

q -Leonardo polynomials

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Abstract: In this study, we define the q -Leonardo Pisano polynomials and q -Leonardo Lucas polynomials of the first and the second kinds, collectively termed q -Leonardo polynomials. We investigate some interesting properties of the new polynomials.

Keywords: q -Leonardo polynomials, q -Fibonacci polynomials, Generating functions.

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1 Introduction

The Fibonacci numbers are a sequence of numbers that have become fundamental in a number of different fields, including mathematics, nature, art and even technology. This sequence is defined by the recurrence relation in which each term is the sum of the two previous ones, starting with 0 and 1. It was introduced by the Italian mathematician Leonardo Fibonacci in his work *Liber Abaci* in 1202.



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The Fibonacci numbers are defined by the following recurrence relation for $n \geq 2$

$$F_n = F_{n-1} + F_{n-2}$$

with initial conditions $F_0 = 0, F_1 = 1$.

In the realm of mathematics, Fibonacci numbers hold considerable importance, exhibiting a close relationship with the golden ratio due to the convergence of the ratio between consecutive Fibonacci numbers as the sequence advances. This characteristic establishes a connection between the Fibonacci sequence and various natural phenomena, encompassing growth patterns, symmetry, and biological structural efficiency. Moreover, the Fibonacci sequence assumes a pivotal role in number theory, particularly in investigations concerning divisibility, modular arithmetic, and Diophantine equations.

Lucas numbers, named after French mathematician François Édouard Anatole Lucas, represent integer sequences bearing a strong resemblance to the Fibonacci sequence. While both sequences follow recurrence relations, Lucas numbers employ distinct initial conditions. The Lucas sequence initiates with the values 2 and 1, with each subsequent term derived from the sum of its two predecessors.

The Lucas numbers are defined by the following recurrence relation for $n \geq 2$

$$L_n = L_{n-1} + L_{n-2},$$

with initial conditions $L_0 = 2, L_1 = 1$.

The mathematical relevance of Lucas numbers extends beyond their association with Fibonacci numbers, as they contribute to the resolution of specific Diophantine equations and find applications in the theory of generating functions. Furthermore, Lucas numbers feature prominently in the examination of Pell's equation and other recursive relationships, underscoring their significance in mathematical research. Accordingly, the Fibonacci and Lucas numbers are determined by

$$F_n = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-j}{j},$$

$$L_n = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-j} \binom{n-j}{j}.$$

These numbers have been of interest to many researchers from past to present. There are many generalizations of these numbers in the literature. One of them is the polynomials containing these numbers. The Fibonacci polynomials were the subject of a study in 1883 by the Belgian mathematician Eugene Charles Catalan and the German mathematician E. Jacobsthal. For $n \geq 2$, the Fibonacci polynomials are defined by

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$$

with the initial values $F_0(x) = 0$ and $F_1(x) = 1$.

For $n \geq 2$, the Lucas polynomials originally studied in 1970 by Bicknell, are defined by

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x)$$

with the initial values $L_0(x) = 2$ and $L_1(x) = x$.

The Leonardo numbers are defined by following recurrence relation for $n \geq 2$

$$Le_n = Le_{n-1} + Le_{n-2} + 1,$$

with initial conditions $Le_0 = Le_1 = 1$ [8]. The relationship between the Leonardo numbers and Fibonacci numbers is

$$Le_n = 2F_{n+1} - 1.$$

Alp and Koçer obtained new identities and matrix representation of the Leonardo numbers [2]. Prasad and Kumari defined the Leonardo polynomials as follows:

$$U_{n+2}(x) = xU_{n+1}(x) + U_n(x) + x,$$

for $n \geq 0$, where $U_0(x) = 1$, $U_1(x) = 2x - 1$. Then the authors investigated their numerous algebraic properties, such as summation formulas, generating functions, Pascal 2-triangle, interrelations with Fibonacci, Lucas, and Chebyshev polynomials, etc., in closed form in [21]. Alp and Kocer defined bivariate Leonardo polynomials and studied relationships between well known polynomials [3]. The Leonardo numbers have been generalized in various ways by different researchers [5, 9, 10, 12, 19, 20, 22–25]. In the literature, various generalizations have been applied to various fields [1, 14, 15, 26]. In [16–18], Kim et al., studied on sums of finite products of balancing polynomials, Lucas-balancing polynomials, and Spivey-type recurrence relations for degenerate Bell and Dowling polynomials. Providing a Spivey-type formula for q -Leonardo polynomials would be very useful.

The field of q -calculus, alternatively referred to as quantum calculus or the calculus of finite differences, holds a pivotal role in contemporary mathematical research owing to its intricate connections with both discrete and continuous mathematical frameworks. This mathematical discipline extends conventional notions from classical calculus by incorporating the q -parameter, thereby offering a novel approach to investigate the characteristics of functions, operators, numbers, and polynomials.

$$[n]_q = \frac{1 - q^n}{1 - q},$$

with $[0]_q = 0$. The q -factorial is defined by

$$[n]! := \begin{cases} 1 & \text{if } n = 0 \\ [n] \cdot [n-1] \cdots [1] & \text{if } n = 1, 2, \dots \end{cases}$$

for $n \in \mathbb{N}$. The q -binomial coefficients are defined by for $k \geq 0$ and $k \leq n$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[n-k]! [k]!},$$

where $\begin{bmatrix} n \\ 0 \end{bmatrix} = 1$ and $\begin{bmatrix} n \\ k \end{bmatrix} = 0$ for $n < k$ [13].

The q -analogues of Fibonacci and Lucas polynomials have been studied by many mathematicians. The q -Fibonacci polynomials are defined by the following recurrence relation for $n \geq 2$

$$\mathbf{F}_n(x, q) = x\mathbf{F}_{n-1}(x, q) + q^{n-2}\mathbf{F}_{n-2}(x, q),$$

with initial conditions $\mathbf{F}_0(x, q) = 0, \mathbf{F}_1(x, q) = 1$. The explicit formula of the q -Fibonacci polynomials is

$$\mathbf{F}_n(x, q) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} q^{k^2} \begin{bmatrix} n-k-1 \\ k \end{bmatrix} x^{n-2k-1}$$

for $n > 0$ [7].

The q -Lucas polynomials of the first kind are defined by the following recurrence relation for $n \geq 2$

$$\mathbf{L}_n(x, q) = x\mathbf{L}_{n-1}(x, q) + q^{n-2}\mathbf{L}_{n-2}(x, q),$$

with initial conditions $\mathbf{L}_0(x, q) = 2, \mathbf{L}_1(x, q) = x$. The explicit formula of the q -Lucas polynomials of the first kind is

$$\mathbf{L}_n(x, q) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} q^{2\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix} \left(1 + \frac{[k]_q}{[n-k]_q} \right) x^{n-2k}$$

for $n > 0$ [6].

