

Arithmetical functions associated with divisibility sequences

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Abstract: This note looks at some aspects of divisibility sequences and generalized integers, including so-called Fermatian numbers, and extensions of ideas of Mollie Horadam and Morgan Ward.

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1 Introduction

This note is complementary to, but different from [13], with some overlapping parts for clarity, and for examples as analogues of classical number theory. The properties of divisibility sequences have been examined by Mollie Horadam [7], Lehmer [9], Pierce [11], Ward [17], and Williams [18].

A sequence of integers, $\{u_n\}$, $n = 1, 2, \dots$, is said to be a divisibility sequence if $u_s \mid u_t$ whenever $s \mid t$. For example, Vorob'ev proved that the sequence of Fibonacci numbers is a divisibility sequence [16]. In particular, Ward proved properties A and B for divisibility sequences, namely,



A: If $c = (a, b)$, then $u_c = (u_a, u_b)$ for all $(u_a, u_b) \in \{u_n\}$;

B: For every prime divisor p and every positive integer a , $u_m \equiv 0 \pmod{p^a}$ when, and only when, $m \equiv 0 \pmod{z}$, where z is the rank of apparition of p^a in $\{u_n\}$.

The two theorems in Ward which arise from these properties and which affect our development here are in turn:

Theorem A: Properties A and B are equivalent to one another, and

Theorem B: The binomial coefficients belonging to every divisibility sequence having Property A or Property B are all rational integers.

We shall consider arithmetical functions which apply to divisibility sequences built on the δ and ρ functions defined by

$$\delta\{m, s\} = \begin{cases} 1, & \text{if } m \text{ divides } s, \\ 0, & \text{if } m \text{ does not divide } s, \end{cases} \quad \text{and} \quad \rho_n\{m, s\} = \begin{cases} 0, & \text{if } j \mid (n, s), 1 < j < m, \\ 1, & \text{otherwise.} \end{cases}$$

2 Rho and Delta functions

These modified functions lead into the section on generalized integers to provide more number theory context for what follows. We note that it replicates some work of Hardy and Wright [3] on Ramanujan's sum and Mollie Horadam's [6] $m\delta(m, s)$ for the same sum. To this end, and for notational convenience, we let the generalized integer delta and rho functions be defined by [12]

$$\begin{aligned} \delta(u_m, u_s) &= \delta\{m, s\} = \begin{cases} 1, & \text{if } u_m \mid u_s. \\ 0, & \text{otherwise.} \end{cases} \\ \rho_{u_n}(u_m, u_s) &= \rho_n\{m, s\} = \begin{cases} 0, & \text{if } \exists u_j : u_j \mid u, \\ 1, & \text{otherwise.} \end{cases} \end{aligned} \tag{2.1}$$

We call $u_d = (u_j, u_k)$ the greatest common divisor of u_j and u_k , $(u_d, u_j, u_k \in \{u_n\})$ if

- (a) $u_d \mid u_j$ and $u_d \mid u_k$, and
- (b) every $u_c \mid u_d$ if $u_c \mid u_j$ and $u_c \mid u_k$, $u_c \in \{u_n\}$.

When $u_1 = (u_j, u_k)$, we say that u_j and u_k are co-primes. A formula for the greatest common divisor is then given by

$$(u_s, u_t) = \max \{u_m \delta\{m, s\} \delta\{m, t\}\} \text{ for } 1 \leq m \leq \min\{s, t\}. \tag{2.2}$$

Proof of (2.2): $\delta\{m, s\} \delta\{m, t\} = 1$ if and only if $u_m \mid u_s$ and $u_m \mid u_t$. So, if $u_m \mid u_s$ and $u_m \mid u_t$, then

$$u_m = u_m \delta\{m, s\} \delta\{m, t\}$$

and the maximum of all these will be the greatest common divisor of u_s and u_t . □

Example 1. For the ordinary integers, $(4, 8) = \max\{1, 2, 4\} = 4$, and we can now say that $\rho_n\{m, s\} \delta\{m, s\} = 1$, if and only if u_m is unity or the smallest prime factor of u_s . The proof of this is that if u_m is unity or the smallest prime factor of u_s , then $\delta\{m, s\} = 1$ because $u_m | u_s$, and $\rho\{m, s\}$ is 1, because there is no $u_j | u_s : 1 < u_j < u_m$. Also, $\rho_n\{m, s\} \delta\{m, s\} = 1$, for all s . But if it is not unity, then either $\rho_n\{m, s\} = 0$, which implies that there exists a $u_j : 1 < u_j < u_m, u_j | u_s$, or $\delta\{m, s\} = 0$, which implies that u_m does not divide u_s , or both. All of these imply that u_m is not unity or is not the smallest prime divisor of u_s .

