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Generalizing Pascal's row sums to complex orders: A Poisson summation approach

Fabrizio Mancini

I.I.S. da Vinci
Civitanova Marche, Italy
e-mail: `mancini.fabrizio@iisdavinci.edu.it`

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Abstract: We generalize Pascal's triangle row sums by defining shifted fractional binomial coefficients with a real scaling parameter and a complex shift. Using the Poisson summation formula, we derive exact summation identities extended via analytic continuation to conditionally convergent regimes. We prove that when the magnitude of the scaling parameter is not greater than 2, the sum collapses to a simple exponential form independent of the shift; otherwise, it generalizes classical series multisections to non-integer steps and complex shifts. These closed forms allow for rapid and exact computation even where symbolic algebra systems fail.

Keywords: Pascal's triangle, Fractional binomial coefficients, Bernoulli numbers with fractional indices, Gamma function, Fourier transform, Poisson summation formula, Analytic continuation, Series multisection.
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1 Introduction

The identities involving binomial coefficients lie at the heart of combinatorial number theory. Among the most fundamental properties of Pascal's triangle is the row sum identity

$$\sum_{k=0}^{n}\binom{n}{k}=2^n,$$

valid for non-negative integers n. The extension of this identity to real or complex upper indices, alongside the study of continuous analogues of binomial coefficients via the Gamma function, has a rich history in mathematical analysis. Foundational works by Pollard and Shisha [7], as well as Boas and Pollard [1], explored the interplay between discrete binomial sums and their integral continuous counterparts. More recently, Salwinski [10] provided a comprehensive overview of the continuous binomial coefficient and its integration properties. However, the evaluation of infinite sums involving these coefficients becomes significantly more challenging when the indices are subjected to non-integer scaling or complex shifts. We tackle this challenge by proposing a broad generalization of the Pascal row sum problem. Motivated by a recent work of Caratelli *et al.* [2], we introduce the *shifted α-fractional binomial coefficient*, denoted by $\binom{n}{k}_{\alpha,\delta}$, which incorporates a real scaling parameter α acting on the arguments $n \in \mathbb{C}$ and $k \in \mathbb{Z}$ and a complex shift δ acting on k. Specifically, we investigate the infinite series

$$S_{\alpha,\delta}(n)=\sum_{k=-\infty}^{+\infty}\binom{n}{k}_{\alpha,\delta}=\sum_{k=-\infty}^{+\infty}\frac{\Gamma(\alpha n+1)}{\Gamma(\alpha k+\delta+1)\Gamma(\alpha(n-k)-\delta+1)}.$$

Analogous series were studied by Osler [6], who established that the sum coincides with its integral counterpart for scaling steps $|\alpha| \leq 1$. Our work significantly extends this result by identifying the sharp boundary $|\alpha| \leq 2$ for the validity of this integral representation and by deriving exact evaluations for larger steps ($|\alpha| > 2$) where the analogy breaks down.

Direct evaluation of this series using standard hypergeometric summation techniques often fails to produce concise closed forms, especially when α is irrational or when the series converges only conditionally. As we demonstrate, even modern symbolic algebra systems struggle in these regimes.

To address these issues, we adopt an analytic approach based on the Poisson summation formula (PSF). We derive an exact evaluation of $S_{\alpha,\delta}(n)$ by analyzing the spectral properties of the underlying function $f(z)=[\Gamma(a+z)\Gamma(b-z)]^{-1}$. Our main result establishes that, for $|\alpha| \leq 2$ (with $\alpha \neq 0$) and independently of δ, the sum collapses to a remarkably simple single-term expression:

$$\sum_{k=-\infty}^{+\infty}\binom{n}{k}_{\alpha,\delta}=\frac{2^{n\alpha}}{|\alpha|}.$$

This identity holds in the region of absolute convergence $\alpha\Re(n) > 0$. Furthermore, under the strict inequality $|\alpha| < 2$, its validity extends via analytic continuation to the domain $\alpha\Re(n) > -1$. This extension covers the full domain of convergence, as the series generally diverges in the conditional strip $-1 < \alpha\Re(n) \leq 0$ for the boundary case $|\alpha| = 2$. This identity recovers the classical Pascal row sum for $\alpha = 1$ (with $\delta = 0$) and extends it to the complex plane. Finally, for $|\alpha| > 2$, we show that the sum generalizes the classical series multisection identities to non-integer steps and complex shifts, producing finite sums of trigonometric terms.

The paper is organized as follows: Section 2 establishes the Fourier transform of f and derives the main summation identity using the PSF. Section 3 details the analytic continuation to the conditionally convergent regime. Section 4 introduces the shifted fractional binomial coefficients and derives the generalizations of Pascal's identity. Finally, Section 5 provides comparisons with evaluations performed by *Mathematica* (v. 14.3).

2 Spectral properties and general summation formula

We begin by deriving the explicit expression for the Fourier transform of the coefficient function.

Lemma 2.1. *Let $\Re(a+b) > 1$. The Fourier transform of the function*

$$f(z) = \frac{1}{\Gamma(a+z)\Gamma(b-z)},$$

defined by $\widehat{f}(\xi) = \int_{-\infty}^{+\infty} f(x)e^{-2\pi i\xi x}\,dx$, is given by

$$\widehat{f}(\xi) = \frac{(2\cos(\pi\xi))^{a+b-2}}{\Gamma(a+b-1)} e^{i\pi\xi(a-b)} \cdot \mathbb{1}_{]-\frac{1}{2},\frac{1}{2}[}(\xi), \tag{1}$$

where $\mathbb{1}_I(\xi)$ denotes the indicator function of the interval I. In particular, $\widehat{f}$ is compactly supported with $\operatorname{supp}(\widehat{f}) = [-\frac{1}{2}, \frac{1}{2}]$.