The q -Lucas polynomials of the second kinds are defined by the following recurrence relation

$$\mathbb{L}_n(x, q) = x\sqrt{q}^{n-1}\mathbb{L}_{n-1}\left(\frac{x}{\sqrt{q}}, q\right) + q^{n-1}\mathbb{L}_{n-2}\left(\frac{x}{\sqrt{q}}, q\right),$$

with initial conditions $\mathbb{L}_0(x, q) = 2, \mathbb{L}_1(x, q) = x$. The explicit formula of the q -Lucas polynomials of the second kind is

$$\mathbb{L}_n(x, q) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix} \left(1 + q^{n-2k} \frac{[k]_q}{[n-k]_q} \right) x^{n-2k}$$

for $n > 0$.

2 Main results

In this section, we introduced the q -analogues of Leonardo polynomials called the q -Leonardo Pisano polynomials. Also, we define q -Leonardo Lucas polynomials of the first kind and q -Leonardo Lucas polynomials of the second kind.

Definition 2.1. *The q -Leonardo Pisano polynomials are defined by*

$$\mathcal{L}_e(x, q) = x\mathcal{L}_{e_{n-1}}(x, q) + q^{n-1}\mathcal{L}_{e_{n-2}}(x, q) + q(x + q^{n-1} - 1), \quad n \geq 2 \quad (1)$$

with initial conditions $\mathcal{L}_{e_0}(x, q) = 2 - q$ and $\mathcal{L}_{e_1}(x, q) = 2x - q$.

Table 1. Special cases of the q -Leonardo Pisano polynomials

x	q	$\mathcal{L}_{e_n}(x, q)$	q -Leonardo Pisano polynomials
x	1^-	$Le_n(x)$	Leonardo Pisano polynomials
1	q	$\mathcal{L}_{e_n}(q)$	q -Leonardo Pisano numbers
1	1^-	Le_n	Leonardo numbers
$2x$	q	$\mathcal{L}_{p_n}(x, q)$	q -Leonardo Pell polynomials
$2x$	1^-	$Lp_n(x)$	Leonardo Pell Polynomials
2	q	$\mathcal{L}_{p_n}(q)$	q -Leonardo Pell numbers
2	1^-	Lp_n	Leonardo Pell numbers

Definition 2.2. The q -Leonardo Lucas polynomials of the first kind are defined by

$$\mathcal{L}_{u_n}(x, q) = x\mathcal{L}_{u_{n-1}}(x, q) + q^{n-2}\mathcal{L}_{u_{n-2}}(x, q) - q(x + q^{n-2} - 1), \quad n \geq 2 \quad (2)$$

with initial conditions $\mathcal{L}_{u_0}(x, q) = 4 + q$ and $\mathcal{L}_{u_1}(x, q) = 2x + q$.

Table 2. Special cases of the q -Leonardo Lucas polynomials of the first kind

x	q	$\mathcal{L}_{u_n}(x, q)$	q -Leonardo Lucas polynomials of the first kind
x	1^-	$\mathcal{L}_{u_n}(x)$	Leonardo Lucas polynomials
1	q	$\mathcal{L}_{u_n}(q)$	q -Leonardo Lucas numbers of the first kind
1	1^-	Lu_n	Leonardo Lucas numbers
$2x$	q	$\mathcal{L}_{p_n}(2x, q)$	q -Leonardo Pell-Lucas polynomials
$2x$	1^-	$\mathcal{L}_{p_n}(2x)$	Leonardo Pell-Lucas polynomials
2	q	$\mathcal{L}_{p_n}(2, q)$	q -Leonardo Pell-Lucas numbers
2	1^-	Lp_n	Leonardo Pell-Lucas numbers

Definition 2.3. The q -Leonardo Lucas polynomials of the second kind are defined by

$$\mathcal{L}_{u_n}(x, q) = x\sqrt{q^{n-1}}\mathcal{L}_{u_{n-1}}\left(\frac{x}{\sqrt{q}}, q\right) + q^{n-1}\mathcal{L}_{u_{n-2}}\left(\frac{x}{q}, q\right) - q\left(x\sqrt{q^{n-1}} + q^{n-1} - 1\right), \quad (3)$$

for $n \geq 2$ with initial conditions $\mathcal{L}_{u_0}(x, q) = 4 + q$ and $\mathcal{L}_{u_1}(x, q) = 2x + q$.

It is clear that for $q \rightarrow 1^-$ $\mathcal{L}_{u_n}(x, q) = \mathcal{L}_{u_n}(x, q)$.

Table 3. Special cases of the q -Leonardo Lucas polynomials of the second kind

x	q	$\mathcal{L}_{u_n}(x, q)$	q -Leonardo Lucas polynomials of the second kind
1	q	$\mathcal{L}_{u_n}(q)$	q -Leonardo Lucas numbers of the second kind
1	1^-	Lu_n	Leonardo Lucas numbers
2	1^-	Lp_n	Leonardo Pell-Lucas numbers

3 Some properties of q -Leonardo polynomials

In this section, we give several properties for q -Leonardo Pisano polynomials, the q -Leonardo Lucas polynomials of the first kind and the q -Leonardo Lucas polynomials of the second kind. And we give the relations between these polynomials and the special cases of these relations.

3.1 Relationship between the q -Leonardo polynomials and q -polynomials

Theorem 3.1. *The relationship between the q -Fibonacci polynomials and the q -Leonardo Pisano polynomials is as follows.*

$$\mathcal{L}_n(x, q) = 2\mathbf{F}_{n+1}(x, q) - q. \quad (4)$$

Proof. Let us use the induction method on n for the proof. For $n = 0$ and $n = 1$, we have, respectively

$$\begin{aligned} \mathcal{L}_0(x, q) &= 2 - q = 2 \cdot 1 - q = 2\mathbf{F}_1(x, q) - q, \\ \mathcal{L}_1(x, q) &= 2x - q = 2 \cdot x - q = 2\mathbf{F}_2(x, q) - q. \end{aligned}$$

Thus, Equation (4) is true. Suppose that Equation (4) holds true for $n = k - 1$ and $n = k$. Then, by using Equation (1), we get

$$\begin{aligned} \mathcal{L}_{k+1}(x, q) &= x\mathcal{L}_k(x, q) + q^k\mathcal{L}_{k-1}(x, q) + q(x + q^k - 1) \\ &= x(2\mathbf{F}_{k+1}(x, q) - q) + q^k(2\mathbf{F}_k(x, q) - q) + qx + q^{k+1} - q \\ &= 2(x\mathbf{F}_{k+1}(x, q) + q^k\mathbf{F}_k(x, q)) - q \\ &= 2\mathbf{F}_{k+2}(x, q) - q \end{aligned}$$

which completes the proof of the assertion (4) by the principle of the mathematical induction on n . \square

If we choose $x = 1$, $q \rightarrow 1^-$ and both $x = 1$ and $q \rightarrow 1^-$ in (Eq. 4), then we obtain

$$\begin{aligned} \mathcal{L}_n(q) &= 2\mathbf{F}_{n+1}(q) - q, \\ \mathcal{L}_n(x) &= 2f_{n+1}(x) - 1, \\ Le_n &= 2F_{n+1} - 1. \end{aligned}$$

Corollary 3.1. *The explicit formula of the q -Leonardo Pisano polynomials is*

$$\mathcal{L}_n(x, q) = 2 \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} q^{k^2} \begin{bmatrix} n - k \\ k \end{bmatrix} x^{n-2k} - q$$

for $n > 0$.

