w_{n-1}^{(k,t)} w_{n-2}^{(k,t)} - (w_n^{(k,t)})^4 = w_{n+2}^{(k,t)} w_{n-2}^{(k,t)} w_{n+1}^{(k,t)} w_{n-1}^{(k,t)} - (w_n^{(k,t)})^4 \\
&= \left[(w_n^{(k,t)})^2 + \mu_{k,t}(-1)^{n+2} F_{k,2}^2\right] \left[(w_n^{(k,t)})^2 + \mu_{k,t}(-1)^{n+1} F_{k,1}^2\right] - (w_n^{(k,t)})^4 \\
&= (w_n^{(k,t)})^4 - \mu_{k,t}(-1)^n (w_n^{(k,t)})^2 + k^2 \mu_{k,t}(-1)^n (w_n^{(k,t)})^2 - k^2 \mu_{k,t}^2 (-1)^{2n} - (w_n^{(k,t)})^4 \\
&= \mu_{k,t}(-1)^n (w_n^{(k,t)})^2 (k^2 - 1) - k^2 \mu_{k,t}^2\,.
\end{aligned}$$

Since $F_{k,1} = 1$ and $F_{k,2} = k$, we verify the results. □

Proposition 5.5 (Convolution's identity)**.** *Let m, n be non-negative integer numbers. Then the (k,t)-Fibonacci–François sequence $\{w_n^{(k,t)}\}_{n\geq 0}$ satisfies the following identity:*

$$w_{m-1}^{(k,t)}\, w_n^{(k,t)} + w_m^{(k,t)}\, w_{n+1}^{(k,t)} = w_1^{(k,t)}\, w_{n+m}^{(k,t)} + w_0^{(k,t)}\, w_{n+m-1}^{(k,t)}\,.$$

Proof. By Equation (21), we have $w_n^{(k,t)} = w_1^{(k,t)} F_{k,n} + w_0^{(k,t)} F_{k,n-1}$. Then

$$\begin{aligned}
&w_{m-1}^{(k,t)} w_n^{(k,t)} + w_m^{(k,t)} w_{n+1}^{(k,t)} = (w_1^{(k,t)} F_{k,m-1} + w_0^{(k,t)} F_{k,m-2})(w_1^{(k,t)} F_{k,n} + w_0^{(k,t)} F_{k,n-1}) \\
&\qquad + (w_1^{(k,t)} F_{k,m} + w_0^{(k,t)} F_{k,m-1})(w_1^{(k,t)} F_{k,n+1} + w_0^{(k,t)} F_{k,n}) \\
&= (w_1^{(k,t)})^2 F_{k,m+n} + w_0^{(k,t)} w_1^{(k,t)} F_{k,m+n-1} + w_0^{(k,t)} w_1^{(k,t)} F_{k,m+n-1} + (w_0^{(k,t)})^2 F_{k,m+n-2} \\
&= w_1^{(k,t)} (w_1^{(k,t)} F_{k,m+n} + w_0^{(k,t)} F_{k,m+n-1}) + w_0^{(k,t)} (w_1^{(k,t)} F_{k,m+n-1} + w_0^{(k,t)} F_{k,m+n-2}) \\
&= w_1^{(k,t)} w_{n+m}^{(k,t)} + w_0^{(k,t)} w_{n+m-1}^{(k,t)}\,,
\end{aligned}$$

as required. □

6 Generating functions

In this section, we determine the generating functions for the (k,t)-Fibonacci–François sequence. More specifically, we present three types of generating functions: the ordinary generating function, the exponential generating function, and the Poisson generating function. We point out that [7] introduced the ordinary generator.

The following result provides the explicit form of the generating function for the (k,t)-Fibonacci–François sequence $\{w_n^{(k,t)}\}_{n\geq 0}$.

Proposition 6.1. *The ordinary generating function for the (k,t)-Fibonacci–François sequence $\{w_n^{(k,t)}\}_{n\geq 0}$ is denoted by $G_{w_n^{(k,t)}}(x)$ and is given by*

$$G_{w_n^{(k,t)}}(x) = \frac{k-2+(k+2+t)x}{1-kx-x^2}.$$

Proof. Take $F_{k,0}^{(i)} = k-2$ and $F_{k,0}^{(i)} = k^2-k+2+t$ in [7, Equation (12)]. □

The exponential generating function $E_{w_n^{(k,t)}}(x)$ for a sequence $\{w_n^{(k,t)}\}_{n\geq 0}$ is represented as a power series given by:

$$E_{w_n^{(k,t)}}(x) = w_0^{(k,t)} + w_1^{(k,t)}x + \frac{w_2^{(k,t)}x^2}{2!} + \cdots + \frac{w_n^{(k,t)}x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{w_n^{(k,t)}x^n}{n!}.$$

The proof of the next result follows directly from Binet's formula.