Example 2. For the Fermatian numbers, $\{\underline{z}_n\}$, when $z = 2$,

$$\rho_6\{2, 6\} = \rho_{63}(3, 63) = 1, \text{ and } \delta\{2, 6\} = \delta(3, 63) = 1, \text{ so } \rho_6\{2, 6\} \delta\{2, 6\} = 1,$$

$$\text{but } \rho_6\{3, 6\} = \rho_{63}(7, 63) = 0, \text{ since } 1 < 3 < 7 \text{ and } 3 | 63, \text{ so } \rho_6\{3, 6\} \delta\{3, 6\} = 0.$$

Also, $\rho_7\{7, 7\} = \rho_{127}\{127, 127\} = 1$, as there is no $\underline{z}_n, 1 < n < 7: \underline{z}_n | 127$, and so $\rho_7\{7, 7\} \delta\{7, 7\} = 1$,

but $\delta\{3, 7\} = \delta(7, 127) = 0$, so $\rho_7\{3, 7\} \delta\{3, 7\} = 0$.

Other analogies can be constructed, such as numbers of divisors and greatest integer function:

$$\sigma_k(u_n) = \sum_{u_d | u_n} u_d^k, \quad (2.3)$$

and

$$\left[\frac{u_s}{u_m} \right] - \left[\frac{u_{s-1}}{u_m} \right] = \delta\{m, s\}, \left[\frac{u_0}{u_m} \right] \quad (2.4)$$

connected by

$$\sum_{m=1}^n \left[\frac{u_n}{u_m} \right] = \sum_{s=1}^n \sigma_o(u_s), \text{ if } \left[\frac{u_{j-1}}{u_j} \right] = 0, \forall j. \quad (2.5)$$

In turn, $\sigma_0(u_n)$ represents the number of divisors u_d of u_n , and when $\{n\} = \{u_n\}, \left[\frac{u_s}{u_m} \right] = \left[\frac{s}{m} \right]$, i.e., the generalized floor function reduces to the ordinary greatest integer (floor) function.

3 Fermatian generalized integers

A typical approach is to start with generalized integers, I_n : generalized primes $p_1 < p_2 < p_3 \dots$ yield generalized integers $I_n = \prod p_i^{a_i}$ [4]; on the other hand, we can begin with divisibility sequences, the elements of which we consider generalized integers and some of which we consider generalized primes, u_p is a generalized prime (modulo an element of a divisibility sequence) if the only divisors of u_p are 1 and u_p [5]. Thus, these generalized primes cannot necessarily be represented as a product of distinct generalized primes. For example, if \underline{z}_n is the n -th Fermatian number of index z , some of the properties of which we owe to Carlitz and Moser [2] are:

$$z_n = \begin{cases} -z^n z_n, & (n < 0) \\ 1 + z + z^2 + \dots + z^{n-1}, & (n > 0) \\ 1, & (n = 0) \end{cases} \quad (3.1)$$

so that

$$\underline{1}_n = n, \quad (3.2)$$

and

$$\underline{1}_n! = n!, \quad (3.3)$$

where

$$\underline{z}_n! = \underline{z}_n \underline{z}_{n-1} \dots \underline{1}_n. \quad (3.4)$$

Table 1. First 9 Fermatian numbers of the first 10 indices

$z \backslash n$	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	1	3	7	15	31	63	127	255	511
3	1	4	13	40	121	364	1093	3280	9841
4	1	5	21	85	341	1365	5461	21845	87381
5	1	6	31	156	781	3906	19531	97656	488281
6	1	7	43	259	1555	9331	55987	335923	2015539
7	1	8	57	400	2801	19608	137257	960800	6725601
8	1	9	73	585	4681	37449	299593	2396745	19173961
9	1	10	91	820	7381	66430	597871	5380840	48427561
10	1	11	111	1111	11111	111111	1111111	11111111	111111111

