Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2026, Volume 32, Number 2, 342–353
DOI: 10.7546/nntdm.2026.32.2.342-353

Some new inequalities for the q-gamma and related functions, II

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Received: 10 December 2025 **Revised:** 1 June 2026
Accepted: 4 June 2026 **Online First:** 4 June 2026

Abstract: As a continuation of [5], we offer new inequalities for Jackson's q-gamma function $\Gamma_q(x)$. For example, we obtain a q-analogue of the famous Jordan inequality for $(\sin x)/x$ for $x \in (0, \pi/2)$. Related inequalities, and other relations, such as the limit relations for the q-gamma constant γ_q, are also pointed out.
Keywords: q-gamma function, q-digamma functions, q-sine function inequalities, Special functions.
2020 Mathematics Subject Classification: 33B15, 26D15, 33E05.

1 Introduction

In what follows, we assume that q is a real number such that $0 < q < 1$. The q-shifted factorials of a real number a are defined by

$$(a;q)_0 = 1,\ (a;q)_n = \prod_{i=0}^{n-1}(1 - a \cdot q^i), (a;q)_\infty = \lim_{n\to\infty}(a;q)_n.$$

For any real number x, we let

$$[x]_q = \frac{1-q^x}{1-q} \tag{1}$$

for which we have $\lim\limits_{x\to\infty}[x]_q = x$. The Jackson q-gamma function is defined by (see [13, 14]):

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty} \cdot (1-q)^{1-x}. \tag{2}$$

It is immediate that

$$\Gamma_q(1) = 1,\ \Gamma_q(x+1) = [x]_q \cdot \Gamma_q(x) \tag{3}$$

and

$$\lim_{q \to 1-} \Gamma_q(x) = \Gamma(x), \tag{4}$$

where $\Gamma(x)$ is the classical Euler-gamma function (see [2]). The Euler digamma function is $\psi(x) = (\log \Gamma(x))' = \Gamma'(x)/\Gamma(x)$ and its q-analogue is the q-digamma function given by

$$\psi_q(x) = (\log \Gamma_q(x))' = \Gamma_q'(x)/\Gamma_q(x). \tag{5}$$

One has the following series representations

$$\begin{aligned} \psi_q(x) &= -\log(1-q) + (\log q) \cdot \sum_{n=0}^{\infty} \frac{q^{n+x}}{1-q^{n+x}} \\ &= -\log(1-q) + (\log q) \cdot \sum_{n=1}^{\infty} \frac{q^{nx}}{1-q^n} \end{aligned} \tag{6}$$

(see [1, 2]).

It follows from (6) that the derivative of the function $\psi_q(x)$ is strictly completely monotonic on $(0, \infty)$, that is

$$(-1)^n (\psi_q'(x))^{(n)} > 0,\ n = 0, 1, 2, \dots \ \ (x > 0). \tag{7}$$

In particular , $\psi_q(x)$ is strictly increasing and concave function on $(0, \infty)$.

As $(\log \Gamma_q(x))'' = \psi_q'(x) > 0$, we get that $\Gamma_q(x)$ is strictly log-convex function on $(0, \infty)$.

In 2005, D. M. Bradley [8] introduced an extension of the Euler gamma constant γ as follows:

$$\gamma_q = \log(1-q) - \frac{\log q}{1-q} \cdot \sum_{i=1}^{\infty} \frac{q^i}{[i]_q} \tag{8}$$

and proved that

$$\lim_{q \to 1-} \gamma_q = \gamma. \tag{9}$$

M. Mahmoud and R. P. Agarwal [15] proved that $\gamma_q = -\psi_q(1)$, and in [5] we have shown that $0 < \gamma_q < 1$ for any $q \in (0, 1)$.