Proposition 6.2. *The exponential generating function for the (k,t)-Fibonacci–François sequence $\{w_n^{(k,t)}\}_{n\geq 0}$ is*

$$E_{w_n^{(k,t)}}(x) = \frac{(k^2-k+2+t-(k-2)\beta)e^{\alpha x} - (k^2-k+2+t-(k-2)\alpha)e^{\beta x}}{\alpha-\beta}$$

where α and β are the distinct roots of the Equation (9).

The Poisson generating function $P_{w_n^{(k,t)}}(x)$ for the sequence $\{w_n^{(k,t)}\}_{n\geq 0}$ is defined as:

$$P_{w_n^{(k,t)}}(x) = \sum_{n=0}^{\infty} \frac{w_n^{(k,t)}x^n}{n!}e^{-x} = e^{-x}E_{w_n^{(k,t)}}(x).$$

This relationship establishes a direct connection between the two generating functions, allowing the Poisson generating function to be derived from its exponential function.

Corollary 6.1. *The Poisson generating function of the (k,t)-Fibonacci–François sequence $\{w_n^{(k,t)}\}_{n\geq 0}$ is as follows:*

$$P_{w_n^{(k,t)}}(x) = \frac{(k^2-k+2+t-(k-2)\beta)e^{(\alpha-1)x} - (k^2-k+2+t-(k-2)\alpha)e^{(\beta-1)x}}{\alpha-\beta},$$

where α and β are the distinct roots of Equation (9).

7 Asymptotic behavior and partial sum

In this section, we will investigate the asymptotic behavior of the quotient $w_{n+q}^{(k,t)}/w_n^{(k,t)}$ for a fixed positive integer q and some partial sums involving $w^{(k,t)}$. The first result shows that the sequence $\{w_{n+q}^{(k,t)}/w_n^{(k,t)}\}_{n\geq 0}$ converges to α as n goes to infinity, and q is a fixed positive integer.

Proposition 7.1. *Let $w_n^{(k,t)}$ be the n-th term of (k,t)-Fibonacci–Fraçois sequence. Then we have*

$$\lim_{n\to\infty}\frac{w_{n+q}^{(k,t)}}{w_n^{(k,t)}}=\alpha^q\,, \tag{33}$$

and

$$\lim_{n\to\infty}\frac{w_{-(n+q)}^{(k,t)}}{w_{-n}^{(k,t)}}=(-\alpha)^q\,, \tag{34}$$

for any positive integer q, where α is a root of Equation (9).

Proof. According to Binet's formula (20), we have

$$\frac{w_{n+q}^{(k,t)}}{w_n^{(k,t)}}=\alpha^q\,\frac{(w_1^{(k,t)}-w_0^{(k,t)}\beta)-(w_1^{(k,t)}-w_0^{(k,t)}\alpha)(\frac{\beta}{\alpha})^{n+q}}{(w_1^{(k,t)}-w_0^{(k,t)}\beta)-(w_1^{(k,t)}-w_0^{(k,t)}\alpha)(\frac{\beta}{\alpha})^{n}}\,.$$

Since $|\beta/\alpha|<1$, it follows that $(\beta/\alpha)^n\to 0$ as $n\to\infty$. Thus,

$$\lim_{n\to\infty}\frac{w_{n+q}^{(k,t)}}{w_n^{(k,t)}}=\alpha^q\,\frac{w_1^{(k,t)}-w_0^{(k,t)}\beta}{w_1^{(k,t)}-w_0^{(k,t)}\beta}=\alpha^q\,,$$

and thus (33) follows.

Using the Equation (27), we can write

$$\frac{w_{-(n+q)}^{(k,t)}}{w_{-n}^{(k,t)}}=(-1)^q\frac{w_{n+q}^{(k,t)}-(k^2+4+2t)F_{k,n+q}}{w_n^{(k,t)}-(k^2+4+2t)F_{k,n}}\,.$$