Proof. The proof follows from the identity

$$\int_{-\infty}^{+\infty} \frac{e^{i\omega x}}{\Gamma(a+x)\Gamma(b-x)} dx = \frac{(2\cos(\frac{\omega}{2}))^{a+b-2}}{\Gamma(a+b-1)} e^{i\frac{\omega}{2}(b-a)} \cdot \mathbb{1}_{]-\pi,\pi[}(\omega),$$

which holds for $|\omega| \neq \pi$ if $\Re(a+b) > 1$, and for all real ω if $\Re(a+b) > 2$ (see [8]). ☐

To apply the PSF, $f(x)$ and $\widehat{f}(\xi)$ must possess adequate decay and regularity.

Lemma 2.2. *Assume that $\Re(a+b) > 2$. Then, both f and $\widehat{f}$ belong to $L^1(\mathbb{R})$ and satisfy the decay condition*

$$|f(x)| + |\widehat{f}(x)| \leq C(1+|x|)^{-1-\epsilon} \qquad \text{for all } x \in \mathbb{R},$$

for some constants $C > 0$ and $\epsilon > 0$.

Proof. Let $\sigma = \Re(a+b)$. By hypothesis, $\sigma > 2$, so we can choose $\epsilon = \sigma - 2 > 0$.

The function $f(x)$ is continuous on $\mathbb{R}$ (as the reciprocal of the Gamma function is entire), hence bounded on any compact set.

We analyze its asymptotic behavior for $|x| \to +\infty$. We rewrite $f(x)$ for large positive x by applying the Euler reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$:

$$f(x) = \frac{\sin(\pi(b-x))}{\pi} \frac{\Gamma(1-b+x)}{\Gamma(a+x)}.$$

For $x \in \mathbb{R}$, the magnitude of the sine term is bounded by $\cosh(\pi\Im(b))$. By combining this bound with the standard asymptotic estimate $\frac{\Gamma(x+\alpha)}{\Gamma(x+\beta)} \sim x^{\alpha-\beta}$, we ensure the existence of a constant $C_0 > 0$ such that

$$|f(x)| \leq C_0|x|^{1-\Re(a+b)} = C_0|x|^{-1-\epsilon},$$

for large positive x. The analysis for $x \to -\infty$ is symmetric (rewriting the $\Gamma(a+x)$ term).

Combining boundedness near the origin with the decay at infinity, we guarantee the existence of a constant $C_1 > 0$ for which

$$|f(x)| \leq C_1(1 + |x|)^{-1-\epsilon} \quad \text{for all } x \in \mathbb{R}.$$

Integrating this bound:

$$\int_{-\infty}^{+\infty} |f(x)|\, dx \leq \int_{-\infty}^{+\infty} C_1(1 + |x|)^{-1-\epsilon}\, dx < +\infty.$$

Thus, $f \in L^1(\mathbb{R})$.

From Lemma 2.1, $\operatorname{supp}(\widehat{f}) = [-\frac{1}{2}, \frac{1}{2}]$. Being a continuous function (for $\Re(a+b) > 2$) with compact support, $\widehat{f}$ is necessarily bounded and belongs to $L^1(\mathbb{R})$. Regarding the decay estimate, since $\widehat{f}$ is bounded and vanishes outside $]-\frac{1}{2}, \frac{1}{2}[$ and $(1 + |x|)^{1+\epsilon}$ is bounded on the compact support of $\widehat{f}$, there exists a constant $C_2 > 0$ such that

$$|\widehat{f}(x)| \leq C_2(1 + |x|)^{-1-\epsilon} \quad \text{for all } x \in \mathbb{R}.$$

Summing the estimates for f and $\widehat{f}$ with $C = C_1 + C_2$, the proof is complete.

□

We now derive the general summation formula for any scaling parameter $\alpha \neq 0$.

Proposition 2.1 (General Summation Identity). *Let $a, b \in \mathbb{C}$ with $\Re(a+b) > 2$. For any non-zero real number α, the following identity holds:*

$$\sum_{k=-\infty}^{+\infty} \frac{1}{\Gamma(a + k\alpha)\Gamma(b - k\alpha)} = \frac{1}{|\alpha|\Gamma(a + b - 1)} \sum_{\substack{m \in \mathbb{Z} \\ |m| < |\alpha|/2}} \left(2\cos\left(\frac{\pi m}{\alpha}\right)\right)^{a+b-2} e^{i\frac{\pi m}{\alpha}(a-b)}. \tag{2}$$

Proof. We apply the Poisson summation formula to the function $h(x) = f(\alpha x)$. The Fourier transform of $h(x)$ is given by the scaling property:

$$\widehat{h}(\xi) = \frac{1}{|\alpha|}\widehat{f}\left(\frac{\xi}{\alpha}\right).$$

By Lemma 2.2, the PSF holds pointwise (see Grafakos [4], Theorem 3.1.17):

$$\sum_{k \in \mathbb{Z}} f(k\alpha) = \sum_{m \in \mathbb{Z}} \widehat{h}(m) = \frac{1}{|\alpha|} \sum_{m \in \mathbb{Z}} \widehat{f}\left(\frac{m}{\alpha}\right).$$

The term $\widehat{f}(m/\alpha)$ is non-zero only when the argument lies within the interval $]-\frac{1}{2}, \frac{1}{2}[$, i.e., when $|m| < \frac{|\alpha|}{2}$. Inserting the explicit expression for $\widehat{f}$ from (1) for these valid integers m yields the stated identity. □

3 Analytic continuation

While the previous result requires $\Re(a+b) > 2$, i.e., the absolute convergence of the series, we now extend the validity of the general summation identity (2) to the region of conditional convergence via analytic continuation.