Proof. Using Equation (4) and explicit formula for the q -Fibonacci polynomials, we may write

$$\begin{aligned} \mathcal{L}_n(x, q) &= 2\mathbf{F}_{n+1}(x, q) - q \\ &= 2 \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} q^{k^2} \begin{bmatrix} n - k \\ k \end{bmatrix} x^{n-2k} - q. \end{aligned} \quad \square$$

Theorem 3.2. *The relationship between the q -Lucas polynomials of the first kind and q -Leonardo Lucas polynomials is*

$$\mathcal{L}u_n(x, q) = 2\mathbf{L}_n(x, q) + q. \quad (5)$$

Proof. Let us use the induction method on n for the proof. For $n = 0$ and $n = 1$, we have

$$\begin{aligned} \mathcal{L}u_0(x, q) &= 4 + q = 2 \cdot 2 + q = 2\mathbf{L}_0(x, q) + q, \\ \mathcal{L}u_1(x, q) &= 2x + q = 2 \cdot x + q = 2\mathbf{L}_1(x, q) + q. \end{aligned}$$

Thus Equation (5) is true. Suppose that Equation in (5) holds true for $n = k$ and $n = k + 1$. Then, by using Equation (2), we get

$$\begin{aligned} \mathcal{L}u_{k+2}(x, q) &= x\mathcal{L}u_{k+1}(x, q) + q^k\mathcal{L}u_k(x, q) - q(x + q^k - 1) \\ &= x(2\mathbf{L}_{k+1}(x, q) + q) + q^k(2\mathbf{L}_k(x, q) + q) - qx - q^{k+1} + q \\ &= 2(x\mathbf{L}_{k+1}(x, q) + q^k\mathbf{L}_k(x, q)) + q \\ &= 2\mathbf{L}_{k+2}(x, q) + q. \end{aligned} \quad \square$$

Theorem 3.3. *The relationship between the q -Lucas polynomials of the second kind and q -Leonardo Lucas polynomials is*

$$\mathcal{L}u_n(x, q) = 2\mathbb{L}_n(x, q) + q. \quad (6)$$

Proof. It is proved similarly to Theorem 3.2 using the principle of the mathematical induction on n . □

If we choose $x = 1$, $q \rightarrow 1^-$ and both $x = 1$ and $q \rightarrow 1^-$ in (5) and (6), we obtain

$$\begin{aligned} \mathcal{L}u_n(q) &= 2\mathbf{L}_n(q) + q, \\ \mathcal{L}u_n(q) &= 2\mathbb{L}_n(q) + q, \\ \mathcal{L}u_n(x) &= 2\ell_n(x) + 1, \\ \mathcal{L}u_n &= 2L_n + 1. \end{aligned}$$

Corollary 3.2. *Explicit formulas of the q -Leonardo Lucas polynomials of the first kind are*

$$\mathcal{L}u_n(x, q) = 2 \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} q^{2\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix} \left(1 + \frac{[k]}{[n-k]} \right) x^{n-2k} + q \quad (7)$$

and

$$\mathcal{L}u_n(x, q) = 2 \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix} \left(1 + q^{n-2k} \frac{[k]}{[n-k]} \right) x^{n-2k} + q \quad (8)$$

for $n > 0$.

3.2 Negative indexed q -Leonardo polynomials

q -Leonardo Pisano polynomials can be extended backward to negative indices by rearranging the defining recurrence

$$\mathcal{L}_{e_{n-2}}(x, q) = q^{1-n} \mathcal{L}_{e_n}(x, q) - xq^{1-n} \mathcal{L}_{e_{n-1}}(x, q) - q^{2-n} (x + q^{n-1} - 1). \quad (9)$$

Theorem 3.4. *Negative indexed q -Leonardo Pisano polynomials are $\mathcal{L}_{e_{-1}}(x, q) = -q$ and*

$$\mathcal{L}_{e_{-n}}(x, q) = (-1)^n \left(q^{n-1} \mathcal{L}_{e_{n-2}} \left(x\sqrt{q^{n-1}}, q \right) + q^n \right) - q \quad (10)$$

for $n \geq 2$.

Proof. Using the recurrence relation (9) for $n = 1$, we obtain

$$\begin{aligned} \mathcal{L}_{e_{-1}}(x, q) &= \mathcal{L}_{e_1}(x, q) - x\mathcal{L}_{e_0}(x, q) - q(x + q^0 - 1) \\ &= 2x - q - x(2 - q) - q(x + 1 - 1) \\ &= -q. \end{aligned}$$

We use the induction method on n . For $n = 0$, using (9) we obtain

$$\mathcal{L}_{e_{-2}}(x, q) = \mathcal{L}_{e_0}(x, q) - x\mathcal{L}_{e_{-1}}(x, q) - q(x + q^{-1} - 1) = q.$$

and the right-hand side of (10) becomes

$$(-1)^2 \left(q^{2-1} \mathcal{L}_{e_{2-2}} \left(x\sqrt{q^{2-1}}, q \right) + q^2 \right) - q = q(2 - q) + q^2 - q = q = \mathcal{L}_{e_{-2}}(x, q).$$

For $n = -1$ in Eq. (9),

$$\begin{aligned} \mathcal{L}_{e_{-3}}(x, q) &= q^2 \mathcal{L}_{e_{-1}}(x, q) - xq^2 \mathcal{L}_{e_{-2}}(x, q) - q^3(x + q^{-2} - 1) \\ &= q^2(-q) - xq^2(q) - q^3(x + q^{-2} - 1) \\ &= -2q^2x - q \end{aligned}$$

and using (10) we obtain

$$\begin{aligned} (-1)^3 \left(q^{3-1} \mathcal{L}_{e_{3-2}} \left(x\sqrt{q^{3-1}}, q \right) + q^3 \right) - q &= (-1)^3 (q^2(2(xq) - q) + q^3) - q \\ &= -2q^2x - q \\ &= \mathcal{L}_{e_{-3}}(x, q). \end{aligned}$$

