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The homothetical-Hopf transformation of Pythagorean quadruples

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Abstract: This note introduces a transformation of Pythagorean quadruples by using the composition between the Hopf map and a homothety of the space $\mathbb{R}^3$. Both the real algebra of complex numbers and the algebra of quaternions are used in this construction. Three examples are detailed, the first one concerning the well-known twin Pythagorean quadruple $(1, 2, 2, 3)$. The trigonometric parametrization of the Euclidean unit sphere $S^2 \subset \mathbb{E}^3$ allows us to prove that this transformation does not produce twin Pythagorean quadruples. A matrix approach for our transformation is also presented.
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1 Pythagorean quadruples and their homothetical-Hopf transformation

For a natural number $n \geq 2$ let $\mathbb{E}^n := (\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ be the n-dimensional *Euclidean space*:

$$\begin{cases} \langle \bar{x}, \bar{y} \rangle := x^1 y^1 + \ldots + x^n y^n, \quad \|\bar{x}\|^2 := \langle \bar{x}, \bar{x} \rangle \geq 0, \\ \bar{x} = (x^1, \ldots, x^n), \quad \bar{y} = (y^1, \ldots, y^n). \end{cases} \tag{1.1}$$

For the real number $R > 0$ let $S^{n-1}(R)$ be the $(n-1)$-dimensional *sphere*:

$$S^{n-1}(R) := \{\bar{x} \in \mathbb{R}^n; \|\bar{x}\| = R\} = \{\bar{x} \in \mathbb{R}^n; \|\bar{x}\|^2 = R^2\} \tag{1.2}$$

with the usual *unit sphere* S^{n-1} when $R = 1$. Being a hypersurface in $\mathbb{E}^n$ the sphere $S^{n-1}(R)$ is itself a Riemannian manifold; in fact a *space form* since it has the constant sectional curvature $+1$.

The starting point of this short note is the well-known *Riemannian submersion*:

$$H : S^3 \subset \mathbb{C} \times \mathbb{C} \to S^2\left(\frac{1}{2}\right) \subset \mathbb{C} \times \mathbb{R}, \quad H(z, w) := \left(z\overline{w}, \frac{1}{2}(|z|^2 - |w|^2)\right) \tag{1.3}$$

called *Hopf fibration*; see [6]. Here $\overline{w}$ denotes the complex conjugate of $w \in \mathbb{C}$.

Our interest consists of *Pythagorean quadruples*, namely $\square = (A, B, C, D) \in (\mathbb{N}^*)^4$ satisfying:

$$D^2 = A^2 + B^2 + C^2. \tag{1.4}$$

Denotes $\mathcal{PQ}$ their set; such a quadruple is called *primitive* if the greatest common divisor of its elements is 1. If $\mathbb{R}^4$ is endowed with the Minkowski-Lorentz product:

$$< \bar{x}, \bar{y} >_{ML} := x^1 y^1 + ... + x^3 y^3 - x^4 y^4$$

then $\square$ is a *null vector* with integer components. Let us call this fact as being *the physical interpretation* of the Pythagorean quadruples.

The relationship between Pythagorean quadruples and the Hopf map is provided by the quaternion algebra $\mathbb{H} := \mathbb{R}^4$ which is exactly the ambient space of S^3 with a special product; see again [6]. More precisely, the parametrization of any primitive $\square \in \mathcal{PQ}$ is ([7]):

$$\begin{cases} A := m^2 + n^2 - p^2 - q^2, B := 2(mq + np), C := 2(nq - mp), D := m^2 + n^2 + p^2 + q^2, \\ 0 < m, n, p, q \in \mathbb{N}, \quad 1 = \gcd(m, n, p, q), \quad m + n + p + q = \text{odd}. \end{cases} \tag{1.5}$$

Recall also that a non-zero real number λ defines the *homothety with a ratio* λ as:

$$h^n_\lambda : \mathbb{E}^n \to \mathbb{E}^n, \quad \bar{x} \to \lambda \bar{x}, \tag{1.6}$$

which implies that h^n_λ is a bijection map having the inverse $h^n_{1/\lambda}$. In fact, the set of all homotheties $\{h^n_*\}$ acts on $\mathbb{E}^n$ as the commutative group $(\mathbb{R}^*, \cdot)$.

