Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2026, Volume 32, Number 2, 335–341
DOI: 10.7546/nntdm.2026.32.2.335-341

A proof of Spence's formula using the reciprocity law for Dedekind sums

Steven Brown

48 rue Pottier, 78150 Le Chesnay Rocquencourt, France
e-mail: steven.brown.math@gmail.com

Received: 28 February 2026 **Revised:** 24 May 2026
Accepted: 30 May 2026 **Online First:** 3 June 2026

Abstract: In 1963, Edward Spence published a proof of the following:

With ϕ being Euler's totient function, if $n > 1$ is an integer, and if

$$0 < a_1 < \cdots < a_{\phi(n)} < n,$$

are the positive integers less than n, coprime with n, then

$$\sum_{j=1}^{\phi(n)} j a_j = \frac{\phi(n)}{24}\left(8n\phi(n) + 6n + 2\phi(m)(-1)^{\omega(m)} - 2^{\omega(m)}\right),$$

where m is the square-free part of n, and $\omega(m)$ is the number of prime factors of m.

Spence's proof relies on an ingenious observation considering Nagell's totient function. Later in 1971, Lucien Van Hamme provided an alternative proof of the result using Fourier analysis and previous work from Hubert Delange in 1968. In this paper, I propose another proof of the formula using the reciprocity law for Dedekind sums. If the formula is of interest on its own, it also plays a role in the analysis of the distribution of the a_j as suggested by the work from Hubert Delange.
Keywords: Spence formula, Dedekind sums, Arithmetical functions.
2020 Mathematics Subject Classification: 11A99, 11F20, 11A25.

Copyright © 2026 by the Author. This is an Open Access paper distributed under the terms and conditions of the Creative Commons Attribution 4.0 International License (CC BY 4.0). https://creativecommons.org/licenses/by/4.0/

1 A proof of Spence formula

We will prove:

Theorem 1.1. *With ϕ being Euler's totient function, if $n > 1$ is an integer, and if*

$$0 < a_1 < \cdots < a_{\phi(n)} < n,$$

are the positive integers less than n, coprime with n, then

$$\sum_{j=1}^{\phi(n)} ja_j = \frac{\phi(n)}{24}\left(8n\phi(n) + 6n + 2\phi(m)(-1)^{\omega(m)} - 2^{\omega(m)}\right), \tag{1}$$

where m is the square-free part of n, and $\omega(m)$ is the number of prime factors of m.

This formula has first been found and proven by Spence in 1963 in [6]. It was subsequently proved by Lucien Van Hamme in 1971 in [7]. In this paper, we propose another proof of the formula using the reciprocity law for Dedekind sums.

Proof. For any real number x, we note $\mathbf{E}(x)$ its integer part and $\{x\}$ its fractional part. We denote by $\mu(n)$ the Möbius function. Let

$$\theta_n(x) := \sum_{\substack{1\le k\le x\\ \gcd(k,n)=1}} 1 = \sum_{1\le k\le x}\sum_{\substack{d|k\\ d|n}} \mu(d) = \sum_{d|n}\mu(d)\mathbf{E}\left(\frac{x}{d}\right) \tag{2}$$

be the number of positive integers coprime with n and less than or equal to x. For any positive integer n, let $t(n)$ be the set of totatives[1] of n.

$$\sum_{j=1}^{\phi(n)} ja_j = \sum_{a\in t(n)} \theta_n(a)a. \tag{3}$$

Setting

$$\nu_n(x) := \sum_{d|n}\mu(d)\left\{\frac{x}{d}\right\} \tag{4}$$

and noticing that

$$\theta_n(x) + \nu_n(x) = x\frac{\phi(n)}{n}, \tag{5}$$

we get

$$\sum_{a\in t(n)}\theta_n(a)a = \frac{\phi(n)}{n}\sum_{a\in t(n)}a^2 - \sum_{a\in t(n)}\nu_n(a)a. \tag{6}$$

Noticing that

$$\begin{aligned}\sum_{a\in t(n)}a^2 &= \sum_{a=1}^{n}\sum_{\substack{d|a\\ d|n}}\mu(d)a^2 = \sum_{d|n}\mu(d)\sum_{a=1}^{n/d}(da)^2\\ &= \frac{1}{6}\sum_{d|n}\mu(d)\left(2\frac{n^3}{d} + 3n^2 + nd\right) = \frac{\phi(n)}{6}\left(2n^2 + (-1)^{\omega(n)m}\right),\end{aligned}$$

it remains to prove the following proposition. □

[1] Or totitive as introduced by James Joseph Sylvester in 1879, see [3, Chapter V]. See also [5, Chapter 3] for more recent results.