Suppose that (10) holds for $n = t$ and $n = t + 1$ then

$$\mathcal{L}_{e_{-t}}(x, q) = (-1)^t \left(q^{t-1} \mathcal{L}_{e_{n-2}} \left(x\sqrt{q^{t-1}}, q \right) + q^t \right) - q, \quad (11)$$

$$\mathcal{L}_{e_{-(t+1)}}(x, q) = (-1)^{t+1} \left(q^t \mathcal{L}_{e_{t-1}} \left(x\sqrt{q^t}, q \right) + q^{t+1} \right) - q. \quad (12)$$

If we use (9), we get

$$\begin{aligned}
 \mathcal{L}_{e-(t+2)}(x, q) &= q^{1+t} \mathcal{L}_{e-t}(x, q) - xq^{1+t} \mathcal{L}_{e-(t+1)}(x, q) - q^{2+t} (x + q^{-(t+1)} - 1) \\
 &= q^{1+t} \left[(-1)^t \left(q^{t-1} \mathcal{L}_{e-t-2} \left(x\sqrt{q^{t-1}}, q \right) + q^t \right) - q \right] \\
 &\quad - xq^{1+t} \left[(-1)^{t+1} \left(q^t \mathcal{L}_{e-t-1} \left(x\sqrt{q^t}, q \right) + q^{t+1} \right) - q \right] \\
 &\quad - q^{2+t} (x + q^{-t-1} - 1) \\
 &= (-1)^t q^{1+t} \left(\underbrace{q^{t-1} \mathcal{L}_{e-t-2} \left(x\sqrt{q^{t-1}}, q \right) + xq^t \mathcal{L}_{e-t-1} \left(x\sqrt{q^t}, q \right)}_A \right) \\
 &\quad + (-1)^t q^{1+2t} + (-1)^t xq^{2+2t} - q
 \end{aligned}$$

where

$$\begin{aligned}
 A &= (-1)^t q^{1+t} \left(2 \sum_{k=1}^{\lfloor \frac{t-1}{2} \rfloor} q^{k^2} x^{t-2k} \begin{bmatrix} t-1-k \\ k-1 \end{bmatrix} q^{\frac{t(t-2k+1)-2k}{2}} - q^n \right) \\
 &\quad + (-1)^t q^{1+t} \left(2 \sum_{k=0}^{\lfloor \frac{t-1}{2} \rfloor} q^{k^2} x^{t-2k} \begin{bmatrix} t-1-k \\ k \end{bmatrix} q^{\frac{t(t-2k+1)-2k}{2}} q^k - xq^{t+1} \right) \\
 &\quad - (-1)^t q^{1+t} (q^t + xq^{t+1}) \\
 &= (-1)^t q^{1+t} \left(2 \sum_{k=1}^{\lfloor \frac{t}{2} \rfloor} q^{k^2} x^{t-2k} q^{\frac{t(t-2k+1)-2k}{2}} \begin{bmatrix} t-k \\ k \end{bmatrix} \right) - (-1)^t q^{1+t} (q^t + xq^{t+1}) \\
 &= (-1)^t q^{1+t} \left(\mathcal{L}_e \left(x\sqrt{q^{t+1}}, q \right) + q - q^t - xq^{t+1} \right).
 \end{aligned}$$

Hence

$$\begin{aligned}
 \mathcal{L}_{e-(t+2)}(x, q) &= q^{1+t} (-1)^t \left[\mathcal{L}_e \left(x\sqrt{q^{t+1}}, q \right) + q - q^n - xq^{t+1} \right] \\
 &\quad + (-1)^t q^{1+2t} + (-1)^t xq^{2+2t} - q \\
 &= (-1)^{t+2} \left(q^{t+1} \mathcal{L}_e \left(x\sqrt{q^{t+1}}, q \right) + q^{t+2} \right) - q. \quad \square
 \end{aligned}$$

If we choose $q \rightarrow 1^-$ and both $x = 1$ and $q \rightarrow 1^-$ in (10), then

$$\begin{aligned}
 \mathcal{L}_{e-n}(x) &= (-1)^n (\mathcal{L}_{e-n-2}(x) + 1) - 1, \\
 Le_{-n} &= (-1)^n (Le_{n-2} + 1) - 1.
 \end{aligned}$$

The q -Leonardo Lucas polynomials of the first kind can be extended backward to negative indices by rearranging the defining recurrence

$$\mathcal{L}_{u_{n-2}}(x, q) = q^{2-n} \mathcal{L}_n(x, q) - xq^{2-n} \mathcal{L}_{n-1}(x, q) + q^{3-n} (x + q^{n-2} - 1). \quad (13)$$

Theorem 3.5. *Negative indexed q -Leonardo Lucas polynomials of the first kind are*

$$\mathcal{L}_{u_{-n}}(x, q) = (-1)^n \sqrt{q^n} \mathcal{L}_{u_n}(x\sqrt{q^n}, q) + (-1)^{n+1} \sqrt{q^{n+2}} + q. \quad (14)$$

Proof. We use the induction methods on n . For $n = 1$ using (13), we obtain

$$\begin{aligned}\mathcal{L}_{u_{-1}}(x, q) &= q^1 \mathcal{L}_{u_1}(x, q) - xq^1 \mathcal{L}_{u_0}(x, q) + q^2 (x + q^{-1} - 1) \\ &= q(2x + q) - xq(4 + q) + q^2 (x + q^{-1} - 1) \\ &= q - 2qx\end{aligned}$$

and the right-hand side of (14) for $n = 1$, then

$$(-1)^1 \sqrt{q^1} \mathcal{L}_{u_1}(x\sqrt{q^1}, q) + (-1)^{1+1} \sqrt{q^{1+2}} + q = -2xq + q = \mathcal{L}_{u_{-1}}(x, q).$$

For $n = 0$, we have

$$\begin{aligned}\mathcal{L}_{u_{-2}}(x, q) &= q^2 \mathcal{L}_{u_0}(x, q) - xq^2 \mathcal{L}_{u_{-1}}(x, q) + q^3 (x + q^{-2} - 1) \\ &= q^2(4 + q) - xq^2(q - 2qx) + q^3(x + q^{-2} - 1) \\ &= 2q^3x^2 + 4q^2 + q.\end{aligned}$$

For $n = 2$ and using RHS (14), we obtain

$$(-1)^2 \sqrt{q^2} \mathcal{L}_{u_2}(x\sqrt{q^2}, q) + (-1)^3 \sqrt{q^4} + q = 2x^2q^3 + 4q^2 + q = \mathcal{L}_{u_{-2}}(x, q).$$