2 q-Jordan and Kober inequalities

The famous Jordan inequality states that for any $x \in (0, \pi/2)$ one has

$$\frac{\sin x}{x} > \frac{2}{\pi} \tag{10}$$

We note that (10) has applications in number theory, too (see, e.g., [20]). Kober's inequality states that

$$\cos x > 1 - \frac{2x}{\pi}. \tag{11}$$

For new proofs, extensions and generalizations of (10) and (11), see, e.g., [23–26].

In 2001, R. W. Gosper [11] defined his q-trigonometric functions as follows:

$$\sin_q(\pi x) = q^{(x-1/2)^2} \cdot \frac{(q^{2x};q^2)_\infty \cdot (q^{2-2x};q^2)_\infty}{(q;q^2)_\infty^2} \tag{12}$$

$$\cos_q(\pi x) = q^{x^2} \cdot \frac{(q^{1-2x};q^2)_\infty \cdot (q^{2x+1};q^2)_\infty}{(q;q^2)_\infty^2} \tag{13}$$

for $0 < q < 1$. It can be seen that

$$\cos_q(x) = \sin_q\left(\frac{\pi}{2} - x\right). \tag{14}$$

Moreover, one has

$$\sin_q(x) = \frac{\vartheta_1(x,p)}{\vartheta_2(p)},\ \cos_q(x) = \frac{\vartheta_2(x,p)}{\vartheta_2(p)} \tag{15}$$

where p is implicitly defined by the equation

$$\log p \cdot \log q = \pi^2$$

and ϑ_1, ϑ_2 are the classical Jacobi theta functions

$$\begin{aligned} \vartheta_1(x,q) &= 2\sum_{n=0}^{\infty}(-1)^n q^{(n+1/2)^2} \cdot \sin[(2n+1)x], \\ \vartheta_2(x,q) &= 2\sum_{n=0}^{\infty} q^{(n+1/2)^2} \cdot \cos[(2n+1)x]. \end{aligned} \tag{16}$$

For certain other sums of q-trivgonometric and related functions, see [7]. The following important connection between the q-gamma function and q-sine function holds true (see [16, 17]).

Lemma 1. *We have that*

$$\sin_q(\pi x) = q^{x(x-1)+\frac{1}{4}} \cdot \Gamma_{q^2}^2\left(\frac{1}{2}\right) \cdot \frac{1}{\Gamma_{q^2}(x)\Gamma_{q^2}(1-x)}, \tag{17}$$

which is a generalization of Euler's reflection formula

$$\sin(\pi x) = \frac{\pi}{\Gamma(x)\Gamma(1-x)}. \tag{18}$$

Lemma 2. *The function*

$$f(x) = [x]_q \cdot \Gamma_q(x)\Gamma_q(1-x),\ x \in (0,1) \tag{19}$$

is strictly increasing.

Proof. By (3) one has $f(x) = \Gamma_q(x+1)\Gamma_q(1-x)$, thus $\log f(x) = \log\Gamma_q(x+1) + \log\Gamma_g(1-x)$. A simple derivative gives

$$\frac{f'(x)}{f(x)} = \psi_q(1+x) - \psi_q(1-x) \tag{20}$$

by (5). Since $1+x > 1-x$ from $x > 0$, and using the fact the function $\psi_q(t)$ is strictly increasing for $t > 0$, by (20) we get that $f'(x) > 0$, so $f(x)$ is strictly increasing. □

Now, the q-Jordan inequality may be stated as follows:

Theorem 1. *For $x \in (0, \frac{1}{2}]$ one has*

$$\frac{\sin_q(\pi x)}{[x]_{q^2}} \geq \frac{q^{x(x-1)+1/4}}{\left[\frac{1}{2}\right]_{q^2}}. \tag{21}$$

Proof. By Lemma 2 one has $[x]_{q^2} \cdot \Gamma_{q^2}(x)\Gamma_{q^2}(1-x) \leq \left[\frac{1}{2}\right]_{q^2} \left(\Gamma_{q^2}\left(\frac{1}{2}\right)\right)^2$ for $x \in (0, \frac{1}{2}]$ and from identity (17) we immediately get relation (21). □

Remark 1. *By letting $y = \pi x$, we get*

$$\frac{\sin_q(y)}{\left[\frac{y}{\pi}\right]_{q^2}} \geq \frac{q^{\frac{y}{\pi}\left(\frac{y}{\pi}-1\right)+1/4}}{\left[\frac{1}{2}\right]_{q^2}} \tag{22}$$

for $y \in (0, \frac{\pi}{2}]$.