Also follows from Binet's formula that

$$\begin{aligned}&\frac{w_{n+q}^{(k,t)}-(k^2+4+2t)F_{k,n+q}}{w_n^{(k,t)}-(k^2+4+2t)F_{k,n}}\\ =\;&\alpha^q\,\frac{(w_1^{(k,t)}-w_0^{(k,t)}\beta)-(w_1^{(k,t)}-w_0^{(k,t)}\alpha)(\frac{\beta}{\alpha})^{n+q}-(k^2+4+2t))(1-(\frac{\beta}{\alpha})^{n+q})}{(w_1^{(k,t)}-w_0^{(k,t)}\beta)-(w_1^{(k,t)}-w_0^{(k,t)}\alpha)(\frac{\beta}{\alpha})^{n}-(k^2+4+2t))(1-(\frac{\beta}{\alpha})^{n})}\,.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\left[\frac{w_{n+q}^{(k,t)}-(k^2+4+2t)F_{k,n+q}}{w_n^{(k,t)}-(k^2+4+2t)F_{k,n}}\right]=\alpha^q\,\frac{(w_1^{(k,t)}-w_0^{(k,t)}\beta)-(k^2+4+2t)}{(w_1^{(k,t)}-w_0^{(k,t)}\beta)-(k^2+4+2t)}=\alpha^q\,,$$

then, multiplying both sides by $(-1)^q$, it concludes the proof. □

In what follows, we can immediately establish the next result using fundamental tools from the calculus of limits, along with (33) and (34).

Corollary 7.1. *Let $w_n^{(k,t)}$ be the n-th term of the (k,t)-Fibonacci–François sequence. Then we have*

$$\lim_{n\to\infty}\frac{w_n^{(k,t)}}{w_{n+q}^{(k,t)}}=\left(\frac{1}{\alpha}\right)^q,$$

and

$$\lim_{n\to\infty}\frac{w_{-n}^{(k,t)}}{w_{-(n+q)}^{(k,t)}}=\left(-\frac{1}{\alpha}\right)^q,$$

for any integer q, where α is a root of Equation (9)*.*

Now, we investigate the properties and identities associated with the partial sums of the $n+1$ terms of the (k,t)-Fibonacci–François sequence, providing insights into their pattern and applications.

We begin by presenting three key results concerning the partial sums of the (k,t)-Fibonacci–François sequence. First, we present the sum of the first $n+1$ terms.

Proposition 7.2. *For all non-negative integers n, then*

$$\sum_{i=0}^{n} w_i^{(k,t)}=\frac{1}{k}[w_n^{(k,t)}+w_{n+1}^{(k,t)}-2k-t]\ ,$$

where $\{w_n^{(k,t)}\}_{n\geq 0}$ is the k-Fibonacci–François sequence.

Proof. According to Equation (8) we have the following equations:

$$\begin{array}{rcl} w_0^{(k,t)} &=& w_2^{(k,t)}-kw_1^{(k,t)},\\ &\vdots& \\ w_{n-1}^{(k,t)} &=& w_{n+1}^{(k,t)}-kw_n^{(k,t)}\,. \end{array}$$

By adding both sides of these equations, we have

$$\sum_{i=0}^{n-1} w_i^{(k,t)}=\sum_{i=2}^{n+1} w_i^{(k,t)}-k\sum_{i=1}^{n} w_i^{(k,t)}\,,$$

that is,

$$k\sum_{i=1}^{n} w_i^{(k,t)}=w_n^{(k,t)}+w_{n+1}^{(k,t)}-w_0^{(k,t)}-w_1^{(k,t)},$$

By adding both sides of these equations by $kw_0^{(k,t)}$ and using that $w_0^{(k,t)}=k-2$ and $w_1^{(k,t)}=k^2-k+2+t$, we conclude the result, since $(k-1)(k-2)-(k^2-k+2+t)=-2k-t$. ☐

The sum of the terms at even indices of the (k,t)-Fibonacci–François sequence can be expressed as:

Proposition 7.3. *For all non-negative integers n, we have*

$$\sum_{i=0}^{\frac{n}{2}} w_{2i}^{(k,t)} = \frac{1}{k}\left[w_{n+1}^{(k,t)} - k - t - 2\right], \quad \textit{if } n \textit{ is even},$$

$$\sum_{i=0}^{\frac{n-1}{2}} w_{2i}^{(k,t)} = \frac{1}{k}\left[w_{n}^{(k,t)} - k - t - 2\right], \quad \textit{if } n \textit{ is odd},$$

where $\{w_n^{(k,t)}\}_{n\geq 0}$ is the (k,t)-Fibonacci–François sequence.