Usually, the study of Pythagorean quadruples is based on techniques from abstract algebra and number theory; now we return to the geometrical source of this notion through the above preparatory tools. Hence, the first main result of this work is:

Theorem 1.1. *Any Pythagorean quadruple $\square$ yields a new one:*

$$h^3_2 \circ H(\square) := (A^2, AC + BD, |AB - CD|, D^2) \in \mathcal{PQ}. \tag{1.7}$$

Proof. The use of $\mathbb{H}$ in producing the parametrization (1.5) inspires us to associate a quaternion $q(\Box)$ as follows:

$$q(\Box) := D + (A\bar{i} + B\bar{j} + C\bar{k}) = \mathrm{Re}(q(\Box)) + \mathrm{Im}(q(\Box)) \in \mathbb{H}. \tag{1.8}$$

The motivation of this choice is given by the fact that the square of $q(\Box)$ in $\mathbb{H}$ is a *pure quaternion*:

$$Re\left((q(\Box))^2\right) := D^2 - A^2 - B^2 - C^2 = 0 \tag{1.9}$$

from the definition of $\Box$; a similar choice is performed in [4] when working with the plane $\pi : Ax + By + Cz + D = 0$ in $\mathbb{E}^3$. From the expression (1.8) we find two complex numbers which are useful for the application of the Hopf map:

$$\begin{cases} q(\Box) = (D + A\bar{i}) + (B + C\bar{i})\bar{j}, \\ z(\Box) := \frac{1}{D\sqrt{2}}(D + A\bar{i}) \in \mathbb{C}, \quad w(\Box) := \frac{1}{D\sqrt{2}}(B + C\bar{i}) \in \mathbb{C}. \end{cases} \tag{1.10}$$

The fact that $\Box$ is a Pythagorean quadruple assures that $(z(\Box), w(\Box))$ belongs to the sphere S^3.

A direct computation of the composition $h^3_2 \circ H$ gives:

$$h^3_2 \circ H(z(\Box), w(\Box)) = \frac{1}{D^2}((AC + BD) + (AB - CD)\bar{i}, A^2) \in S^2, \tag{1.11}$$

which is the claimed conclusion. We point out that the direct algebraic computation confirms the validity of our use of the Hopf map:

$$\begin{aligned} (A^2) + (AC + BD)^2 + (AB - CD)^2 &= A^4 + A^2C^2 + B^2D^2 + A^2B^2 + C^2D^2 \\ &= A^2(A^2 + B^2 + C^2) + D^2(B^2 + D^2) \\ &= A^2D^2 + D^2(B^2 + C^2) \\ &= D^4. \end{aligned}$$

In order to explain also (1.5) we use the complex numbers $\alpha = \alpha(\Box) := m + n\bar{i}$, $\beta = \beta(\Box) := q + p\bar{i}$ to derive:

$$\Box = (|\alpha|^2 - |\beta|^2, 2\mathrm{Re}(\alpha\overline{\beta}), 2\mathrm{Im}(\alpha\overline{\beta}), |\alpha|^2 + |\beta|^2). \tag{1.12}$$

This completes the proof. $\Box$

Definition 1.1. *The map $H^2 : \mathcal{PQ} \to \mathcal{PQ}$ provided by the theorem will be called the homothetical-Hopf transformation.*

Example 1.2. The *minimal* primitive Pythagorean quadruple is:

$$\begin{cases} minimal = (A = 1, B = C = 2, D = 3, m = n = q = 1, p = 0), \\ z(minimal) = \frac{1}{3\sqrt{2}}(3 + \bar{i}), \quad w(minimal) = \frac{\sqrt{2}}{3}(1 + \bar{i}) = \frac{2}{3}e^{\frac{\pi}{4}\bar{i}}, \\ \alpha(minimal) = \sqrt{2}e^{\frac{\pi}{4}\bar{i}}, \quad \beta(minimal) = 1 \in \mathbb{R}. \end{cases} \tag{1.13}$$