Now, letting $z = \frac{\pi}{2} - z$ in (22), we get the q-Kober inequality:

Theorem 2.

$$\frac{\cos_q(z)}{\left[\frac{1}{2} - \frac{z}{\pi}\right]_{q^2}} \geq \frac{q^{\left(\frac{1}{2}-\frac{z}{\pi}\right)\left(-\frac{1}{2}-\frac{z}{\pi}\right)+\frac{1}{4}}}{\left[\frac{1}{2}\right]_{q^2}} \tag{23}$$

for $z \in (0, \pi/2]$.

3 Limit relations for the q-gamma constant

In 1976, K. Demys [9], by using the Riemann zeta function theory, has proved the following limit relation:

$$\gamma = \lim_{x\to\infty}\left(x - \Gamma\left(\frac{1}{x}\right)\right), \tag{24}$$

where γ is the classical Euler constant,

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right),$$

and $\Gamma(x)$ is the classical Euler gamma function.

It is immediate that, relation (24) can be rewritten as

$$\gamma = \lim_{x\to 0}\left(\frac{1}{x} - \Gamma(x)\right) = \lim_{x\to 0}\left(\frac{1-\Gamma(x+1)}{x}\right), \tag{25}$$

by using the fact that $x\Gamma(x) = \Gamma(x+1)$.

Now, by rewriting the right side of (25) as $\frac{1-\Gamma_q(x+1)}{x}$, as $\Gamma_q(1) = 1$, by L'Hospital's rule

$$\lim_{x\to 0}\frac{1-\Gamma_q(x+1)}{x} = \lim_{x\to 0}\frac{-\Gamma_q'(x+1)}{1} = -\lim_{x\to 0}\Gamma_q'(1) = \psi_q'(1) = \gamma_q;$$

and remarking that $\Gamma_q(x+1) = [x]_q\Gamma_q(x)$, we get the following:

Theorem 3.

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{[x]_q}{x}\cdot\Gamma_q(x)\right)=\gamma_q, \tag{26}$$

which is an extension of (25) *for the* q*-theory.*

Clearly, (26) can be rewritten as

$$\lim_{x\to\infty}\left(x-x\left[\frac{1}{x}\right]_q\cdot\Gamma_q\left(\frac{1}{x}\right)\right)=\gamma_q \tag{27}$$

which is an extension of the original form (24).

On the other hand, remark that $(e^{\Gamma_q(x)})'=\Gamma_q'(x)\cdot e^{\Gamma_q(x)}$, so the limit of $\frac{e^{\Gamma_q(x)}-e}{x-1}$, as $x\to 1$; again by L'Hospital's rule, is $\Gamma_q'(1)e=-e\gamma_q$, so we get:

Theorem 4. *One has*

$$\lim_{x\to 1}\frac{e^{\Gamma_q(x)-1}-1}{x-1}=-\gamma_q. \tag{28}$$

We note, that, a proof of (24), based on Euler's original definition of the Gamma function and some combinatorial identities, can be found in a paper by H. Gould [12].