Proposition 3.1. *Let α be a non-zero real number and $a, b \in \mathbb{C}$. The identity*

$$\sum_{k=-\infty}^{+\infty} \frac{1}{\Gamma(a+k\alpha)\Gamma(b-k\alpha)} = \frac{1}{|\alpha|\Gamma(a+b-1)} \sum_{\substack{m\in\mathbb{Z} \\ |m|<|\alpha|/2}} \left(2\cos\left(\frac{m\pi}{\alpha}\right)\right)^{a+b-2} e^{i\frac{m\pi}{\alpha}(a-b)}$$

holds in the extended domain $\Re(a+b) > 1$ if α is not an even integer.

Proof. Let $S(a,b)$ denote the series on the LHS and $G(a,b)$ denote the finite sum on the RHS. We established in Proposition 2.1 that $S(a,b) = G(a,b)$ on the non-empty open set $D_1 = \{(a,b) \in \mathbb{C}^2 : \Re(a+b) > 2\}$. The function $G(a,b)$ is entire, thus holomorphic in the connected open set $D = \{(a,b) \in \mathbb{C}^2 : \Re(a+b) > 1\}$. To apply the identity theorem (see, e.g., [5]), it suffices to show that the series $S(a,b)$ (of entire functions) converges uniformly on compact subsets of D, thus defining a holomorphic function on D.

Without loss of generality, we assume $\alpha > 0$. Indeed, since the summation runs over $k \in \mathbb{Z}$, the sum is invariant under the substitution $k \to -k$, which corresponds to changing the sign of α. Thus, we can restrict our analysis to positive α.

We analyze the asymptotic behavior of the general term $c_k = [\Gamma(a+k\alpha)\Gamma(b-k\alpha)]^{-1}$ for large $|k|$. We focus on the case for large positive k ($k \geq k_0 > 1$); the analysis for large negative k is symmetric (rewriting $\Gamma(a+k\alpha)$ instead of $\Gamma(b-k\alpha)$).

Let $K \subset D$ be a compact subset. Then, there exists $\sigma_0 > 1$ such that $\Re(a+b) \geq \sigma_0$ for all $(a,b) \in K$. Using the Euler reflection formula, we write

$$c_k = \frac{\sin(\pi(b-k\alpha))}{\pi} \frac{\Gamma(1-b+k\alpha)}{\Gamma(a+k\alpha)}.$$

Using the asymptotic expansion of the Gamma ratio, we have

$$\frac{\Gamma(1-b+k\alpha)}{\Gamma(a+k\alpha)} = (k\alpha)^{1-a-b}\left(1 + O\left(\frac{1}{k}\right)\right).$$

Substituting this into c_k yields

$$c_k = \frac{1}{\pi}\sin(\pi(b-k\alpha))(k\alpha)^{1-a-b} + R_k,$$

where $|R_k| \leq Ck^{-\sigma_0}$ for all $(a,b) \in K$. Thus, the error term series $\sum_{k\geq k_0} R_k$ converges absolutely and uniformly on K. It remains to analyze the convergence of the series $\sum_{k\geq k_0} v_k u_k$, where

$$v_k = \sin(\pi b - \pi k\alpha), \qquad u_k = (k\alpha)^{1-a-b}.$$

Let N be a positive integer, $N \geq k_0$. Let $V_N = \sum_{k=k_0}^{N} v_k$. The behavior of the partial sums V_N depends critically on the parity of α. Indeed, since

$$\sum_{k=1}^{N} \sin(\pi b - \pi k\alpha) = \frac{\sin(\pi N\alpha/2)}{\sin(\pi\alpha/2)} \cdot \sin\left(\pi b - (N+1)\frac{\pi\alpha}{2}\right),$$

there exists $M_K > 0$ such that

$$|V_N| = \left|\sum_{k=k_0}^{N} \sin(\pi b - \pi k\alpha)\right| \leq M_K,$$

for all $(a,b) \in K$ and all $N \geq k_0$ if and only if $\alpha \notin 2\mathbb{Z}$. To prove uniform convergence of the series $\sum_{k\geq k_0} v_k u_k$ when $\alpha \notin 2\mathbb{Z}$, we use the summation by parts technique. Using the identity $v_k = V_k - V_{k-1}$ (with $V_{k_0-1} = 0$), we rewrite the partial sum as (Abel's identity)

$$\sum_{k=k_0}^{N} v_k u_k = V_N u_N + \sum_{k=k_0}^{N-1} V_k(u_k - u_{k+1}).$$

For the sequence of partial sums to converge uniformly as $N \to +\infty$, two conditions must be met uniformly on K:

1. the boundary term vanishes, i.e. $\lim_{N\to+\infty} V_N u_N = 0$;
2. the series $\sum_{k\geq k_0} V_k(u_k - u_{k+1})$ converges.

Since $|V_k|$ is uniformly bounded by M_K, it suffices to prove that $|u_N| \to 0$ uniformly on K as $N \to +\infty$ and that the series $\sum_{k\geq k_0} |u_k - u_{k+1}|$ converges uniformly on K.