Sometimes a Pythagorean quadruple with $B = C$ is called a *twin*; see [8, p. 10]. We compute:

$$\begin{cases} h_2 \circ H(minimal) = \frac{1}{9}(8 - 4\bar{i}, 1), \quad H^2(minimal) = (1, 8, 4, 9), \\ \langle minimal, H^2(minimal)\rangle_{ML} = -2, \\ z(H^2(minimal)) = \frac{1}{9\sqrt{2}}(9 + \bar{i}), \quad w(H^2(minimal)) = \frac{2\sqrt{2}}{9}(2 + \bar{i}). \end{cases} \tag{1.14}$$

A new application of H^2 to $(1,4,8,9) = (m = 1, n = q = 2, p = 0)$ gives the Pythagorean quadruple $(1,44,68,81) = (m = 4, n = 5, p = 2, q = 6)$. Applying H^2 for the third time, we obtain $(1,3632,5464,6561) = (m = 40, n = 41, p = 12, q = 56)$. $\Box$

Remark 1.3. i) Denoting $\Box = q(m,n,p,q)$ and $H^2(\Box) = q(M,N,P,Q)$ it results:

$$\begin{cases} M = p^2 + q^2, \quad N = m^2 + n^2, \quad \alpha(H^2(\Box)) = |\beta(\Box)|^2 + |\alpha(\Box)|^2\bar{i}, \\ (M^2 + N^2)Q = M(mq + np + mp - nq) + N(mq + np + nq - mp). \end{cases} \tag{1.15}$$

ii) An important map in the study of quaternions is *the conjugation*:

$$q \to \bar{q} := \mathrm{Re}(q) - \mathrm{Im}(q). \tag{1.16}$$

The effect of this map on the quaternion $q(\Box)$ from (1.8) means the following transformation:

$$z(\Box) \to \overline{z(\Box)}(in\, \mathbb{C}), \quad w(\Box) \to -w(\Box). \tag{1.17}$$

iii) Our Theorem 1.1. reveals a strong relationship between the Riemannian nature of the Hopf map and the pseudo-Riemannian nature of the Minkowski–Lorentz metric appeared in the physical interpretation of Pythagorean quadruples. It follows in this way the two faces of a coin titled "Pythagorean quadruple". $\Box$

Example 1.4. Let us consider now a non-twin Pythagorean quadruple. For example:

$$\Box = (3,6,2,7), \quad m = p = q = 1, n = 2 \tag{1.18}$$

gives: $H^2(\Box) = (9,48,4,49)$ corresponding to $M = Q = 2$, $P = 4$ and $N = 5$.

Example 1.5. A very interesting class of Pythagorean quadruples is produced by Pythagorean triples via the Exercise 3.17 from [9, p. 73]: If $\triangle := (u,v,w) \in (\mathbb{N}^*)^3$ is a *Pythagorean triple*, then:

$$(uv)^4 + (vw)^4 + (wu)^4 = (w^4 - u^2v^2)^2. \tag{1.19}$$

It follows that a proper $\triangle$ (for which we suppose $u < v < w$) yields the Pythagorean quadruple:

$$\Box(\triangle) := (A = (wu)^2, B = (vw)^2, C = (uv)^2, D = w^4 - u^2v^2 = w^4 - A). \tag{1.20}$$

Concretely, the classical $\triangle_{\min} = (3,4,5)$ gives:

$$\begin{cases} \Box(\triangle_{\min}) = (A = 15^2 = 225, B = 20^2 = 400, C = 12^2 = 144, D = 13 \cdot 37 = 481), \\ m = 8, \quad n = 17, \quad p = q = 16. \end{cases} \tag{1.21}$$

We leave it as an exercise to compute the image H^2 of this huge Pythagorean quadruple.

2 The special case of twin Pythagorean quadruples

First of all, we point out that if the point $P(x_0, y_0, z_0)$ of the unit sphere S^2 satisfies $y_0 = z_0$, then P also belongs to the ellipse:

$$E : x^2 + 2y^2 = 1 \tag{2.1}$$

with the eccentricity $e = \frac{1}{\sqrt{2}}$; this means that E is a *self-complementary ellipse*, a special class of ellipses studied in [3]. The plane $\pi : y - z = 0$ in the Euclidean space $\mathbb{E}^3$ is the plane through the origin $O(0,0,0)$ and having the unit normal vector $\bar{N} = e(0,1,-1) \in S^2$.