Remark 2. *In* [5, Lemma 9], *it is proved that* $\psi_q\left(\frac{1}{2}\right)<2\psi_q(1)$, *thus as* $\psi_q(1)=-\gamma_q$, *we get:*

$$\gamma_q>-\frac{1}{2}\psi_q\left(\frac{1}{2}\right). \tag{29}$$

4 Inequalities for q-gamma and q-digamma functions

Lemma 3. 1) *Let* $f:[0,\infty)\to(0,\infty)$ *be a differentiable, log-convex function, and let* $a\geq 1$. *The the function* g *defined by* $g(x)=(f(x))^a/f(ax)$ *decreases in its domain. In particular, if* $0\leq x\leq y$, *one has the inequalities*

$$\frac{\big(f(y)\big)^a}{f(ay)}\leq\frac{\big(f(x)\big)^a}{f(ax)}\leq\big(f(0)\big)^{a-1}. \tag{30}$$

If $0<a\leq 1$, *then* g *is an increasing function, and the inequalities in* (30) *are reversed.*

2) *Let* f *be differentiable, log-concave, and* $a\geq 1$. *Then the function* g *defined in* 1) *is increasing, and the inequalities* (30) *are reversed. For* $0<a\leq 1$, *then* g *is decreasing.*

Proof. Result 1) is stated and proved in [19]. For the proof of 2), define $\beta(x)=f'(x)/f(x)=(\log f(x))'$, which is decreasing, so that $\beta(x)\geq\beta(ax)$ for $a\geq 1$. Now, as $\frac{g'(x)}{g(x)}=(\log g(x))'=a\cdot[\beta(x)-\beta(ax)]\geq 0$, we get $g'(x)\geq 0$, so $g(x)$ is increasing.

From $0\leq x\leq y$ we get $g(0)\leq g(x)\leq g(y)$, so the inequalities are valid in reversed order. □

Theorem 5. *For* $x\in[0,1]$ *and* $a\geq 1$ *one has*

$$\frac{1}{\Gamma_q(1+a)}\leq\frac{(\Gamma_q(1+x))^a}{\Gamma_q(1+ax)}. \tag{31}$$

For $0 \le x \le y$ *one has for* $a \ge 1$

$$\frac{(\Gamma_q(1+y))^a}{\Gamma_q(1+ay)} \le \frac{(\Gamma_q(1+x))^a}{\Gamma_q(1+ax)}. \tag{32}$$

For $0 < a \le 1$, *then inequalities* (31) *and* (32) *are reversed.*

Proof. The function $f(x) = \Gamma_q(x+1)$is log-convex, so applying Lemma 3, 1), the inequalities (31) and (32) are valid. For $y = 1$, relation (32) implies (31). □

Remark 3. *For the Euler gamma function, inequality* (31) *appeared in the author's paper* [23].

Theorem 6. *For* $x \in [0,1]$ *and* $a \ge 1$ *one has*

$$\frac{1}{\Gamma_q(2+a)^{1/(a+1)}} \ge \frac{(\Gamma_q(2+x))^{a/(x+1)}}{(\Gamma_q(2+ax))^{1/(ax+1)}}. \tag{33}$$

For $0 < a \le 1$, *then inequalities is reversed.*

Proof. In [5, Theorem 4] it is proved that the function $f(x) = (\Gamma_q(x+2))^{1/(x+1)}$ is log-concave. Then applying Lemma 3, 2), and letting $y = 1$, we get inequality (33). □

Theorem 7. *For* $x \in [0,1]$ *and* $a \ge 1$ *one has*

$$\frac{(\Gamma_q(x+2))^{a/(x+1)}}{(\Gamma_q(ax+2))^{1/(ax+1)}} \cdot \frac{ax+1}{(x+1)^a} \ge \frac{a+1}{2^a} \frac{1}{(\Gamma_q(a+2))^{1/(a+1)}}. \tag{34}$$

Proof. In [5, Theorem 5], it is proved that the function

$$f(x) = \frac{(\Gamma_q(x+2))^{1/(x+1)}}{(x+1)}$$

is log-convex on $(0, \infty)$.