1. For any $(a,b) \in K$, we have

$$|u_k(a,b)| = \alpha^{1-\Re(a+b)}\, k^{1-\Re(a+b)} \leq C'\, k^{1-\sigma_0}.$$

Since $\sigma_0 > 1$, $|u_N| \to 0$ uniformly on K as $N \to +\infty$.

2. Let $U : \mathbb{R}_{>0} \to \mathbb{C}$, $U(x) = (\alpha x)^{1-a-b}$. For any $(a,b) \in K$, by the mean value inequality, we have

$$\begin{aligned}
|u_k - u_{k+1}| &= |U(k) - U(k+1)| \\
&\leq \sup_{x\in[k,k+1]} |U'(x)| \\
&= |1-a-b|\alpha^{1-\Re(a+b)} \sup_{x\in[k,k+1]} x^{-\Re(a+b)} \\
&\leq C_K k^{-\sigma_0}.
\end{aligned}$$

As $\sigma_0 > 1$, by the Weierstrass M-test the series of differences $\sum_k |u_k - u_{k+1}|$ converges uniformly on K.

Both conditions 1 and 2 are satisfied, thus the series $\sum_k v_k u_k$ converges uniformly on compact subsets of D. Hence, the series $S(a,b)$ defines a holomorphic function on D. Since $S(a,b)$ coincides with $G(a,b)$ on the non-empty open set $D_1 \subset D$, by the identity theorem it must coincide with $G(a,b)$ on all of D. ☐

Remark 3.1. *When α is an even integer, the general summation identity* (2) *still holds if a and b are integers, regardless of any convergence constraint, as the series eventually terminates in this case.*

4 Generalizing Pascal row sum identity

Definition 4.1. *For any $\alpha \in \mathbb{R}$, $k \in \mathbb{Z}$, and $\delta \in \mathbb{C}$, we define the* Shifted α-Fractional Binomial Coefficient *as the meromorphic function of $n \in \mathbb{C}$ given by:*

$$\binom{n}{k}_{\alpha,\delta} := \frac{\Gamma(\alpha n + 1)}{\Gamma(\alpha k + \delta + 1)\Gamma(\alpha(n-k) - \delta + 1)}. \tag{3}$$

The coefficient is defined at removable singularities via analytic continuation.

Remark 4.1. *Singularities in* (3) *arise only at the poles of the numerator, i.e., when αn is a negative integer. These are removable if a corresponding pole occurs in the denominator, which is always guaranteed if $\alpha, \delta \in \mathbb{Z}$. However, for arbitrary real α and complex δ, singularities (poles) may persist. In the following theorems, the restriction to the domain $\alpha\Re(n) > -1$ (and subsets thereof) excludes all potential poles and defines the natural region of convergence for the infinite series under investigation.*

The unshifted α-fractional binomial coefficients $\binom{n}{k}_\alpha := \binom{n}{k}_{\alpha,0}$ naturally arise when expanding the defining relation of the *fractional index Bernoulli numbers $B_{\alpha,n}$*, generated by the function

$$\frac{t^\alpha}{E_\alpha(t^\alpha) - 1} = \sum_{n=0}^{\infty} B_{\alpha,n} \frac{t^{\alpha n}}{\Gamma(n\alpha + 1)}, \quad \alpha > 0,$$

where $E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)}$ is the one-parameter Mittag-Leffler function (see [2, 9]). Indeed, by applying the Cauchy product formula to this identity, we find that these numbers satisfy the recurrence relation

$$\begin{cases} B_{\alpha,0} = \Gamma(\alpha+1), \\ \sum_{k=0}^{n} \binom{n+1}{k}_\alpha B_{\alpha,k} = 0, \quad n \geq 1, \end{cases}$$

or, equivalently,

$$\begin{cases} B_{\alpha,0} = \Gamma(\alpha+1), \\ B_{\alpha,n} = -\frac{1}{\binom{n+1}{1}_\alpha} \sum_{k=0}^{n-1} \binom{n+1}{k}_\alpha B_{\alpha,k}, \quad n \geq 1. \end{cases}$$

This result highlights that the coefficients $\binom{n}{k}_\alpha$ provide the structural extension of the binomial convolution kernel to the fractional setting, preserving the algebraic form of the classical Bernoulli recurrence.

We are now in a position to prove the following generalizations of Pascal's identity.

Theorem 4.1 (Generalized Pascal's Identity). *Let $\alpha \in \mathbb{R} \setminus \{0\}$ and $\delta \in \mathbb{C}$. Then, the identity*

$$\sum_{k=-\infty}^{+\infty} \binom{n}{k}_{\alpha,\delta} = \frac{2^{n\alpha}}{|\alpha|} \left[1 + 2 \sum_{0<m<|\alpha|/2} \cos^{n\alpha}\left(\frac{m\pi}{\alpha}\right) \cos\left(\frac{m\pi}{\alpha}(n\alpha - 2\delta)\right)\right] \tag{4}$$

holds whenever $\alpha\Re(n) > 0$. The validity of the identity extends to the domain $\alpha\Re(n) > -1$ if α is not an even integer.