Secondly, for a twin quadruple we have the complex number:

$$w(\Box) = \frac{B}{D} e^{\frac{\pi}{4}\bar{i}}. \tag{2.2}$$

Thirdly, the homotethical-Hopf transformation does not preserve the twin property (which reads as $m(q+p) = n(q-p)$) since for such a Pythagorean quadruple its image is:

$$H^2(\Box) = (A^2, B(D+A) > B(D-A), D^2). \tag{2.3}$$

Fourthly, we use the well-known parametrization of the sphere S^2 with the angles $\varphi, \theta \in [0, \frac{\pi}{2})$ to express the first three components of the Pythagorean quadruple $\Box$:

$$A = D\sin\varphi, \quad B = D\cos\varphi\cos\theta, \quad C = D\cos\varphi\sin\theta. \tag{2.4}$$

The twin property gives $\theta = \frac{\pi}{4}$. By denoting $\Box = \Box(\varphi, \theta)$ it follows H^2 as being a transformation on pairs of angles:

$$H^2 : (\varphi, \theta) \to (\Phi, \Theta). \tag{2.5}$$

A direct computation gives the relationships between these pairs of angles:

$$\begin{cases} \sin\Phi = \sin^2\varphi \to 0 < \Phi < \varphi, \\ \tan\Theta = \frac{|\sin\varphi - \tan\theta|}{1 + \sin\varphi\tan\theta}. \end{cases} \tag{2.6}$$

The last relation provides a new proof of the third remark above; for $\theta = \frac{\pi}{4}$ it results:

$$\tan\Theta = \frac{1 - \sin\varphi}{1 + \sin\varphi} < 1. \tag{2.7}$$

The degenerate case $\varphi = 0$ (i.e., $A = 0$) allowing $\tan\Theta = 1$ yields the Diophantine equation $2B^2 = D^2$ with the degenerate solution $(A = B = C = 0 = D)$.

If we allow some of the components (A, B, C) to be negative, then we can find a twin image $H^2(\Box)$ starting from a rational pair $(\sin\varphi, \tan\theta) \in \mathbb{Q} \times \mathbb{Q}$ only through the cases:

1) $\sin\varphi_1 = -\frac{1}{2}, \quad \tan\theta_1 = \frac{1}{3}$; hence $\varphi_1 = -\frac{\pi}{6}$ and $\theta_1 = 18.43°$;

2) $\sin\varphi_2 = -\frac{1}{3}, \quad \tan\theta_2 = \frac{1}{2}$; hence $\varphi_2 = -19.47°$ and $\theta_2 = 26.57°$ giving $\theta_1 + \theta_2 = \frac{\pi}{4}$.

In the first case, from $D = -2A$ it results $(2B)^2 + (2C)^2 = 3A^2$ with the only integer solution $(0,0,0)$ while the second case gives the Pythagorean quadruple $\lambda(-1,2,2,3)$ which, following Example 1.2, does not produce a twin one. In conclusion, we prove the second main result of this study:

Theorem 2.1. *The image of the map H^2 on the subset of proper Pythagorean quadruples does not contain twin Pythagorean quadruples.*

3 The matrix approach to H^2

A second motivation for the choice of notation H^2 is that the homotethical-Hopf transformation can be thought as a *quadratic* one. There exist *linear* transformations on $\mathcal{PQ}$, see [8]. Recall that a similar fact holds for Pythagorean triples; see [1] or [2].

In this section, we treat the transformation H^2 with the tools of the matrix calculus. More precisely, supposing the order $0 < A < B < C < D$, we have

$$H^2(\square) = (A^2, AC + BD, CD - AB, D^2),$$

which means:

$$H^2(\square) = (A, B, C, D) \cdot M(\square), \quad M(\square) := \begin{pmatrix} A & \frac{C}{2} & -\frac{B}{2} & 0 \\ 0 & \frac{D}{2} & -\frac{A}{2} & 0 \\ 0 & \frac{A}{2} & \frac{D}{2} & 0 \\ 0 & \frac{B}{2} & \frac{C}{2} & D \end{pmatrix} \in M_4(\mathbb{R}). \tag{3.1}$$