Applying Lemma 3, 1) with $y = 1$ and $0 \le x \le 1$, we get relation (34) by remarking that $\Gamma_q(2) = 1$. □

Theorem 8. *For* $x \in [0,1]$ *and* $a \ge 1$ *one has*

$$\frac{\big(\psi_q(x+2)\big)^a}{\psi_q(ax+2)} \ge \frac{\Big(-\gamma_q - \frac{q\log q}{1-q}\Big)}{\psi_q(a+2)}. \tag{35}$$

Proof. The function $f(x) = \psi_q(x+2)$ is concave on base of (7). It is well-known that concave functions are also log-concave. Then, using Lemma 3, 2) and remarking that

$$\psi_q(2) = \psi_q(1) - \frac{q\log q}{1-q} = -\gamma_q - \frac{q\log q}{1-q}$$

(see (9)), as for $0 \le y \le x$, one has

$$\frac{\big(\psi_q(x+2)\big)^a}{\psi_q(ay+2)} \ge \frac{\big(\psi_q(y+2)\big)^a}{\psi_q(ay+2)},$$

from which for $y = 0$ we get inequality (35). □

Theorem 9. *For $x \in [0,1]$ and $a \geq 1$ one has*

$$\frac{1}{\big(\Gamma_q(a+1)\big)^{a+1}} \leq \frac{\big(\Gamma_q(x+1)\big)^{a(x+1)}}{\big(\Gamma_q(ax+1)\big)^{a+1}}. \tag{36}$$

Proof. By an immediate computation we get that for $a(x) = (\Gamma_q(x))^x$ one has $(\log a(x))'' = 2\psi_q(x) + x\psi_q'(x) > 0$, for $x \geq 1$, by [1, Lemma 3.4]. Put now $f(x) = (\Gamma(x+1))^{x+1}$. Clearly, $x+1 \geq 1$ for $x \geq 0$, and using the log-convexity of this function, by Lemma 3, 1) and putting $y = 1$, after an easy computation we get (36). □

The following Lemma gives the q-Raabe formula:

Lemma 4. *For any $t \geq 0$ one has*

$$\int_t^{t+1} \log \Gamma_q(x)dx = \left(\frac{1}{2} - t\right)\log(1-q) - \frac{1}{\log q}\cdot \mathrm{Li}_2(q^t) + \log(q;q)_\infty, \tag{37}$$

where

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

Proof. Formula (37) was first proved independently by M. E. Bachraoui [3], and M. Mahmoud and R. P. Agarwal [15].

In 2020, M. E. Bachraoui and the author [6] have provided a new elegant proof for (37). □

Theorem 10. 1) *For any $x > 0$ one has*

$$\log \Gamma_q(x) > \left(\frac{1}{2} - x\right)\log(1-q) - \frac{1}{\log q}\cdot \mathrm{Li}_2(q^x) + \log(q;q)_\infty - \frac{1}{2}\log[x]_q. \tag{38}$$

2) *For $x > 1/2$ one has*

$$\log \Gamma_q(x) < (1-x)\log(1-q) - \frac{1}{\log q}\cdot \mathrm{Li}_2(q^{x-1/2}) + \log(q;q)_\infty. \tag{39}$$

Proof. The function $\log \Gamma_q(x)$ is strictly convex for $x > 0$. Now, apply the classical Hadamad inequalities (see, e.g., [18, 21]) for $\log \Gamma_q(x)$:

$$\log \Gamma_q\left(\frac{a+b}{2}\right) < \frac{1}{b-a}\int_a^b \log \Gamma_q(t)dt < \frac{\log \Gamma_q(a) + \log \Gamma_q(b)}{2}. \tag{40}$$

Let $a = t$, $b = t+1$; and apply relation (37). Since $\Gamma_q(t+1) = [t]_q \cdot \Gamma_q(t) = \frac{1-q^t}{1-q} \cdot \Gamma_q(t)$; after simple computations, from (37) and the right-hand side of (40), for $t = x$ we get inequality (38). By applying the left-hand side of (40), combined with (37), and by putting $x = t + \frac{1}{2}$, we get immediately (39). □