Proof. By Definition 4.1, we write the sum as

$$\sum_{k=-\infty}^{+\infty} \binom{n}{k}_{\alpha,\delta} = \Gamma(n\alpha+1) \sum_{k=-\infty}^{+\infty} \frac{1}{\Gamma(1+\delta+k\alpha)\,\Gamma(n\alpha+1-\delta-k\alpha)}.$$

Let $a = 1+\delta$ and $b = n\alpha + 1 - \delta$. We apply Proposition 2.1. This proposition requires the absolute convergence condition $\Re(a+b) > 2$, which translates to $\alpha\Re(n) > 0$. Substituting in (2) and grouping the terms for m and $-m$ yields

$$\begin{aligned}\sum_{k=-\infty}^{+\infty} \binom{n}{k}_{\alpha,\delta} &= \frac{1}{|\alpha|} \sum_{\substack{m\in\mathbb{Z}\\ |m|<|\alpha|/2}} \left(2\cos\left(\frac{m\pi}{\alpha}\right)\right)^{n\alpha} e^{-\frac{m\pi}{\alpha}(n\alpha-2\delta)i} \\ &= \frac{2^{n\alpha}}{|\alpha|}\left[1 + 2\sum_{0<m<|\alpha|/2} \cos^{n\alpha}\left(\frac{m\pi}{\alpha}\right)\cos\left(\frac{m\pi}{\alpha}(n\alpha-2\delta)\right)\right].\end{aligned}$$

For the second part of the theorem, we use Proposition 3.1. Thanks to the analytic continuation established via Abel's summation, this proposition guarantees validity in the domain $\Re(a+b) > 1$, which translates to $\alpha\Re(n) > -1$, if $\alpha \notin 2\mathbb{Z}$. ☐

Remark 4.2. *As noted in Remark 3.1, if α is an even integer, the identity* (4) *remains valid provided that the series terminates. In our context, this occurs when αn is a non-negative integer and δ is an integer. Under these conditions, only finitely many binomial coefficients are non-zero, reducing the infinite series to a finite sum. Consequently, convergence is trivial and the identity holds regardless of the general convergence constraints.*

Equation (4) simplifies significantly when $|\alpha| \leq 2$.

Theorem 4.2 (Collapsed Pascal Identity)**.** *Let α be a non-zero real number such that $|\alpha| \leq 2$ and $\delta \in \mathbb{C}$. Then, the identity*

$$\sum_{k=-\infty}^{+\infty} \binom{n}{k}_{\alpha,\delta} = \frac{2^{n\alpha}}{|\alpha|} \tag{5}$$

holds whenever $\alpha\Re(n) > 0$, independently of δ. The validity of the identity extends to the domain $\alpha\Re(n) > -1$ if $|\alpha| < 2$.

Proof. It follows straightforwardly from Theorem 4.1, since the set of integers m satisfying $0 < m < \frac{|\alpha|}{2}$ is empty for $|\alpha| \leq 2$, causing the summation term to vanish. ☐

Remark 4.3. *Similar to the discussion above, in the boundary case $|\alpha| = 2$, if αn is a non-negative integer and δ is an integer, the series terminates and the identity $\sum \binom{n}{k}_{\alpha,\delta} = 2^{n\alpha-1}$ holds trivially.*

When $n \in \mathbb{Z}$ and $\alpha > 0$, considering the array of shifted fractional binomial coefficients $\binom{n}{k}_{\alpha,\delta}$, Theorems 4.1 and 4.2 imply that we can evaluate the sum of the elements of:

- each positive row ($n > 0$);
- *any* row $n \geq 1 - m$ when $\alpha = \frac{1}{m}$ with $m \in \mathbb{Z}_{\geq 1}$.

Here is a representation of the unshifted "half-fractional" array $\binom{n}{k}_{\frac{1}{2}}$ for $-1 \le n \le 7$:

$n=-1$	$\cdots\ \frac{3}{8}\ 0\ -\frac{1}{2}\ 0\ \mathbf{1}\ \mathbf{1}\ 0\ -\frac{1}{2}\ 0\ \frac{3}{8}\ \cdots$
0	$\cdots\ \frac{2}{5\pi}\ 0\ -\frac{2}{3\pi}\ 0\ \frac{2}{\pi}\ \mathbf{1}\ \frac{2}{\pi}\ 0\ -\frac{2}{3\pi}\ 0\ \frac{2}{5\pi}\ \cdots$
1	$\cdots\ \frac{1}{16}\ 0\ -\frac{1}{8}\ 0\ \frac{1}{2}\ \mathbf{1}\ \mathbf{1}\ \frac{1}{2}\ 0\ -\frac{1}{8}\ 0\ \frac{1}{16}\ \cdots$
2	$\cdots\ \frac{4}{35\pi}\ 0\ -\frac{4}{15\pi}\ 0\ \frac{4}{3\pi}\ \mathbf{1}\ \frac{4}{\pi}\ \mathbf{1}\ \frac{4}{3\pi}\ 0\ -\frac{4}{15\pi}\ 0\ \frac{4}{35\pi}\ \cdots$
3	$\cdots\ \frac{3}{128}\ 0\ -\frac{1}{16}\ 0\ \frac{3}{8}\ \mathbf{1}\ \frac{3}{2}\ \frac{3}{2}\ \mathbf{1}\ \frac{3}{8}\ 0\ -\frac{1}{16}\ 0\ \frac{3}{128}\ \cdots$
4	$\cdots\ \frac{16}{315\pi}\ 0\ -\frac{16}{105\pi}\ 0\ \frac{16}{15\pi}\ \mathbf{1}\ \frac{16}{3\pi}\ 2\ \frac{16}{3\pi}\ \mathbf{1}\ \frac{16}{15\pi}\ 0\ -\frac{16}{105\pi}\ 0\ \frac{16}{315\pi}\ \cdots$
5	$\cdots\ \frac{3}{256}\ 0\ -\frac{5}{128}\ 0\ \frac{5}{16}\ \mathbf{1}\ \frac{15}{8}\ \frac{5}{2}\ \frac{5}{2}\ \frac{15}{8}\ \mathbf{1}\ \frac{5}{16}\ 0\ -\frac{5}{128}\ 0\ \frac{3}{256}\ \cdots$
6	$\cdots\ \frac{32}{1155\pi}\ 0\ -\frac{32}{315\pi}\ 0\ \frac{32}{35\pi}\ \mathbf{1}\ \frac{32}{5\pi}\ 3\ \frac{32}{3\pi}\ 3\ \frac{32}{5\pi}\ \mathbf{1}\ \frac{32}{35\pi}\ 0\ -\frac{32}{315\pi}\ 0\ \frac{32}{1155\pi}\ \cdots$
7	$\cdots\ \frac{7}{1024}\ 0\ -\frac{7}{256}\ 0\ \frac{35}{128}\ \mathbf{1}\ \frac{35}{16}\ \frac{7}{2}\ \frac{35}{8}\ \frac{35}{8}\ \frac{7}{2}\ \frac{35}{16}\ \mathbf{1}\ \frac{35}{128}\ 0\ -\frac{7}{256}\ 0\ \frac{7}{1024}\ \cdots$

