

A note on h -fold signed sumset in the set of integers

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Abstract: Let h and k be positive integers, and let $A = \{a_0, a_1, \dots, a_{k-1}\}$ be a finite of k integers. The h -fold signed sumset, denoted by $h_{\pm}A$, is defined as

$$h_{\pm}A := \left\{ \sum_{i=0}^{k-1} \lambda_i a_i : \lambda_i \in \{0, \pm 1, \pm 2, \dots, \pm h\} \text{ for } i = 0, 1, \dots, k-1 \text{ and } \sum_{i=0}^{k-1} |\lambda_i| = h \right\}.$$

Bhanja and Pandey [J. Number Theory, 196 (2019), 340–352] gave an optimal lower bound for the cardinality of $h_{\pm}A$. They also characterized the set A when the cardinality of $h_{\pm}A$ attains the optimal lower bound. In this note, we revisit their results by providing new proofs. We observe that the study of obtaining the optimal lower bound for the cardinality of $h_{\pm}A$, and the structure of the set A when $h_{\pm}A$ attains the optimal lower bound, rather than for an arbitrary set of integers, suffices when A is an arithmetic progression.

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1 Introduction

Let \mathbb{N} be the set of all positive integers and \mathbb{Z} be the set of all integers. Let A be a nonempty finite set of integers. The cardinality of the set A is denoted by $|A|$. By $\max(A)$ and $\min(A)$, we mean the largest and smallest elements of the set A , respectively. For integers α and β , let

$$\begin{aligned}\alpha * A &:= \{\alpha a : a \in A\}, \\ A + \beta &:= \{a + \beta : a \in A\}.\end{aligned}$$

For $\alpha < \beta$, we let $[\alpha, \beta] = \{\alpha, \alpha + 1, \dots, \beta\}$. The greatest common divisor of the integers x_1, x_2, \dots, x_k is denoted by $d(x_1, x_2, \dots, x_k)$. Let $A = \{a_0, a_1, \dots, a_{k-1}\}$ be a nonempty finite set of integers with $a_0 < a_1 < \dots < a_{k-1}$. Define

$$d(A - a_0) := (a_1 - a_0, a_2 - a_0, \dots, a_{k-1} - a_0)$$

and

$$A^{(N)} := \left\{ \frac{a_i - a_0}{d(A - a_0)} : a_i \in A \text{ for } i \in [0, k - 1] \right\}.$$

The set $A^{(N)}$ is called the *normal form* of A . Here $d(A^{(N)}) = 1$ and $\min(A^{(N)}) = 0$.

Let h and k be positive integers. Let $A = \{a_0, a_1, \dots, a_{k-1}\}$ be a nonempty finite set of k integers. The *h -fold sumset* and *h -fold signed sumset*, denoted by hA and $h_{\pm}A$, respectively, are defined as follows:

$$hA := \left\{ \sum_{i=0}^{k-1} \lambda_i a_i : \lambda_i \in [0, h] \text{ for } i = 0, 1, \dots, k - 1 \text{ and } \sum_{i=0}^{k-1} \lambda_i = h \right\}$$

and

$$h_{\pm}A := \left\{ \sum_{i=0}^{k-1} \lambda_i a_i : \lambda_i \in [-h, h] \text{ for } i = 0, 1, \dots, k - 1 \text{ and } \sum_{i=0}^{k-1} |\lambda_i| = h \right\}.$$

It is easy to check that $|hA|$ is translation and dilation invariant of the set A , whereas $|h_{\pm}A|$ is only dilation invariant.

Recently, Bhanja and Pandey [1] gave an optimal lower bound of $|h_{\pm}A|$ where A is a nonempty finite set of integers (called direct problem). They also characterized the set A , a finite set of integers, when $|h_{\pm}A|$ attains the optimal lower bound (called inverse problem).

In this article, we give a new proofs of both direct and inverse results using the fundamental results proved by Nathanson (Theorem 1.1), and Tang and Xing (Theorem 1.2).

The following direct and inverse theorem proved by Nathanson.

Theorem 1.1. [2, Theorem 1.4, Theorem 1.6] *Let $h \geq 1$ and A be a nonempty finite set of integers. Then*

$$|hA| \geq h|A| - h + 1.$$

This lower bound is the best possible. Furthermore, if $|hA|$ attains this lower bound with $h \geq 2$, then A is an arithmetic progression.

In 2021, Tang and Xing [3] proved the following inverse theorem.

Theorem 1.2. [3, Theorem 1.1] *Let $h \geq 2$ and $k \geq 5$ be integers. Let A be a set of integers with $|A| = k$. If $hk - h + 1 < |hA| \leq hk + h - 2$, then*

$$A^{(N)} = [0, k] \setminus \{x\} \text{ for } 1 \leq x \leq k - 1.$$

Moreover, $|hA^{(N)}| = hk$ for $x = 1$ or $k - 1$, and $|hA^{(N)}| = hk + 1$ for $2 \leq x \leq k - 2$.