Next, the sum of the terms with odd indices of the k-Fibonacci difference sequence is given by:

Proposition 7.4. *For all non-negative integers n, then*

$$\sum_{i=0}^{\frac{n}{2}} w_{2i-1}^{(k,t)} = \frac{1}{k}\left[w_{n}^{(k,t)} - k + 2\right], \textit{ if } n \textit{ is even, and}$$

$$\sum_{i=0}^{\frac{n-1}{2}} w_{2i}^{(k,t)} = \frac{1}{k}\left[w_{n+1}^{(k,t)} - k + 2\right], \textit{ if } n \textit{ is odd.}$$

A direct consequence of the previous results is the result presented below.

Proposition 7.5. *Let $\{w_n^{(k,t)}\}_{n\geq 0}$ be the k-Fibonacci–François sequence. For all non-negative integers m, we have the following formulas:*

(a) $\displaystyle\sum_{i=0}^{m}(-1)^i w_i^{(k,t)} = \frac{1}{k}[w_m^{(k,t)} - w_{m+1}^{(k,t)} - 2k - t]$, *if m is odd, and*

(b) $\displaystyle\sum_{i=0}^{m}(-1)^i w_i^{(k,t)} = \frac{1}{k}[w_{m+1}^{(k,t)} - w_m^{(k,t)} - t - 4]$, *if m is even.*

To finalize, we derive an expression for the sum of the squares of the first $n+1$ terms of the k-Fibonacci difference sequence, relating it to the classical Fibonacci sequence.

Proposition 7.6. *The sum of the squares of the first $n+1$ terms of the (k,t)-Fibonacci–François sequence is given by*

$$\sum_{i=0}^{n}(w_i^{(k,t)})^2 = \frac{1}{k}[w_n^{(k,t)} \cdot w_{n+1}^{(k,t)} + k(w_0^{(k,t)})^2 - w_0^{(k,t)}\, w_1^{(k,t)}]$$

for all non-negative integers n.

Proof. To begin, observe that for $n \geq 2$, the following identity holds:

$$w_n^{(k,t)} w_{n+1}^{(k,t)} - w_{n-1}^{(k,t)} w_n^{(k,t)} = w_n^{(k,t)}(w_{n+1}^{(k,t)} - w_{n-1}^{(k,t)}) = k(w_n^{(k,t)})^2\,.$$

Thus, we find:

$$
\begin{aligned}
k(w_2^{(k,t)})^2 &= w_2^{(k,t)}w_3^{(k,t)} - w_1^{(k,t)}w_2^{(k,t)},\\
k(w_3^{(k,t)})^2 &= w_3^{(k,t)}w_4^{(k,t)} - w_2^{(k,t)}w_3^{(k,t)},\\
&\vdots\\
k(w_n^{(k,t)})^2 &= w_n^{(k,t)}w_{n+1}^{(k,t)} - w_{n-1}^{(k,t)}w_n^{(k,t)}.
\end{aligned}
$$

Adding both sides of these equations yields:

$$k(w_2^{(k,t)})^2 + k(w_3^{(k,t)})^2 + \cdots + k(w_{n-1}^{(k,t)})^2 + k(w_n)^2 = w_n^{(k,t)}w_{n+1}^{(k,t)} - w_1^{(k,t)}w_2^{(k,t)}.$$

Since $w_2^{(k,t)} = w_0^{(k,t)} + kw_1^{(k,t)}$, it follows

$$
\begin{aligned}
k\sum_{j=0}^{n}(w_j^{(k,t)})^2 &= k(w_0^{(k,t)})^2 + k(w_1^{(k,t)})^2 + w_n^{(k,t)}w_{n+1}^{(k,t)} - w_1^{(k,t)}w_2^{(k,t)}\\
&= k(w_0^{(k,t)})^2 - w_0^{(k,t)}w_1^{(k,t)} + w_n^{(k,t)}w_{n+1}^{(k,t)},
\end{aligned}
$$

which completes the proof. □

8 Conclusion

In the present work, we have studied the (k,t)-Fibonacci–François sequence associated with the generalized two-parameter François sequence, and original results, including the classical Tagiuri–Vajda identity and other related results, have thus been established. We have seen that the sequence belongs to the family of k-Fibonacci sequences defined by specific initial terms. Then, we explore several classical identities from different perspectives. Following the standard framework, we also derive certain consequences from Binet's formula including some expressions for the negative indices. Still in this context, we can write the (k,t)-Fibonacci–François sequence as a linear combination of the terms of the k-Fibonacci and k-Lucas sequences, enabling us to extract a wide variety of properties and identities for the (k,t)-Fibonacci–François numbers, as well as some expressions for the negative subscripts of the sequences. This allowed us to establish expressions for the limit of the quotient of successive terms and to evaluate some partial sums. In future work, we intend to apply these results to understand the generalized two-parameter François sequences.

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