Suppose that Equation (16) holds for $n = t$ and $n = t + 1$, then

$$\begin{aligned}\mathcal{L}_{u_{-t}}(x, q) &= (-1)^t \sqrt{q^t} \mathcal{L}_{u_t}(x\sqrt{q^t}, q) + (-1)^{t+1} \sqrt{q^{t+2}} + q, \\ \mathcal{L}_{u_{-(t+1)}}(x, q) &= (-1)^{t+1} \sqrt{q^{t+1}} \mathcal{L}_{u_{t+1}}(x\sqrt{q^{t+1}}, q) + (-1)^{t+2} \sqrt{q^{t+3}} + q.\end{aligned}$$

If we use (13), we obtain

$$\begin{aligned}\mathcal{L}_{u_{-(t+2)}}(x, q) &= q^{2+t} \mathcal{L}_{u_{-t}}(x, q) - xq^{2+t} \mathcal{L}_{u_{-(t+1)}}(x, q) + q^{3+t} (x + q^{-t-2} - 1) \\ &= q^{2+t} (-1)^t \left\{ \sqrt{q^t} \mathcal{L}_{u_t}(x\sqrt{q^{t+1}}, q) - \sqrt{q^t} + q \right\} + q^{3+t} \\ &\quad + q^{2+t} (-1)^t \left\{ x\sqrt{q^t} \sqrt{q} \mathcal{L}_{u_{t+1}}(x\sqrt{q^{t+1}}, q) - x\sqrt{q^{t+1}} \sqrt{q} \right\} - xq^{t+3} \\ &\quad + xq^{t+3} + q - q^{3-t} \\ &= q^{2+t} (-1)^t \left\{ \sqrt{q^t} \mathcal{L}_{u_t}(x\sqrt{q^{t+1}}, q) + x\sqrt{q^t} \sqrt{q} \mathcal{L}_{u_{t+1}}(x\sqrt{q^{t+1}}, q) \right\} \\ &\quad - q^{2+t} (-1)^t \sqrt{q^t} \{-\sqrt{q} - xq\} + q \\ &= (-1)^{t+2} \sqrt{q^{t+2}} \mathcal{L}_{u_{t+2}}(x\sqrt{q^{t+2}}, q) + (-1)^{t+3} \sqrt{q^{t+4}} + q. \quad \square\end{aligned}$$

The q -Leonardo Lucas polynomials of the second kind can be extended backward to negative indices by rearranging the defining recurrence

$$\begin{aligned}\mathcal{L}_{u_{n-2}}(x, q) &= q^{1-n} \mathcal{L}_{u_n}(qx, q) - xq^{2-n} (\sqrt{q})^{n-1} \mathcal{L}_{u_{n-1}}(\sqrt{q}x, q) \\ &\quad + q^{2-n} (x\sqrt{q^{n+1}} + q^{n-1} - 1)\end{aligned} \tag{15}$$

Theorem 3.6. *Negative indexed q -Leonardo Lucas polynomials of the second kind are*

$$\mathcal{L}_{u_{-n}}(x, q) = (-1)^n \sqrt{q^n} \mathcal{L}_{u_n}(x\sqrt{q^n}, q) + (-1)^{n+1} \sqrt{q^{n+2}} + q. \tag{16}$$

Proof. We use the induction method on n . For $n = 1$ using (15), we obtain

$$\begin{aligned}\mathcal{L}_{u_{-1}}(x, q) &= q^0 \mathcal{L}_{u_1}(qx, q) - xq(\sqrt{q})^0 \mathcal{L}_{u_0}(\sqrt{q}x, q) + q(xq + q^0 - 1) \\ &= q - 2qx\end{aligned}$$

and the right-hand side (16) for $n = 1$, then becomes

$$\begin{aligned}(-1)^1 \sqrt{q} \mathcal{L}_{u_1}(x\sqrt{q}, q) + (-1)^2 \sqrt{q^3} + q &= (-1) \sqrt{q} (2x\sqrt{q} + q) + \sqrt{q^3} + q \\ &= q - 2qx \\ &= \mathcal{L}_{u_{-1}}(x, q)\end{aligned}$$

For $n = 2$, we get

$$\mathcal{L}_{u_{-2}}(x, q) = q \mathcal{L}_{u_0}(qx, q) - xq^2 \sqrt{q}^{-1} \mathcal{L}_{u_{-1}}(\sqrt{q}x, q) + q^2(x\sqrt{q} + q^{-1} - 1) = 2q^3x^2 + 5q$$

and using the right-hand side of (16), we obtain

$$(-1)^2 \sqrt{q^2} \mathcal{L}_{u_2}(x\sqrt{q^2}, q) + (-1)^3 \sqrt{q^4} + q = 2x^2q^3 + 5q = \mathcal{L}_{u_{-2}}(x, q).$$

Suppose that Equation (16), holds for $n = t$ and $n = t + 1$, then

$$\mathcal{L}_{u_{-t}}(qx, q) = (-1)^t \sqrt{q^t} \mathcal{L}_{u_t}(xq\sqrt{q^t}, q) + (-1)^{t+1} \sqrt{q^{t+2}} + q.$$

If we use (15), we obtain

$$\begin{aligned}\mathcal{L}_{u_{-(t+2)}}(x, q) &= q^{1+t} \mathcal{L}_{u_{-t}}(qx, q) - xq^{2+t} \sqrt{q}^{-(t+1)} \mathcal{L}_{u_{-(t+1)}}(x\sqrt{q}, q) + q^{t+2}(x\sqrt{q}^{-t+1} + q^{-t-1} - 1) \\ &= (-1)^t \sqrt{q^{t+2}} (xq\sqrt{q^t} \mathcal{L}_{u_{t+1}}(x\sqrt{q^{t+2}}, q) + q^t \mathcal{L}_{u_t}(xq\sqrt{q^t}, q) - xq^2 \sqrt{q^t} - q^{t+1}) + q \\ &= (-1)^{t+2} \sqrt{q^{t+2}} \mathcal{L}_{u_{t+2}}(xq\sqrt{q^t}, q) + (-1)^{t+3} \sqrt{q^{t+4}} + q. \quad \square\end{aligned}$$

If we choose $q \rightarrow 1^-$ and both $x = 1$ and $q \rightarrow 1^-$ in (14) and (16), then we obtain

$$\begin{aligned}\mathcal{L}_{u_{-n}}(x) &= (-1)^n (\mathcal{L}_{u_n}(x) - 1) + 1, \\ Lu_{-n} &= (-1)^n (Lu_n - 1) + 1.\end{aligned}$$

Interestingly, the negative index terms of q -Leonardo Lucas polynomials of the first and second kinds are expressed in terms of each other.