Theorem 11. *One has for any $0 < a < b$*

$$e^{(b-a)\cdot\psi_q\left(\frac{a+b}{2}\right)} > \frac{\Gamma_q(b)}{\Gamma_q(a)} > e^{(b-a)\left[\frac{\psi_q(a)+\psi_q(b)}{2}\right]}. \tag{41}$$

Proof. Let $f(x) = (\log\Gamma_q(x))' = \psi_q(x)$ in reverse inequalities of (40). As

$$\int_a^b (\log\Gamma_q(x)'dx = \log\Gamma_q(b) - \log\Gamma_q(a) = \log\frac{\Gamma_q(b)}{\Gamma_q(a)},$$

after simple computations, we get inequalities (41). We have used that $\psi_q(x)$ is concave, from (7). □

Remark 4. *By putting* $a = x$, $b = x+1$, *since* $\Gamma_q(x+1) = [x]_q \cdot \Gamma_q(x)$, *and* $\psi_q(x+1) = \psi_q(x) - \frac{q^x \log q}{q-1}$, *from* (41) *we can write:*

$$e^{\psi_q\left(x+\frac{1}{2}\right)} > [x]_q > e^{\psi_q(x) - \frac{q^x \log_q}{2(q-1)}}, \tag{42}$$

which written equivalently give the following relations:

$$\psi_q(x) < \log\frac{1-q^x}{1-q} + \frac{q^x\log q}{2(q-1)},\ x > 0, \tag{43}$$

$$\psi_q(x) > \log\frac{1-q^{x-1/2}}{1-q},\ x > \frac{1}{2}. \tag{44}$$

Remark 5. *The following refinement of the right-hand side Hadamard inequality is proved in* [22] *and* [25, p. 271].

Let $f : [a,b] \to \mathbb{R}$ *convex(concave) and differentiable. Then one has*

$$\frac{1}{b-a}\int_a^b f(x)dx \underset{(\geq)}{\leq} \frac{1}{2}\left[\frac{af(b)+bf(a)}{a+b} + f\left(\frac{a^2+b^2}{a+b}\right)\right] \underset{(\geq)}{\leq} \frac{f(a)+f(b)}{2}. \tag{45}$$

By using (37) *and* (45) *for* $f(x) = \Gamma_q(x)$, *one can obtain a refinement of* (38).

Remark 6. *Remarking that* $\frac{a^2+b^2}{a+b} = \frac{a}{a+b}\cdot a + \frac{b}{a+b}\cdot b = \lambda a + \mu b$, *where* $\lambda = \frac{a}{a+b}$, $\mu = \frac{b}{a+b}$, *and by* $\lambda, \mu > 0$, $\lambda + \mu = 1$; *and using the fact that* $\log\Gamma_q(x)$ *is convex, we can write*

$$\log\Gamma_q\left(\frac{a^2+b^2}{a+b}\right) \leq \frac{a\log\Gamma_q(a) + b\log\Gamma_q(b)}{a+b},$$

so we get the inequality:

$$\left[\Gamma_q\left(\frac{a^2+b^2}{a+b}\right)\right]^{a+b} \leq \left(\Gamma_q(a)\right)^a \cdot \left(\Gamma_q(n)\right)^b;\ a,b > 0. \tag{46}$$

There is equality only for $a = b$.

Remark 7. *By using* (45) *for the concave function* $f(x) = \psi_q(x) = (\log\Gamma_q(x))'$, *we get*

$$\frac{1}{b-a}\log\frac{\Gamma_q(b)}{\Gamma_q(a)} > \frac{1}{2}\left[\frac{a\psi_q(b)+b\psi_q(a)}{a+b} + \psi_q\left(\frac{a^2+b^2}{a+b}\right)\right] > \frac{\psi_q(a)+\psi_q(b)}{2}. \tag{47}$$