A simple corollary we obtain from the above theorem.

Corollary 1.1. *Let $h \geq 2$ and $k \geq 5$ be integers. Let A be a set of integers with $|A| = k$. If $hk - h + 1 < |hA| \leq hk + h - 2$, then $|hA| \geq hk$.*

2 Direct and inverse theorems for $h_{\pm}A$ in integers

First, we prove the direct and inverse theorem for $h_{\pm}A$ when the set A is a finite arithmetic progression of positive integers.

Lemma 2.1. *Let k , a , and d be positive integers. Let $A = \{a, a + d, \dots, a + (k - 1)d\}$. Then*

$$|2_{\pm}A| \geq \begin{cases} 4k - 2, & d = 2a; \\ 4k, & d \neq 2a. \end{cases}$$

Furthermore, $|2_{\pm}A| = 4k - 2$ if and only if $d = 2a$.

Proof. Note that $2A$ and $2(-A)$ are disjoint subsets of $2_{\pm}A$. Using Theorem 1.1, we have

$$|2_{\pm}A| \geq |2A| + |2(-A)| \geq 4k - 2.$$

It is easy to see that $|2_{\pm}A| = 4k - 2$ if $d = 2a$. Assume $d \neq 2a$. Then $\{-d, d\} \subseteq 2_{\pm}A \setminus (2A \cup 2(-A))$. Therefore

$$|2_{\pm}A| \geq |2A| + |2(-A)| + 2 \geq 4k.$$

This completes the proof of the lemma. □

Lemma 2.2. *Let $h \geq 3$ and $k \geq 3$ be integers. Let $A = \{a, a + d, \dots, a + (k - 1)d\}$, where $a, d \in \mathbb{N}$. Then*

$$|h_{\pm}A| \geq \begin{cases} 2hk - h + 1, & d = 2a; \\ 2hk - h + 3, & d \neq 2a. \end{cases}$$

Furthermore, $|h_{\pm}A| = 2hk - h + 1$ if and only if $d = 2a$.

Proof. If $d = 2a$, then $A = a * \{1, 3, 5, \dots, 2k - 1\}$. Therefore $h_{\pm}A \subseteq [-h(2k - 1)a, h(2k - 1)a]$ and $h_{\pm}A$ contains only odd(even) multiples of a if h is odd(even). Thus,

$$|h_{\pm}A| = 2hk - h + 1.$$

Let $d \neq 2a$. We use induction on h to obtain the required lower bound. Let $h = 3$. Then

$$3A = \{3a + id : i \in [0, 3k - 3]\} \subseteq 3_{\pm}A \quad (1)$$

and

$$3(-A) = \{-3a - id : i \in [0, 3k - 3]\} \subseteq 3_{\pm}A, \quad (2)$$

where $3A \cap 3(-A) = \emptyset$. It is easy to verify that $|3_{\pm}A| \geq 6k$ if one of the following holds:

1. $d = a$,
2. $d = 4a$,
3. $a = 2d$,
4. $2a = 3d$.

Now, assume $d \neq a$, $d \neq 2a$, $d \neq 4a$, $a \neq 2d$, and $3d \neq 2a$. The set

$$S = \{-2a - (a+d), -3a, -2a + (a+d), 2a - (a+d), -2a + (a+2d), 2a - (a+2d), 3a, 2a + (a+d)\}$$

contains 8 non-zero elements of $3_{\pm}A$. Also

$$|S \cap 3A| = |S \cap 3(-A)| = 2. \quad (3)$$

Therefore, (1), (2), and (3) imply that

$$|3_{\pm}^{\wedge}A| \geq |3A| + |3(-A)| + |S| - 4 = 3k - 2 + 3k - 2 + 8 - 4 = 6k.$$

Thus, we have

$$|3_{\pm}^{\wedge}A| \geq 6k.$$

This proves that the result holds for $h = 3$. Assume the result holds for $h - 1 \geq 3$. Now, we show that the result also holds for $h \geq 4$.