3.3 Some identities for q -Leonardo Pisano polynomials

Theorem 3.7. (Cassini's Identity) For $n \geq 2$, the following identity holds:-

$$\begin{aligned}\mathcal{L}_{e_{n+1}}(x, q) \mathcal{L}_{e_{n-1}}\left(\frac{x}{\sqrt{q}}, q\right) - \sqrt{q} \mathcal{L}_{e_n}(x, q) \mathcal{L}_{e_n}\left(\frac{x}{\sqrt{q}}, q\right) \\ + q \left(\mathcal{L}_{e_{n+1}}(x, q) - \sqrt{q} \mathcal{L}_{e_n}(x, q) - \sqrt{q} \mathcal{L}_{e_n}\left(\frac{x}{\sqrt{q}}, q\right) + \mathcal{L}_{e_{n-1}}\left(\frac{x}{\sqrt{q}}, q\right) \right) \\ = 4(-1)^n q^{\frac{n^2+1}{2}} - q^2(1 - \sqrt{q}).\end{aligned}\tag{17}$$

Proof. In [11], the author obtained Cassini formula for the q -Fibonacci polynomials as follows.

$$F_{n+2}(x, q) F_n\left(\frac{x}{\sqrt{q}}, q\right) - \sqrt{q} F_{n+1}(x, q) F_{n+1}\left(\frac{x}{\sqrt{q}}, q\right) = (-1)^{n+1} (\sqrt{q})^{n^2+1}.\tag{18}$$

Using the relation between q -Fibonacci polynomials and q -Leonardo Pisano polynomials in (18), we have

$$\left(\frac{\mathcal{L}_{n+1}(x,q)+q}{2}\right)\left(\frac{\mathcal{L}_{n-1}\left(\frac{x}{\sqrt{q}},q\right)+q}{2}\right) - \sqrt{q}\left(\frac{\mathcal{L}_n(x,q)+q}{2}\right)\left(\frac{\mathcal{L}_n\left(\frac{x}{\sqrt{q}},q\right)+q}{2}\right) = (-1)^{n+1}(\sqrt{q})^{n^2+1}.$$

Then we obtain

$$\begin{aligned} \mathcal{L}_{n+1}(x,q)\mathcal{L}_{n-1}\left(\frac{x}{\sqrt{q}},q\right) - \sqrt{q}\mathcal{L}_n(x,q)\mathcal{L}_n\left(\frac{x}{\sqrt{q}},q\right) + q\mathcal{L}_{n+1}(x,q) + q\mathcal{L}_{n-1}\left(\frac{x}{\sqrt{q}},q\right) \\ - q\sqrt{q}\mathcal{L}_n(x,q) - q\sqrt{q}\mathcal{L}_n\left(\frac{x}{\sqrt{q}},q\right) \\ = 4(-1)^{n+1}(\sqrt{q})^{n^2+1} + q^2(\sqrt{q}-1). \end{aligned} \quad \square$$

If we choose $q \rightarrow 1^-$ and both $x = 1$ and $q \rightarrow 1^-$ in the sequence in (17), then we obtain

$$\begin{aligned} \mathcal{L}_{n+1}(x)\mathcal{L}_{n-1}(x) - \mathcal{L}_n^2(x) + \mathcal{L}_{n+1}(x) - 2\mathcal{L}_n(x) + \mathcal{L}_{n-1}(x) = 4(-1)^{n+1}, \\ L_{n+1}^2 - L_n L_{n+2} = L_n - L_{n-1} + 4(-1)^{n+1}. \end{aligned}$$

Theorem 3.8. (Honsberger's Identity) For $n \geq 1$, the following identity holds:▮

$$\begin{aligned} 2\sqrt{q}^{-kn}\mathcal{L}_{n+k}(x,q) = \mathcal{L}_n(x,q)\mathcal{L}_k\left(\frac{x}{\sqrt{q}^n},q\right) + \sqrt{q}^{k+n-1}\mathcal{L}_{n-1}(x,q)\mathcal{L}_{k-1}\left(\frac{x}{\sqrt{q}^{n+1}},q\right) \\ + q\left(\mathcal{L}_n(x,q) + \mathcal{L}_k\left(\frac{x}{\sqrt{q}^n},q\right)\right) + \sqrt{q}^{k+n+1}\left(\mathcal{L}_{n-1}(x,q) + \mathcal{L}_{k-1}\left(\frac{x}{\sqrt{q}^{n+1}},q\right)\right) \\ + q^2\left(1 + \sqrt{q}^{k+n-3} - 2\sqrt{q}^{-kn-2}\right). \end{aligned} \quad (19)$$

Proof. In [11], the author obtained Honsberger's identity for the q -Fibonacci polynomials as follows

$$\sqrt{q}^{-kn}F_{n+k+1}(x,q) = F_{n+1}(x,q)F_{k+1}\left(\frac{x}{\sqrt{q}^n},q\right) + \sqrt{q}^{k+n-1}F_n(x,q)F_k\left(\frac{x}{\sqrt{q}^{n+1}},q\right). \quad (20)$$

Using the relation between q -Fibonacci polynomials and q -Leonardo Pisano polynomials in (20), we have

$$\begin{aligned} \sqrt{q}^{-kn}\left(\frac{\mathcal{L}_{n+k}(x,q)+q}{2}\right) = \left(\frac{\mathcal{L}_n(x,q)+q}{2}\right)\left(\frac{\mathcal{L}_k\left(\frac{x}{\sqrt{q}^n},q\right)+q}{2}\right) \\ + \sqrt{q}^{k+n-1}\left(\frac{\mathcal{L}_{n-1}(x,q)+q}{2}\right)\left(\frac{\mathcal{L}_{k-1}\left(\frac{x}{\sqrt{q}^{n+1}},q\right)+q}{2}\right). \end{aligned}$$

Then we obtain

$$\begin{aligned} 2\sqrt{q}^{-kn}\mathcal{L}_{n+k}(x,q) = \mathcal{L}_n(x,q)\mathcal{L}_k\left(\frac{x}{\sqrt{q}^n},q\right) + \sqrt{q}^{k+n-1}\mathcal{L}_{n-1}(x,q)\mathcal{L}_{k-1}\left(\frac{x}{\sqrt{q}^{n+1}},q\right) \\ + q\left(\mathcal{L}_n(x,q) + \mathcal{L}_k\left(\frac{x}{\sqrt{q}^n},q\right)\right) + \sqrt{q}^{k+n+1}\left(\mathcal{L}_{n-1}(x,q) + \mathcal{L}_{k-1}\left(\frac{x}{\sqrt{q}^{n+1}},q\right)\right) \\ + q^2\left(1 + \sqrt{q}^{k+n-3} - 2\sqrt{q}^{-kn-2}\right). \end{aligned} \quad \square$$