Remark 8. *In paper* [10] *it is proved that if* $f : [a,b] \to (0,\infty)$ *is log-convex, then*

$$\frac{1}{b-a}\int_a^b f(t)dt \leq L(f(a), f(b)), \tag{48}$$

where

$$L(u,v) = \frac{u-v}{\log u - \log v}$$

is the logarithmic mean $(L(u,u) = u)$. *If* f *is strictly log-convex, then the inequality is strict. By letting* $f(t) = \Gamma_q(t)$; $a = x$; $b = x+1$, *and using the left-hand side in Hadamard inequality, we get*

$$\Gamma_q(x + \frac{1}{2}) < (\Gamma_q(x))([x]_q - 1)/\log[x]_q \tag{49}$$

In what follows, we will use discrete inequalities for convex functions, with applications to q-gamma and digamma functions. Let I be a real interval.

Lemma 5. 1) *If* $f : I \subset \mathbb{R} \to \mathbb{R}$ *is a convex function, then*

$$f\Big(\frac{x_1 + x_2 + \ldots + x_n}{n}\Big) \leq \frac{f(x_1) + \cdots + f(x_n)}{n} \quad (x_i \in I, i = \overline{1,n}). \tag{50}$$

If f *is strictly convex, then the inequality is strict.*

2) *If* f *is differentiable, and has an increasingly, strictly positive derivative, then*

$$\frac{f(x_1) + \cdots + f(x_n)}{n} \leq (\geq), f\Big(\frac{x_1 f_1'(x_1) + \cdots + x_n f'(x_n)}{f_1'(x_1) + \cdots + f_n'(x_n)}\Big), \tag{51}$$

for $x_i \in I$ $(i = \overline{1,n})$.

Proof. (50) is the classical Jensen inequality for convex functions, while (50) is a particular case of Slater's inequality (see [8]). We will give here the simple proof of (51).

Since $fi'(x_i) > 0$, $i = 1, 2, \ldots, n$; then it is immediate that

$$\begin{aligned} \min\{x_1, \ldots, x_n\} \cdot (f_1'(x_1) + \cdots + f'(x_n)) &\leq x_1 f_1'(x_1) + x_n f'(x_n) \\ &\leq \max\{x_1, \ldots, x_n\}(f_1'(x_1) + \cdots + f'(x_n)), \end{aligned}$$

thus

$$y = \frac{x_1 f_1'(x_1) + \cdots + x_n f'(x_n)}{f_1'(x_1) + \cdots + f_n'(x_n)} \in I. \tag{52}$$

Since f' is increasing and $f' > 0$, one has

$$f(y) \geq f(x_i) + (y - x_i) \cdot f'(x_i) \tag{53}$$

for any $x_i \in I$, $y \in I$. If $y = x_i$, there is equality in (51), while for $y \neq x_i$; let, e.g., $y > x_i$. Then by the Lagrange mean-value theorem $f(y) - f(x_i) = (y - x_i)f'(\xi_i)$, $y > \xi_i > x_i$. By $f'(\xi_i) \geq f'(x_i)$, inequality (53) follows. For $y < x_i$ the proof is similar. Now, letting in (53) the y given by (52), after summation for $i = 1, 2, \ldots, n$, inequality (51) follows.

If f' is strictly increasing, the inequality is strict. □

Remark 9. *In Slater's general result* f *is supposed to be convex with* $f_+'(x) > 0$ *and in place of* $f'(x_i)$ *there are* $f_+'(x_i)$, *i.e., the derivative at the right side (see* [28]*).*

Theorem 12. *Let x_0 be the unique root of $\psi_q(x)=0$ on $(0,\infty)$. If $x_i \in \left(\frac{1}{x_0+1}, \frac{1}{2}\right]$ $(i = 1, 2, \ldots, n)$, then*

$$\left(\prod_{i=1}^{n} \Gamma_q\left(\frac{x_i'}{x_i}\right)\right) \leq (X_{n,q})^n, \tag{54}$$

where

$$X_{n,q} = \frac{\displaystyle\sum_{i=1}^{n} \frac{1}{x_i}\psi_q\left(\frac{x_i'}{x_i}\right)}{\displaystyle\sum_{i=1}^{n} \frac{1}{x_1^2}\psi_q\left(\frac{x_i'}{x_i}\right)} \tag{55}$$

and $x_i' = 1 - x_i$.