Let $A' = A \setminus \{a + (k - 1)d\}$. Note that

$$(h - 1)_{\pm}A' + (a + (k - 1)d)$$

and

$$\{(h - i - 1)(a + (k - 2)d) + (i + 1)(a + (k - 1)d) : i \in [1, h - 1]\}$$

are disjoint subsets of $h_{\pm}A$. Also, by induction hypothesis, we have

$$|(h - 1)_{\pm}A' + (a + (k - 1)d)| = |(h - 1)_{\pm}A'| \geq 2(k - 1)(h - 1) - (h - 1) + 3 = 2kh - 2k - 3h + 6.$$

Now, we construct extra $2k + h - 2$ elements of $h_{\pm}A$ to complete the proof. It is easy to see that

$$\begin{aligned} & \min((h - 1)_{\pm}A' + (a + (k - 1)d)) \\ &= -(h - 1)(a + (k - 2)d) + a + (k - 1)d \\ &= -(h - 2)(a + (k - 2)d) - (a + (k - 4)d) + a + (k - 3)d \\ &> -(a + (k - 1)d) - (h - 3)(a + (k - 2)d) - (a + (k - 4)d) + a + (k - 3)d \\ &> -2(a + (k - 1)d) - (h - 4)(a + (k - 2)d) - (a + (k - 4)d) + a + (k - 3)d \\ &\vdots \end{aligned}$$

$$\begin{aligned}
&> -(h-2)(a+(k-1)d) - (a+(k-4)d) + a+(k-3)d \\
&= -(h-2)(a+(k-1)d) - (a+(k-3)d) + a+(k-2)d \\
&> -(h-3)(a+(k-1)d) - (a+(k-2)d) - (a+(k-3)d) + a+(k-4)d \\
&> -(h-3)(a+(k-1)d) - (a+(k-2)d) - (a+(k-3)d) + a+(k-5)d \\
&\vdots \\
&> -(h-3)(a+(k-1)d) - (a+(k-2)d) - (a+(k-3)d) + a \\
&> -(h-3)(a+(k-1)d) - (a+(k-2)d) - (a+(k-3)d) - a \\
&> -(h-3)(a+(k-1)d) - (a+(k-2)d) - (a+(k-3)d) - (a+d) \\
&\vdots \\
&> -(h-3)(a+(k-1)d) - (a+(k-2)d) - (a+(k-3)d) - (a+(k-3)d) \\
&> -(h-3)(a+(k-1)d) - 2(a+(k-2)d) - (a+(k-3)d) \\
&> -(h-3)(a+(k-1)d) - 3(a+(k-2)d) \\
&> -(h-2)(a+(k-1)d) - 2(a+(k-2)d) \\
&> -(h-1)(a+(k-1)d) - (a+(k-2)d) \\
&> -h(a+(k-1)d).
\end{aligned}$$

This completes the proof of the lemma. □

Theorem 2.1. *Let $h \geq 3$ and $k \geq 3$ be positive integers. Let A be a set of k integers with $A \cap (-A) = \emptyset$. Then*

$$|h_{\pm}A| \geq 2hk - h + 1.$$

*This lower bound is the best possible. Moreover, if $|h_{\pm}A| = 2hk - h + 1$, then $A \cup (-A) = a * \{-(2k-1), \dots, -3, -1, 1, 3, \dots, 2k-1\}$ for some positive integer a .*

Proof. Let $B = (-A \cup A) \cap \mathbb{N}$. Since $A \cap (-A) = \emptyset$, we have $h_{\pm}A = h_{\pm}B$. Note that $hB \cap h(-B) = \emptyset$ and $hB \cup h(-B) \subseteq h_{\pm}A$. Therefore

$$|h_{\pm}A| \geq 2|hB|. \tag{4}$$

Also, by Theorem 1.1, we have

$$|hB| \geq hk - h + 1.$$

Now, consider the following cases:

1. If $|hB| \geq hk + h - 1$, then

$$|h_{\pm}A| \geq 2|hB| \geq 2hk + 2h - 2 > 2hk - h + 1. \tag{5}$$

2. If $hk - h + 1 < |hB| \leq hk + h - 2$, then by Corollary 1.1, we have

$$|hB| \geq hk.$$

In this case also

$$|h_{\pm}A| \geq 2|hB| \geq 2hk > 2hk - h + 1.$$

3. If $|hB| = hk - h + 1$, then by Theorem 1.1, we have

$$B = \{a, a + d, \dots, a + (k - 1)d\}.$$

Using Lemma 2.2, we get that $|h_{\pm}A| \geq |h_{\pm}B| \geq 2hk - h + 1$.

Therefore, in each case, we have $|h_{\pm}A| \geq 2hk - h + 1$. Now, if $|h_{\pm}A| = 2hk - h + 1$, then it follows from above that $|hB| = hk - h + 1$ and $|h_{\pm}B| = 2hk - h + 1$. Using Theorem 1.1 and Lemma 2.2, we get that $A \cup (-A) = a * \{-(2k - 1), \dots, -1, 1, \dots, 2k - 1\}$. This completes the proof of the theorem. \square

As a special case of the above theorem we obtain the following theorem proved by Bhanja and Pandey.