If we choose $q \rightarrow 1^-$ and both $k = n$ and $q \rightarrow 1^-$ in Equation (19), then we obtain:

$$\begin{aligned} 2\mathcal{L}_{n+k}(x) = \mathcal{L}_n(x)\mathcal{L}_k(x) + \mathcal{L}_{n-1}(x)\mathcal{L}_{k-1}(x) + \mathcal{L}_n(x) + \mathcal{L}_k(x) + \mathcal{L}_{n-1}(x) + \mathcal{L}_{k-1}(x), \\ 2L_{n+k} = L_n L_k + L_{n-1} L_{k-1} + L_{n+1} + L_{k+1} - 2. \end{aligned}$$

3.4 Relations between q -Leonardo polynomials

Theorem 3.9. For $n \geq 2$, the following equality gives the relationships between the q -Leonardo Pisano polynomials and q -Leonardo Lucas polynomials of the first kind:

$$\sqrt{q}^n \mathcal{L}u_n(x, q) = \mathcal{L}e_n(x\sqrt{q}, q) + q^{n-1} \mathcal{L}e_{n-2}\left(\frac{x}{\sqrt{q}}, q\right) + q(1 + q^{n-1} + \sqrt{q}^n). \quad (21)$$

Proof. Using the explicit formula for q -Leonardo Lucas polynomials of the first kind, we may write

$$\begin{aligned} \mathcal{L}u_n(x, q) &= q + 2 \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} q^{2\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix} \left(1 + \frac{[k]_q}{[n-k]_q}\right) x^{n-2k} \\ &= q + \sqrt{q^{-n}} 2 \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix} (x\sqrt{q})^{n-2k} + 2 \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} q^{k^2} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} x^{n-2k} q^{-k} \\ &= q + \sqrt{q^{-n}} \mathcal{L}e_n(x\sqrt{q}, q) + \sqrt{q}^{1-n} + \sqrt{q}^{n-2} \left(\mathcal{L}e_{n-2}\left(\frac{x}{\sqrt{q}}, q\right) + q\right). \end{aligned}$$

Then we have

$$\sqrt{q}^n \mathcal{L}u_n(x, q) = \mathcal{L}e_n(x\sqrt{q}, q) + q^{n-1} \mathcal{L}e_{n-2}\left(\frac{x}{\sqrt{q}}, q\right) + q(1 + q^{n-1} + \sqrt{q}^n). \quad \square$$

If we choose $q \rightarrow 1^-$ and both $x = 1$ and $q \rightarrow 1^-$ in the sequence in (21), we obtain:

$$\begin{aligned} \mathcal{L}u_n(x) &= \mathcal{L}e_n(x) + \mathcal{L}e_{n-2}(x) + 3, \\ Lu_n &= Le_n + Le_{n-2} + 3. \end{aligned}$$

Theorem 3.10. For $n \geq 0$, the following identity holds:

$$(x^2 + 4) \mathcal{L}e_n(x, q) = x \mathcal{L}u_{n+1}(x, q) + 2\sqrt{q}^n \mathcal{L}u_n\left(\frac{x}{\sqrt{q}}, q\right) - q(x^2 + x + 2\sqrt{q}^n + 4) \quad (22)$$

If we choose $q \rightarrow 1^-$ and both $x = 1$ and $q \rightarrow 1^-$ in the sequence in (22), we have

$$(x^2 + 4) \mathcal{L}e_n(x) = x \mathcal{L}u_{n+1}(x) + 2 \mathcal{L}u_n(x) - (x^2 + x + 6)$$

and

$$5Le_n = Lu_{n+1} + 2Lu_n - 8.$$

Theorem 3.11. For $n \geq 2$, the following equality gives the relationships between the q -Leonardo Pisano polynomials and q -Leonardo Lucas polynomials of the second kind

$$\mathcal{L}u_n(x, q) = \mathcal{L}e_n(x, q) + q^{n-1} \mathcal{L}e_{n-2}(x, q) + q(q^{n-1} + 2). \quad (23)$$

Proof. Using the explicit formula for q -Leonardo Lucas polynomials of the second kind, we may write

$$\begin{aligned} \mathcal{L}u_n(x, q) &= q + 2 \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix} \left(1 + q^{n-2k} \frac{[k]_q}{[n-k]_q}\right) x^{n-2k} \\ &= \mathcal{L}e_n(x, q) + 2q + 2q^{n-1} \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} q^{k^2} \begin{bmatrix} n-k-2 \\ k \end{bmatrix} x^{n-2k-2} \\ &= \mathcal{L}e_n(x, q) + q^{n-1} \mathcal{L}e_{n-2}(x, q) + q(q^{n-1} + 2). \quad \square \end{aligned}$$

If we choose $q \rightarrow 1^-$ and both $x = 1$ and $q \rightarrow 1^-$ in (23), we have

$$\begin{aligned}\mathcal{L}u_n(x) &= \mathcal{L}e_n(x) + \mathcal{L}e_{n-2}(x) + 3, \\ Lu_n &= Le_n + Le_{n-2} + 3.\end{aligned}$$

Theorem 3.12. For $n \geq 0$, the following identity holds:

$$(x^2 + 2q^{n+1} + 2q^n) \mathcal{L}e_n(x, q) = \mathcal{L}u_{n+2}(x, q) + q^n \mathcal{L}u_n(x, q) - q^{n+1}(3 + 2q) - q(1 + x^2). \quad (24)$$

Proof. If we use the relationships between the q -Leonardo Pisano polynomials and q -Leonardo Lucas polynomials of the second kind in Equation (23), then we have

$$\begin{aligned}\mathcal{L}u_{n+2}(x, q) &= \mathcal{L}e_{n+2}(x, q) + q^{n+1} \mathcal{L}e_n(x, q) + q^{n+2} + 2q \\ &= x \mathcal{L}e_{n+1}(x, q) + 2q^{n+1} \mathcal{L}e_n(x, q) + q(x + 2q^{n+1} + 1)\end{aligned}$$

and

$$\begin{aligned}q^n \mathcal{L}u_n(x, q) &= q^n (\mathcal{L}e_n(x, q) + q^{n-1} \mathcal{L}e_{n-2}(x, q) + q^n + 2q) \\ &= 2q^n \mathcal{L}e_n(x, q) - xq^n \mathcal{L}e_{n-1}(x, q) + q^{n+1}(3 - x)\end{aligned}$$

then

$$\begin{aligned}\mathcal{L}u_{n+2}(x, q) + q^n \mathcal{L}u_n(x, q) &= (2q^{n+1} + 2q^n) \mathcal{L}e_n(x, q) + x(\mathcal{L}e_{n+1}(x, q) - q^n \mathcal{L}e_{n-1}(x, q)) \\ &\quad + q(x + 2q^{n+1} + 1) + q^{n+1}(3 - x) \\ &= (2q^{n+1} + 2q^n) \mathcal{L}e_n(x, q) + x(x \mathcal{L}e_n(x, q) + q(x + q^n - 1)) \\ &\quad + q(x + 2q^{n+1} + 1) + q^{n+1}(3 - x) \\ &= (2q^{n+1} + 2q^n + x^2) \mathcal{L}e_n(x, q) + q(x^2 + 1) + q^{n+1}(2q + 3)\end{aligned}$$