If we take the columns, and assume their regularity, we see that they are, not surprisingly given their constructions, sequences in the Online Encyclopedia of Integer Sequences, as shown in Table 2; this permits further cross-order investigations of similarities among these sequences. For instance,

$$\begin{aligned} \{a_n\} &= \{\{\underline{z}_2\} + \{\underline{z}_3\}\} \text{ with } a_n = n^2 + 1 \text{ is A002522 of Sloane [14], and} \\ \{a_n\} &= \{\{\underline{z}_3\} + \{\underline{z}_4\}\} \text{ with } a_n = n^3 - 4n^2 + 6n - 2 \text{ is A188377 [15].} \end{aligned} \quad (3.5)$$

Table 2. Fermatian vertical sequences in Sloane

$\{\underline{z}_n\}$	$\{\underline{z}_1\}$	$\{\underline{z}_2\}$	$\{\underline{z}_3\}$	$\{\underline{z}_4\}$	$\{\underline{z}_5\}$	$\{\underline{z}_6\}$	$\{\underline{z}_7\}$
Sloane A...	000012	00027	002061	53698	53699	53700	53716

If we look at the Fermatians of index 2: $\underline{2}_6 = 63 = (\underline{2}_2)^2 \underline{2}_3$, so it is not prime, but $\underline{2}_8 = 255 = \underline{2}_4 17$, which, in a strict sense, is neither prime nor composite in the Fermatian set of generalized integers, although $17 = 2 + 15 = \underline{1}_2 + \underline{2}_4$ and $255 = 15 \times 17$. The sequence of Fermatian numbers, though, is a divisibility sequence, because if $s \mid t$, then $\underline{q}_s \mid \underline{q}_t$, since $t = sx$, and

$$\frac{\underline{q}_t}{\underline{q}_s} = \frac{1 - q^t}{1 - q^s} = \frac{1 - q^{sx}}{1 - q^s} = \frac{1 - q^s}{1 - q^s} (1 + q^s + q^{2s} + \dots + q^{s(x-1)}) = \underline{q}_x^s,$$

as required.

4 Concluding comments

Finally, we apply some of the preceding ideas to generalized Hurwitz series as another example [cf. 1, 10]. Ordinary Hurwitz series have the form $\sum_{n=0}^{\infty} a_n t^n / n!$ where a_n are integers [8]. The ordinary exponential series is an example of a Hurwitz series. We shall call a series of the form $\sum_{n=0}^{\infty} a_n t^n / u_n!$, where the u_n are arbitrary integers, a generalized Hurwitz series (GH-series).

The Cauchy product of this series and another GH-series, $\sum_{n=0}^{\infty} b_n t^n / u_n!$, is also a GH-series of the form

$$\sum_{n=0}^{\infty} \sum_{m=0}^n \left\{ \begin{matrix} n \\ m \end{matrix} \right\} a_m b_{n-m} t^n / u_n!,$$

in which the generalized binomial coefficients are integers for divisibility sequences with Property A. This is a special use of the term ‘‘product’’ explained by Ward.

GH-series can be changed into other GH-series by the series operators D_x and I_x which can be applied to any function of x regular at $x = 0$ and transform it into another function regular at $x = 0$.

$$\begin{aligned} D_x \sum_{n=0}^{\infty} a_n \frac{x^n}{u_n!} &= \sum_{n=0}^{\infty} a_{n+1} \frac{x^n}{u_n!} \\ \left| I_x \sum_{n=0}^{\infty} a_n \frac{x^n}{u_n!} \right|_0^t &= \sum_{n=0}^{\infty} a_{n-1} \frac{x^n}{u_n!}. \end{aligned} \tag{4.1}$$

For a series without a constant term, $H_1(t) = \sum_{n=1}^{\infty} a_n t^n / u_n!$, it follows from an analogue of the function of a function rule that

$$D_x H_1^k(x) = u_k H_1^{k-1}(x) D_x H_1(x). \tag{4.2}$$

Then

$$H_1^k(t) = \left| I_x D_x H_1^k(x) \right|_0^t = \left| I_x u_k H_1^{k-1}(x) D_x H_1(x) \right|_0^t$$

and

$$\frac{1}{u_k!} H_1^k(t) = \left| I_x \frac{H_1^{k-1}(x)}{u_{k-1}!} D_x H_1(x) \right|_0^t.$$

Thus, by induction on k , it can be proved that $\frac{1}{u_k!} H_1^k(t)$ is a GH-series for all $k \geq 1$.

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