Proof. In [5] it is proved that the function $f(x) = \log \Gamma_q\left(\frac{1-x}{x}\right)$ is convex on $(0, \frac{1}{2}]$. Remark that $f'(x) = -\frac{1}{x^2} \cdot \psi_q\left(\frac{1-x}{x}\right) > 0$ as $\frac{1-x}{x} < x_0$ can be rewritten as $x > \frac{1}{x_0+1}$; and $\psi_q\left(\frac{1-x}{x}\right) < 0$. Clearly $\frac{1}{x_0+1} < \frac{1}{2}$, as $x_0 > 1$.

Now, using the notation $x_i' = 1 - x_i$, after elementary computations we get (54). □

Remark 10. *In* [5]*, by using* (50) *it is proved that*

$$\left(\prod_{i=1}^{n} \Gamma_q\left(\frac{x_i'}{x_i}\right)\right)^{1/n} \geq \Gamma_q\left(\frac{A_n'}{A_n}\right), \tag{56}$$

where A_n is the arithmetic mean of x_i, and $A_n' = 1 - A_n$. This is a Ky Fan type inequality for Γ_q. Now, by (56) *and* (54) *we get also the inequality*

$$\Gamma_q\left(\frac{A_n'}{A_n}\right) \leq X_{n,q} \ \text{ for } \ x_i \in \left(\frac{1}{x_0+1}, \frac{1}{2}\right]. \tag{57}$$

Remark 11. *If f' is strictly increasing, but $f'(x) < 0$, then* (51) *holds true also with the "$\leq$" sign, as in the right-hand side of* (51) *one can replace $x_i f_i'(x_i)$ with $-x_i' f_i'(x_i)$ in the numerator, and $f_i'(x_i)$ with $-f_i'(x_i)$ in the denominator. Let $f(x) = \log \Gamma_q(x)$ in this inequality. Then we get*

$$\left(\prod_{i=1}^{n} \Gamma_q(x_i)\right) \leq \left(\sum_{i=1}^{n} x_i \psi(x_i) \Big/ \sum_{i=1}^{n} \psi(x_i)\right)^n. \tag{58}$$

Let $x_i = \frac{i}{n}$, with $(i,n) = 1$ and $i = 1, \ldots, \varphi(n)$; where $\varphi(n)$ is the Euler totient function. A closed formula for $\displaystyle\prod_{\substack{i=1 \\ (i,n)=1}}^{n} \Gamma\left(\frac{i}{n}\right)$ was proved by Sándor and Tóth (see [27]*), while the q-gamma generalization analogue was given in* [4]*. Applying* (51) *in this case, we get*

$$\prod_{\substack{i=1 \\ (i,n)=1}}^{n} \Gamma_q\left(\frac{i}{n}\right) \leq \frac{\displaystyle\sum_{\substack{i=1 \\ (i,n)=1}}^{n} \frac{i}{n}\psi_q\left(\frac{i}{n}\right)}{\displaystyle\sum_{\substack{i=1 \\ (i,n)=1}}^{n} \psi_q\left(\frac{i}{n}\right)}, \tag{59}$$

which is inverse to an inequality stated in [5]*. We note that, as*

$$\sum_{i=1}^{n-1} \psi_q\left(\frac{i}{n}\right) = (n-1)\psi_q(1) - n \log \frac{1-q}{1-q^{1/n}} \tag{60}$$

(see (56)*), with the methods of* [27]*, we can obtain a formula for* $\sum_{\substack{i=1 \\ (i,n)=1}}^{n} \psi_q(i/n)$*, but we omit the details here.*

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