Theorem 2.2. [1, Theorem 2.3, Theorem 2.4] *Let $h \geq 3$ and $k \geq 3$ be integers. Let A be a set of k positive integers. Then*

$$|h_{\pm}A| \geq 2hk - h + 1.$$

*Moreover, if $|h_{\pm}A| = 2hk - h + 1$, then $A = a * \{1, 3, 5, \dots, 2k - 1\}$ for some positive integer a .*

Theorem 2.3. *Let h and k be positive integers. Let A be a set of k integers such that $A \cap (-A) = \emptyset$. Then*

$$|h_{\pm}A| \geq 2hk - 2h + 2.$$

*This lower bound is the best possible for $h \leq 2$. Moreover, if $h \geq 2$ and $|h_{\pm}A| = 2hk - 2h + 2$, then $h = 2$ and $A \cup (-A) = a * \{-(2k - 1), \dots, -1, 1, \dots, 2k - 1\}$ for some positive integer a .*

Proof. Let $B = (-A \cup A) \cap \mathbb{N}$. Then $h_{\pm}B = h_{\pm}A$. Note that $hB \cap h(-B) = \emptyset$ and $hB \cup h(-B) \subseteq h_{\pm}A$. Therefore, by Theorem 1.1, we have

$$|h_{\pm}A| \geq 2|hB| \geq 2hk - 2h + 2. \quad (6)$$

Further, if $h \geq 2$ and $|h_{\pm}A| = 2hk - 2h + 2$, then by Theorem 2.1, we get that $h = 2$. Also, from (6), we get that $|2B| = 2k - 1$. Again, using Theorem 1.1, we get that

$$B = \{a, a + d, a + 2d, \dots, a + (k - 1)d\}, \text{ where } a, d \in \mathbb{N}.$$

This implies that $2_{\pm}A = 2_{\pm}B = \{2a + id : i \in [0, 2k - 2]\} \cup \{-2a - id : i \in [0, 2k - 2]\}$. Since $d \in 2_{\pm}A$, we have $d = 2a$. Hence, $A \cup (-A) = a * \{-(2k - 1), \dots, -1, 1, \dots, 2k - 1\}$ for some positive integer a . \square

Using the above theorem, we can obtain the following theorem proved by Bhanja and Pandey.

Theorem 2.4. [1, Theorem 2.1, Theorem 2.2] *Let h and k be positive integers. Let A be a set of k positive integers. Then*

$$|h_{\pm}A| \geq 2hk - 2h + 2.$$

*This lower bound is the best possible for $h \leq 2$. Moreover, if $h \geq 2$ and $|h_{\pm}A| = 2hk - 2h + 2$, then $h = 2$ and $A = a * \{1, 3, 5, \dots, 2k - 1\}$ for some positive integer a .*

Theorem 2.5. Let h and $k \geq 3$ be positive integers. Let A be a set of k integers such that $-A \cap A = \{0\}$. Then

$$|h_{\pm}A| \geq 2hk - 2h + 1.$$

Moreover, if $h \geq 2$ and $|h_{\pm}A| = 2hk - 2h + 1$, then $A \cup (-A) = a * [-k + 1, k - 1]$ for some positive integer a .

Proof. Let $B = (-A \cup A) \cap \mathbb{N}_0$. Then $h_{\pm}B = h_{\pm}A$. Note that $hB \cap h(-B) = \{0\}$ and $hB \cup h(-B) \subseteq h_{\pm}A$. Therefore, by Theorem 1.1, we get that

$$|h_{\pm}A| \geq 2|hB| - 1 \geq 2hk - 2h + 1. \quad (7)$$

Further, if $|h_{\pm}A| = 2hk - 2h + 1$, then (7) implies that $|hB| = hk - h + 1$. Again, using Theorem 1.1, we get that $B = a * [0, k - 1]$ for some positive integer a . Hence,

$$A \cup (-A) = a * [-(k - 1), k - 1]. \quad \square$$

As a special case, we can obtain the following theorem.

Theorem 2.6. [1, Theorem 2.5, Theorem 2.6] Let h and $k \geq 3$ be positive integers. Let A be a set of k nonnegative integers with $0 \in A$. Then

$$|h_{\pm}A| \geq 2hk - 2h + 1.$$

Moreover, if $h \geq 2$ and $|h_{\pm}A| = 2hk - 2h + 1$, then $A = a * [0, k - 1]$ for some positive integer a .

References

- [1] Bhanja, J., & Pandey, R. K. (2019). Direct and inverse theorems on signed sumsets of integers. *Journal of Number Theory*, 196, 340–352.
- [2] Nathanson, M. B. (1996). *Additive Number Theory: Inverse Problems and the Geometry of Sumsets*. Springer.
- [3] Tang, M., & Xing, Y. (2021). Some inverse results of sumsets. *Bulletin of the Korean Mathematical Society*, 58(2), 305–313.