The 4×4 matrix $M(\square)$ has the trace and the determinant as functions only of A and D:

$$\mathrm{Tr}(M(\square)) = A + \frac{3D}{2}, \quad \det M(\square) = \frac{AD}{4}(A^2 + D^2) \tag{3.2}$$

and its characteristic polynomial is:

$$P_\lambda(\square) = (\lambda - A)(\lambda - D)\left[\lambda^2 - D\lambda + \frac{A^2 + D^2}{4}\right], \tag{3.3}$$

which implies that the eigenvalues of the $M(\square)$ are both real and complex:

$$0 < \lambda_1(\square) = A < \lambda_2(\square) = D \in \mathbb{R}, \lambda_{3,4}(\square) = \frac{1}{2}(D \pm A\bar{i}) \in \mathbb{C}, \sqrt{2}\lambda_3(\square) = D \cdot z(\square). \tag{3.4}$$

Also, the middle 2×2 block of the matrix $M(\square)$ suggests the special orthogonal matrix:

$$\frac{1}{\sqrt{A^2 + D^2}} \begin{pmatrix} D & -A \\ A & D \end{pmatrix} = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \in SO(2), \quad \psi = \psi(\square). \tag{3.5}$$

In fact, the occurrence of the matrix $M(\square)$ is natural if we consider the differential point of view. Namely, treating H^2 as a smooth map from $\mathbb{R}^4$ to $\mathbb{R}^4$, its Jacobian matrix of the partial derivatives is:

$$\begin{pmatrix} 2A & 0 & 0 & 0 \\ C & D & A & B \\ -B & -A & D & C \\ 0 & 0 & 0 & 2D \end{pmatrix},$$

which is the double of the transposed matrix $M(\square)^T$.

Example 3.1. Revisiting Example 1.2, we get:

$$\begin{cases} \mathrm{Tr}(minimal) = \frac{11}{2}, \quad \det(minimal) = \frac{15}{2}, \\ \cos\psi(minimal) = \frac{3}{\sqrt{10}}, \quad \sin\psi(minimal) = \frac{1}{\sqrt{10}}, \end{cases} \tag{3.6}$$

and we point out that the angle $\frac{\pi}{2} - \psi(minimal) = 71.57°$ is exactly the angle t_2 from Theorem 3.1 from [5].

It is worth remarking that the fourth degree polynomial $P_\lambda(\square)$ from (3.3) defines a plane quartic curve:

$$\mathcal{C}(\square) : y^2 = P_x(\square). \tag{3.7}$$

For our example we get:

$$\mathcal{C}(minimal) : y^2 = (x-1)(x-3)\left(x^2 - 3x + \frac{5}{2}\right), \tag{3.8}$$

which contains four integer points: $(1, 0)$, $(3, 0)$, $(5, \pm 10)$.

4 Conclusions

In this paper, the composition between the Hopf map and a homothety of the space $\mathbb{R}^3$ is used to produce new Pythagorean quadruples from old ones. The famous twin Pythagorean quadruple $(1, 2, 2, 3)$ serves as the first example. We prove that this transformation does not result in twin Pythagorean quadruples by using the trigonometrical parametrization of the unit sphere S^2. In addition to Euclidean and Riemannian geometry, the elementary tools of linear algebra are considered in order to obtain a better image of this very interesting subject.

If we are looking for a possible generalization, we consider the general quadratic Diophantine equation:

$$(x^1)^2 + \cdots + (x^n)^2 = (x^{n+1})^2.$$

It follows a rational point $\left(\frac{x^1}{x^{n+1}}, \ldots, \frac{x^n}{x^{n+1}}\right) \in S^{n-1}$ and we point out that the Hopf map admits extensions to higher dimensions:

$$S^7 \to S^4, \quad S^{15} \to S^8.$$

The total spaces of these bundles, namely S^7 and S^{15} are the unit spheres in the Euclidean spaces $\mathbb{E}^8$ and $\mathbb{E}^{16}$, which means the occurence of the algebra of octonions and sedenions, respectively, as succesive members to $\mathbb{H}$ in the iterated Cayley–Dickson process. It remains an open problem what are their possible applications to the given Diophantine equation.

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