For instance, according to (5), the sum of the row $n = 6$ is 16, while the sum of the negative row $n = -1$ is $\sqrt{2}$.

When $n \in \mathbb{N}$, $\alpha = 1$ and $\delta = 0$, Theorem 4.2 retrieves the familiar row sum property of Pascal's triangle ($\sum_k \binom{n}{k} = 2^n$).

While Theorem 4.2 focuses on the case $|\alpha| \le 2$ where the sum collapses to a single term, Equation (4) of Theorem 4.1 allows for the exact evaluation of binomial sums for larger values of $|\alpha|$, resulting in a finite sum of trigonometric terms. It is worth noting that, even in the oscillatory regime $2 < \alpha \le 4$, where only the $m = 1$ spectral component survives, when $n - 2\delta/\alpha - 1/2 \in \mathbb{Z}$ the sum reduces to the single term $2^{n\alpha}/|\alpha|$.

When α and δ are integers with $0 \le \delta < \alpha$, the generalized Pascal's identity (4) recovers the multisection series identities (setting the row index $N = \alpha n$, see, e.g., [3]):

$$\sum_{k=0}^{+\infty} \binom{N}{\alpha k + \delta} = \frac{1}{\alpha} \sum_{j=0}^{\alpha-1} 2^N \cos^N\left(\frac{\pi j}{\alpha}\right) \cos\left(\frac{\pi j}{\alpha}(N - 2\delta)\right).$$

Our derivation confirms the validity of this algebraic form for complex values of N, interpreting the sum as an infinite series subject to the convergence conditions established in Theorem 4.1.

Finally, we observe that the dependence on δ in (4) is periodic with period α. This reflects the fact that shifting δ by a multiple of α corresponds to a mere integer translation of the summation index k in the definition of the binomial series, which naturally leaves the total sum invariant.

To provide a geometric intuition of the structural change occurring at the critical value $|\alpha| = 2$, we present a 3D visualization of the generalized identity in Figure 1. The plot illustrates the sum $S_{\alpha,\delta}(n)$ for the case $n = 1$, as a function of the scaling parameter α and the real shift δ. The transition from the flat geometry (invariance with respect to δ for $\alpha \le 2$) to the oscillatory behavior (for $\alpha > 2$) is clearly visible, corroborating the spectral analysis derived above.

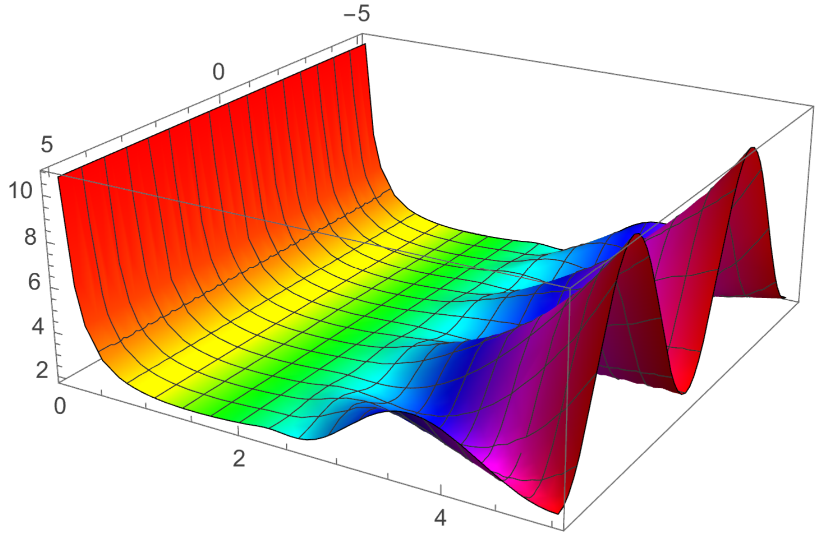

Figure 1. 3D plot of the sum $S_{\alpha,\delta}(1)$ for $0 < \alpha \leq 5$ and $-5 \leq \delta \leq 5$. The visualization highlights the transition at the critical value $\alpha = 2$. For $\alpha \leq 2$, the sum is independent of the shift δ (flat cross-sections), as described by Theorem 4.2. For $\alpha > 2$, oscillations emerge due to the contributions of non-zero spectral components, as delineated by Theorem 4.1.