Proposition 1.1. *For any positive integer n, and m its square-free part,*

$$\sum_{a\in t(n)} \nu_n(a)a = \frac{\phi(n)}{24}\left(-6n + 2(-1)^{\omega(m)}\phi(m) + 2^{\omega(n)}\right). \tag{7}$$

The next section provides intermediate results to prove Proposition 1.1 which is done in the last section.

2 Intermediate results

We provide below some basic propositions that can be skipped or referred to as and when needed.

For positive integers a and b we use the notation $s(b,a)$ as in Rademacher's book [4] to denote the classical Dedekind sum[2]

$$s(b,a) := \sum_{k=1}^{a} \left(\!\left(\frac{kb}{a}\right)\!\right)\left(\!\left(\frac{k}{a}\right)\!\right). \tag{8}$$

with the symbol $(\!(x)\!)$ defined by

$$(\!(x)\!) := \begin{cases} \{x\} - \frac{1}{2}, & \text{if } x \notin \mathbb{Z}, \\ 0, & \text{if } x \in \mathbb{Z}. \end{cases} \tag{9}$$

Lemma 2.1. *For any $x \in \mathbb{R}$,*

$$(\!(x)\!) + (\!(-x)\!) = 0. \tag{10}$$

Proof. If x is an integer, so is $-x$ and the identity is obviously satisfied. If x is not an integer, then $\{x\} + \{-x\} = -\mathbf{E}(x) - \mathbf{E}(-x)$ is an integer which must equal 1 as it lies in $]0,2[$, then

$$(\!(x)\!) + (\!(-x)\!) = \{x\} - \frac{1}{2} + \{-x\} - \frac{1}{2} = 0.$$ □

Proposition 2.1. *Let $n > 1$ be an integer and d be a divisor of n,*

$$\sum_{a\in t(n)} \left(\!\left(\frac{a}{d}\right)\!\right) = 0. \tag{11}$$

Proof. Let $S = \sum_{a\in t(n)} \left(\!\left(\frac{a}{d}\right)\!\right)$. Since $a \mapsto n-a$ is a bijection of $t(n)$, we easily get to

$$2S = \sum_{a\in t(n)} \left(\left(\!\left(\frac{a}{d}\right)\!\right) + \left(\!\left(-\frac{a}{d}\right)\!\right)\right).$$

Considering Lemma 2.1, we conclude that $S = 0$ given that the summands are all equal to 0. □

Proposition 2.2. *Let a and b be positive integers,*

$$\sum_{j=1}^{b-1} \left(\!\left(j\frac{a}{b}\right)\!\right) = 0. \tag{12}$$

[2] In his book, Rademacher assumes a and b to be relatively prime. In the context of this paper, it is convenient to remove this coprimality condition, which poses no issue for $s(a,b)$ to be evaluated. The coprimality condition will be checked whenever required, for example in the case of the application of Dedekind's reciprocity law.

Proof. Let $S = \sum_{j=1}^{b-1} \left(\!\left(j\frac{a}{b} \right)\!\right)$. By considering the sum in reverse order with $j \mapsto b-j$, we get to

$$2S = \sum_{j=1}^{b-1} \left(\left(\!\left(j\frac{a}{b} \right)\!\right) + \left(\!\left(-j\frac{a}{b} \right)\!\right) \right).$$

We conclude that $S = 0$ from Lemma 2.1, as in Proposition 2.1. □

Proposition 2.3. *For any positive integers a, b, and c,*

$$s(ac, bc) = s(a, b). \tag{13}$$

Proof. We have

$$s(ac, bc) = \sum_{k=1}^{bc} \left(\!\left(\frac{ka}{b} \right)\!\right) \left(\!\left(\frac{k}{bc} \right)\!\right).$$

The expression $\left(\!\left(\frac{ka}{b} \right)\!\right) \left(\!\left(\frac{k}{bc} \right)\!\right)$ is equal to 0 when $k = bc$ and when $k = 0$, therefore we can take the sum from 0 to $bc-1$. We may as well write $k = bi + j$ for $0 \le i < c$, and for $0 \le j < b$, so that

$$s(ac, bc) = \sum_{i=0}^{c-1} \sum_{j=0}^{b-1} \left(\!\left(\frac{ja}{b} \right)\!\right) \left(\!\left(\frac{bi+j}{bc} \right)\!\right).$$

Considering that the summand is 0 when $j = 0$, we can take the second sum for j from 1 to $b-1$. In that case $0 < \frac{bi+j}{bc} < 1$, and we can see that

$$\left(\!\left(\frac{bi+j}{bc} \right)\!\right) = \left\{ \frac{bi+j}{bc} \right\} - \frac{1}{2} = \frac{i}{c} + \frac{j}{bc} - \frac{1}{2}.$$