5 Computational verification

In Table 1, the values of the sum $\sum_{k\in\mathbb{Z}} \binom{n}{k}_{\alpha}$ calculated using the collapsed Pascal identity (5) are compared with the evaluations performed by *Mathematica* (v. 14.3). These tests suggest that symbolic solvers still encounter significant difficulties or even fail to produce simple closed forms, particularly when α is irrational or in cases of conditional convergence (except for $n = 0$, where the summands reduce to the Sinc function, recovering a standard Fourier series identity).

Table 2 compares the sum $\sum_{k\in\mathbb{Z}} \binom{n}{k}_{\alpha,\delta}$ calculated using the generalized Pascal identity (4) with *Mathematica*'s output. Again, state-of-the-art symbolic software struggles or fails to produce concise closed forms.

6 Conclusion

In this paper, we have established a unified analytic framework for evaluating infinite sums of shifted fractional binomial coefficients. By leveraging the spectral properties of the coefficient function and the Poisson summation formula, we extended the classical Pascal row sum identity to complex orders, real scaling parameters, and complex shifts. Our derivation identifies two distinct regimes: for scaling parameters $|\alpha| \leq 2$, the sum collapses to a simple exponential form independent of the shift δ, whereas for $|\alpha| > 2$, it yields a finite sum of trigonometric terms, generalizing classical series multisections to non-integer steps and complex shifts.

Future research directions may include applying this spectral method to sums with polynomial weights (i.e., $\sum_k P(k)\binom{n}{k}_{\alpha,\delta}$). Additionally, investigating alternating sums would generalize the identity $\sum_k (-1)^k \binom{n}{k} = 0$ to the fractional setting, offering further insights into the spectral behavior of these coefficients.

α	n	**Conv. (C/A)**	**Formula** (5)	***Mathematica* Output (Timing)**
$1/4$	-3	C	$2 \cdot 2^{1/4}$	N.A.
	-2	C	$2\sqrt{2}$	N.A.
	-1	C	$4/2^{1/4}$	N.A.
	$-1+i$	C	$2^{\frac{7+i}{4}}$	Hypergeom. fns. (82.6 s)
	1	A	$4 \cdot 2^{1/4}$	Hypergeom. fns. (55.4 s)
$1/3$	-2	C	$3/2^{2/3}$	N.A.
	-1	C	$3/2^{1/3}$	N.A.
	1	A	$3 \cdot 2^{1/3}$	$3 \cdot 2^{1/3}$ (54.6 s)
$1/2$	-1	C	$\sqrt{2}$	N.A.
	$-1+i$	C	$2^{\frac{1+i}{2}}$	Hypergeom. fns. (56.3 s)
	0	C	2	2 (0.1 s)
	2	A	4	4 (53.8 s)
	3	A	$4\sqrt{2}$	$4\sqrt{2}$ (54.0 s)
$3/4$	-1	C	$\frac{2}{3} \cdot 2^{1/4}$	N.A.
	1	A	$\frac{4}{3} \cdot 2^{3/4}$	Hypergeom. fns. (56.0 s)
$\sqrt{2}$	1	A	$2^{\sqrt{2}}/\sqrt{2}$	N.A.
	2	A	$2^{\frac{4\sqrt{2}-1}{2}}$	N.A.
$\sqrt{3}$	$1/\sqrt{3}$	A	$2/\sqrt{3}$	Hypergeom. fns. (37.9 s)
	1	A	$2^{\sqrt{3}}/\sqrt{3}$	N.A.
$1/e$	e	A	$2e$	Hypergeom. fns. (34.0 s)
	$-e/2$	C	$e/\sqrt{2}$	N.A.
γ	-1	C	$2^{-\gamma}/\gamma$	N.A.
	1	A	$2^{\gamma}/\gamma$	N.A.
$\log_2 e$	1	A	$e \ln 2$	N.A.
$\pi/2$	1	A	$\frac{2}{\pi} 2^{\frac{\pi}{2}}$	N.A.
	$-1/\pi$	C	$\sqrt{2}/\pi$	N.A.

Table 1. Comparison between the proposed formula (5) and *Mathematica* evaluations.
γ is the Euler–Mascheroni constant. N.A. indicates that the software returned the unevaluated sum. C/A denotes Conditional/Absolute convergence. “Hypergeom. fns.” stands for Hypergeometric functions.