Considering Lemma 2.2, we get to

$$s(ac, bc) = \sum_{i=0}^{c-1} \sum_{j=1}^{b-1} \left(\!\left(\frac{ja}{b} \right)\!\right) \frac{j}{bc} = \sum_{j=1}^{b-1} \left(\!\left(\frac{ja}{b} \right)\!\right) \frac{j}{b}.$$

Considering Lemma 2.2 again to add $-\frac{1}{2}$, and given that $\frac{j}{b} = \left\{ \frac{j}{b} \right\}$ when $0 < j < b$, this is also

$$s(ac, bc) = \sum_{j=1}^{b-1} \left(\!\left(\frac{ja}{b} \right)\!\right) \left(\left\{ \frac{j}{b} \right\} - \frac{1}{2} \right) = \sum_{j=1}^{b-1} \left(\!\left(\frac{ja}{b} \right)\!\right) \left(\!\left(\frac{j}{b} \right)\!\right) = s(a, b). \quad \square$$

Proposition 2.4. *Let $n > 1$ be an integer,*

$$\sum_{d_1 | n} \sum_{d_2 | n} \mu(d_1)\mu(d_2) \frac{d_1 d_2}{n^2} \gcd\left(\frac{n}{d_1}, \frac{n}{d_2} \right)^2 = \frac{2^{\omega(n)} \phi(n)}{n}. \tag{14}$$

Proof. Let $F(n)$ denote the left-hand side of Equation (14). If $n = p^k$ is a prime power, then we have $F(p^k) = 2\phi(n)/n$. Hence it suffices to notice that $n \mapsto F(n)$ is multiplicative. Indeed, take $n = mm'$ with $\gcd(m, m') = 1$. Then d divides n if and only if $d = \delta\delta'$, where δ divides m and δ' divides m', in which situation we have $\gcd(\delta, \delta') = \gcd(m/\delta, m/\delta') = 1$. It follows that if $d_1 = \delta_1\delta_1'$ and $d_2 = \delta_2\delta_2'$ are two such divisors, then

$$f_n(d_1, d_2) := \mu(d_1)\mu(d_2) \frac{d_1 d_2}{n^2} = f_m(\delta_1, \delta_2) f_{m'}(\delta_1', \delta_2')$$

and

$$g_n(d_1,d_2) := \gcd\left(\frac{n}{d_1},\frac{n}{d_2}\right) = g_m(\delta_1,\delta_2)g_{m'}(\delta_1',\delta_2'),$$

that implies $F(mm') = F(m)F(m')$ as desired[3]. □

3 A proof of Proposition 1.1

Proof. For any $d \mid n$ with $d > 1$, we have $\frac{a}{d} \notin \mathbb{Z}$, since d is coprime with a (because n is coprime with a). In that case, $\left\{\frac{a}{d}\right\} = \left(\left(\frac{a}{d}\right)\right) + \frac{1}{2}$. When $d = 1$, $\left\{\frac{a}{1}\right\} = \left(\left(\frac{a}{1}\right)\right) = 0$. Hence

$$\nu_n(a) = \sum_{d\mid n}\mu(d)\left\{\frac{a}{d}\right\} = -\frac{1}{2} + \sum_{d\mid n}\mu(d)\left(\left(\left(\frac{a}{d}\right)\right) + \frac{1}{2}\right).$$

Then we have

$$\sum_{a\in t(n)}\nu_n(a)a = -\frac{1}{2}\frac{n\phi(n)}{2} + \sum_{d\mid n}\mu(d)\sum_{a\in t(n)}a\left(\left(\frac{a}{d}\right)\right).$$

Let us observe that $a \in t(n)$ may be rewritten $n\left(\left(\left(\frac{a}{n}\right)\right) + \frac{1}{2}\right)$ since in that case $\left\{\frac{a}{n}\right\} = \frac{a}{n}$, and $\frac{a}{n} \notin \mathbb{Z}$. After substitution and simplification by means of Proposition 2.1, we get to

$$\sum_{a\in t(n)}\nu_n(a)a = -\frac{n\phi(n)}{4} + n\sum_{d\mid n}\mu(d)\sum_{a\in t(n)}\left(\left(\frac{a}{n}\right)\right)\left(\left(\frac{a}{d}\right)\right).$$

We use P. Nazimov's transformation (see [3, Chapter V] with $f(x) = \left(\left(\frac{x}{n}\right)\right)\left(\left(\frac{x}{d}\right)\right)$ and $m = n$) to get

$$\sum_{a\in t(n)}\left(\left(\frac{a}{n}\right)\right)\left(\left(\frac{a}{d}\right)\right) = \sum_{d'\mid n}\mu(d')\sum_{k=1}^{n/d'}\left(\left(\frac{kd'}{n}\right)\right)\left(\left(\frac{kd'}{d}\right)\right).$$

Hence, recognizing a Dedekind sum, we have

$$\sum_{a\in t(n)}\nu_n(a)a = -\frac{n\phi(n)}{4} + S(n),$$

with

$$S(n) := n\sum_{d\mid n}\sum_{d'\mid n}\mu(d)\mu(d')s\left(\frac{n}{d},\frac{n}{d'}\right).$$

By symmetry with regards the variables d and d', we have

$$2S(n) = n\sum_{d\mid n}\sum_{d'\mid n}\mu(d)\mu(d')\left(s\left(\frac{n}{d},\frac{n}{d'}\right) + s\left(\frac{n}{d'},\frac{n}{d}\right)\right).$$

Let δ be the greatest common divisor of $\frac{n}{d}$ and $\frac{n}{d'}$, then we have from Proposition 2.3:

$$2S(n) = n\sum_{d\mid n}\sum_{d'\mid n}\mu(d)\mu(d')\left(s\left(\frac{n}{\delta d},\frac{n}{\delta d'}\right) + s\left(\frac{n}{\delta d'},\frac{n}{\delta d}\right)\right).$$

[3] Note that, up to a transformation, this is the same function as the one studied by Hubert Delange in [2, pp. 82–83].

Given that $\frac{n}{\delta d}$ and $\frac{n}{\delta d'}$ are relatively prime, the Dedekind reciprocity law (see [4, Equation 4, p. 3]) can be applied under the sum, and

$$2S(n) = n \sum_{d|n} \sum_{d'|n} \mu(d)\mu(d') \left(-\frac{1}{4} + \frac{1}{12} \left(\frac{d'}{d} + \frac{dd'}{n^2}\delta^2 + \frac{d}{d'} \right) \right).$$

We derive Equation (7) of Proposition 1.1 after calculating the contribution of each term in $S(n)$:

- The term in $-\frac{1}{4}$ leads to a zero contribution, since $n > 1$ implies $\sum_{d|n} \mu(d) = 0$ (see [1, Theorem 2.1, p. 25]),
- The two terms, in $\frac{d}{d'}$ and in $\frac{d'}{d}$, are equal by symmetry. They can be calculated as the product of two elementary sums (see for example [1]):

$$\sum_{d|n} \sum_{d'|n} \mu(d)\mu(d') \frac{d'}{d} = \left(\sum_{d|n} \frac{\mu(d)}{d} \right) \left(\sum_{d'|n} \mu(d')d' \right) = \frac{\phi(n)}{n} (-1)^{\omega(m)} \phi(m),$$

- The last term in $\frac{dd'}{n^2}\delta^2$ is calculated in Proposition 2.4. □

4 Conclusion

The proof relies on three key ideas. The first idea is, the reformulation of the problem via Equation (3), which enables to work on the set $t(n)$, and not to worry about ordering the a_j, which is the job naturally done by the function θ_n. The second key idea is, to isolate the terms depending only on the fractional part function, to be able ultimately to manipulate Dedekind sums. The third and main idea is, to get the conditions to apply Dedekind's reciprocity law, in order for the calculations to become possible via closed form formulae. The result is not new, however the method may be applied to get formulae for other sums of that kind, for which no closed form formulae are known yet.

Acknowledgements

I would like to thank William Gasarch for his review of an earlier version of the paper, the anonymous referees for their reviews and their insightful comments, and my wife Natallia for her support.

References

[1] Apostol, T. M. (1976). *Introduction to Analytic Number Theory*. Springer-Verlag, New York.

[2] Delange, H. (1968). Sur la distribution des fractions irréductibles de dénominateur n ou de dénominateur au plus égal à x. *Hommage au Professeur Lucien Godeaux*, 75–89.

[3] Dickson, L. E. (1919). *History of the Theory of Numbers, Vol. I*. Chelsea, New York.

[4] Rademacher, H., & Grosswald, E. (1972). *Dedekind Sums*. The Carus Mathematical Monographs, Volume 16. American Mathematical Society, Washington, D.C.

[5] Sándor, J., & Crstici, B. (2004). *Handbook of Number Theory, II*. Springer Science & Business Media.

[6] Spence, E. (1963). Formulae for sums involving a reduced set of residues modulo n. *Proceedings of the Edinburgh Mathematical Society*, 13(4), 347–349.

[7] Van Hamme, L. (1971). Sur une généralisation de l'indicateur d'Euler. *Bulletins de l'Académie Royale de Belgique*, 57(1), 805–817.