$\boldsymbol{\alpha}$	$\boldsymbol{\delta}$	$\boldsymbol{n}$	**Conv. (C/A)**	**Formula** (4)	***Mathematica* Output (Timing)**
3	0	$\Re(n) > -1/3$	C/A	$\frac{1}{3}(2^{3n} + 2\cos(\pi n))$	$\frac{1}{3}\cosh(n \ln 8) + \frac{1}{3}\sinh(n \ln 8)$ $+\frac{2}{3}\cos(\pi n)$ (0.14 s)
	i	i	C	$\frac{1}{3}(2^{3i} + 2\cosh\frac{\pi}{3})$	Hypergeom. fns. (0.11 s)
4	$2i$	$1/2 + i$	A	2^{4i}	Hypergeom. fns. (0.23 s)
5	0	$\Re(n) > -1/5$	C/A	$\frac{1}{5}2^{5n} + \frac{2}{5}\phi^{5n}\cos(\pi n)$ $+\frac{2}{5}\phi^{-5n}\cos(2\pi n)$	Hypergeom. fns. (0.05 s)
	$-\frac{3}{2} + \frac{5}{2}i$	$-\frac{1}{10} + i$	C	$\frac{1}{5}(2^{-\frac{1}{2}+5i} - 2\phi^{\frac{1}{2}-5i})$	Hypergeom. fns. (0.17 s)
$5/2$	0	$n = 2l,\, l \in \mathbb{N}$	C/A	$\frac{2}{5}\left(2^{5l} + 2\phi^{-5l}\right)$	Hypergeom. fns. (58.6 s)
		$-1/5$	C	$\frac{1}{5}\left(2\sqrt{2+\sqrt{5}} + \sqrt{2}\right)$	N.A.
	$\frac{5}{8}(2l+1),$ $l \in \mathbb{Z}$	$n \in \mathbb{N}$	C/A	$\frac{2}{5}2^{\frac{5}{2}n}$	Hypergeom. fns. (88.0 s)
$12/5$	0	$-1/12$	C	$\frac{5}{12}\left(\sqrt[5]{26 + 15\sqrt{3}} + 1/\sqrt[5]{2}\right)$	N.A.
		$5/6$	A	$\frac{5}{12}(7 - 2\sqrt{3})$	Hypergeom. fns. (91.3 s)
		$5/12$	A	$\frac{5}{12}(4 - \sqrt{3})$	Hypergeom. fns. (41.2 s)
$7/2$	$\frac{7}{8}(2l+1),$ $l \in \mathbb{Z}$	$n \in \mathbb{N}$	C/A	$\frac{2}{7}2^{\frac{7}{2}n}$	Hypergeom. fns. (96.9 s)
$24/5$	0	$5/12$	A	$\frac{5}{24}(9 - 3\sqrt{3} + \sqrt{6} - \sqrt{2})$	Hypergeom. fns. (150.2 s)
$2\sqrt{2}$	0	$5/2$	A	$2^{5\sqrt{2}-\frac{3}{2}}$	N.A.
	$\sqrt{2}$	$1/2$	A	$2^{\sqrt{2}-\frac{3}{2}}$	N.A.
	$\frac{\sqrt{2}}{2}(2l+1),$ $l \in \mathbb{Z}$	$n \in \mathbb{N}$	C/A	$2^{2n\sqrt{2}-\frac{3}{2}}$	N.A.
e	0	$1/e$	A	$\frac{2}{e}(1 + 2\cos^2(\frac{\pi}{e}))$	Hypergeom. fns. (31.5 s)
	$e/4$	1	A	$2^e/e$	N.A.
π	0	$1/\pi$	A	$\frac{2}{\pi}(1 + 2\cos^2(1))$	Hypergeom. fns. (34.0 s)
	1		A		Hypergeom. fns. (31.5 s)
	$1/2$		A	$\frac{2}{\pi}(1 + 2\cos(1))$	Hypergeom. fns. (41.3 s)
	$-\frac{1\pm\pi}{4}$	$-\frac{1}{2\pi}$	C	$\frac{1}{\pi\sqrt{2}}$	N.A.
	$\frac{\pi}{4}(2l+1),$ $l \in \mathbb{Z}$	$n \in \mathbb{N}$	C/A	$2^{n\pi}/\pi$	N.A.
2π	$\frac{\pi}{2}(2l+1),$ $l \in \mathbb{Z}$	$n \in \mathbb{N}$	C/A	$\frac{2^{2n\pi}}{\pi}(\frac{1}{2} - \cos^{2n\pi}(1))$	N.A.

Table 2. Comparison between the proposed formula (4) and *Mathematica* evaluations. $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. N.A. indicates that the software returned the unevaluated sum. C/A denotes Conditional/Absolute convergence. “Hypergeom. fns.” stands for Hypergeometric functions.

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References

[1] Boas, R. P., Jr., & Pollard, H. (1973). Continuous analogues of series. *The American Mathematical Monthly*, 80, 18–25.

[2] Caratelli, D., Natalini, P., & Ricci, P. E. (2023). Fractional Bernoulli and Euler numbers and related fractional polynomials—A symmetry in number theory. *Symmetry*, 15(10). Article ID 1900.

[3] Comtet, L. (1974). *Advanced Combinatorics: The Art of Finite and Infinite Expansions*. D. Reidel Publishing Company.

[4] Grafakos, L. (2008). *Classical Fourier Analysis* (2nd ed.). Springer.

[5] Gunning, R. C., & Rossi, H. (2009). *Analytic Functions of Several Complex Variables*. AMS Chelsea Publishing.

[6] Osler, T. J. (1972). An integral analogue of Taylor's series and its use in computing Fourier transforms. *Mathematics of Computation*, 26(118), 449–460.

[7] Pollard, H., & Shisha, O. (1972). Variations on the binomial series. *The American Mathematical Monthly*, 79, 495–499.

[8] Ramanujan, S. (1920). A class of definite integrals. *Quarterly Journal of Mathematics*, 48, 294–310.

[9] Ricci, P. E., Srivastava, R., & Caratelli, D. (2024). Laguerre-type Bernoulli and Euler numbers and related fractional polynomials. *Mathematics*, 12(3), Article ID 381.

[10] Salwinski, D. (2018). The continuous binomial coefficient: An elementary approach. *The American Mathematical Monthly*, 125, 231–244.