Finally we obtain

$$(x^2 + 2q^{n+1} + 2q^n) \mathcal{L}e_n(x, q) = \mathcal{L}u_{n+2}(x, q) + q^n \mathcal{L}u_n(x, q) - q^{n+1}(3 + 2q) - q(1 + x^2). \quad \square$$

If we choose $x = 1$, $q \rightarrow 1^-$ and both $x = 1$ and $q \rightarrow 1^-$ in the sequence in (24), we obtain:

$$\begin{aligned}(x^2 + 2q^{n+1} + 2q^n) \mathcal{L}e_n(q) &= \mathcal{L}u_{n+2}(q) + q^n \mathcal{L}u_n(q) - q(3q^n + 2q^{n+1} + 2), \\ (x^2 + 4) \mathcal{L}e_n(x) &= \mathcal{L}u_{n+2}(x) + \mathcal{L}u_n(x) - (x^2 + 6), \\ 5Le_n &= Lu_{n+2} + Lu_n - 7.\end{aligned}$$

3.5 Generating functions of q -Leonardo polynomials

We give the generating functions of q -Leonardo Pisano polynomials and q -Leonardo Lucas polynomials of the first and second kind with an operator η_z , where η_z is an operator on functions of z defined by $\eta_z(f(z)) = f(qz)$.

Corollary 3.3. The generating function of q -Leonardo Pisano polynomials is

$$G(z) = \frac{q}{z-1} + \frac{2}{1-xz-z^2\eta_z} \mathbf{1},$$

where $\eta_z(f(z)) = f(qz)$ for any function f .

Proof. Using Equation (4) and the generating functions of the q -Fibonacci polynomials in [4], we have

$$\begin{aligned}
 G(z) &= \sum_{n=0}^{\infty} \mathcal{L}_n(x, q) z^n \\
 &= \sum_{n=0}^{\infty} (2\mathbf{F}_{n+1}(x, q) - q) z^n \\
 &= \frac{2}{z} \sum_{n=1}^{\infty} \mathbf{F}_n(x, q) z^n - q \sum_{n=0}^{\infty} z^n \\
 &= \frac{2}{z} \frac{z}{1 - xz - z^2\eta_z} \mathbf{1} - \frac{q}{1 - z} \\
 &= \frac{q}{z - 1} + \frac{2}{1 - xz - z^2\eta_z} \mathbf{1}. \quad \square
 \end{aligned}$$

Corollary 3.4. *The generating function of the q -Leonardo Lucas polynomials of the first kind is*

$$H(z) = \frac{q}{1 - z} + \frac{4 - 2xz}{1 - xz - z^2\eta_z} \mathbf{1},$$

where $\eta_z(f(z)) = f(qz)$ for any function f .

Proof. Using Equation (5) and the generating functions of the q -Lucas polynomials of the first kind in [11] then the desired is obtained. \square

Corollary 3.5. *The generating function of the q -Leonardo Lucas polynomials of the second kind is*

$$K(z) = 4 - 2q + \frac{q}{1 - z} + \frac{2(2 - xz)}{1 - xz - z^2\eta_z} \mathbf{1},$$

where $\eta_z(f(z)) = f(qz)$ for any function f .

Proof. Using Equation (23) and the generating functions of the q -Leonardo Pisano polynomials, we have

$$\begin{aligned}
 K(z) &= \sum_{k=0}^{\infty} \mathcal{L}u_k(x, q) z^k \\
 &= \mathcal{L}u_0(x, q) + \sum_{k=1}^{\infty} \mathcal{L}u_k(x, q) z^k \\
 &= 4 + q + \sum_{k=1}^{\infty} (2\mathcal{L}e_k(x, q) - x\mathcal{L}e_{k-1}(x, q) + q(3 - x)) z^k \\
 &= 4 - 2q + qx + (2 - xz) \sum_{k=0}^{\infty} \mathcal{L}e_k(x, q) z^k + q(3 - x) \sum_{k=0}^{\infty} z^k \\
 &= 4 - 2q + qx + (2 - xz)G(z) + q(3 - x) \frac{1}{1 - z} \\
 &= 4 - 2q + \frac{q}{1 - z} + \frac{2(2 - xz)}{1 - xz - z^2\eta_z} \mathbf{1}. \quad \square
 \end{aligned}$$

3.6 Summation formulas for q -Leonardo polynomials

In this section, we give several sum properties for q -Leonardo Pisano polynomials, q -Leonardo Lucas polynomials of the first kind, q -Leonardo Lucas polynomials of the second kind. Also we achieve sum properties of the relationships between q -Leonardo polynomials.

Theorem 3.13. *The following identity holds:*

$$\sum_{k=0}^n (x + q^{k+1} - 1) \mathcal{L}_{e_k}(x, q) = \mathcal{L}_{e_{n+1}}(x, q) + q^{n+1} \mathcal{L}_{e_n}(x, q) - q([n+1] + (n+1)(x-1) - 2) - 2 \quad (25)$$

where $[n]$ the q -integer of n .

Proof. Using the recurrence relation of q -Leonardo Pisano polynomials in (1), we have

$$x\mathcal{L}_{e_{n-1}}(x, q) + q^{n-1}\mathcal{L}_{e_{n-2}}(x, q) - \mathcal{L}_{e_n}(x, q) = -q(x + q^{n-1} - 1)$$

and

$$\begin{aligned} x\mathcal{L}_{e_1}(x, q) + q^1\mathcal{L}_{e_0}(x, q) - \mathcal{L}_{e_2}(x, q) &= q(1-x) - q^2 \\ x\mathcal{L}_{e_2}(x, q) + q^2\mathcal{L}_{e_1}(x, q) - \mathcal{L}_{e_3}(x, q) &= q(1-x) - q^3 \\ &\vdots \\ x\mathcal{L}_{e_n}(x, q) + q^n\mathcal{L}_{e_{n-1}}(x, q) - \mathcal{L}_{e_{n+1}}(x, q) &= q(1-x) - q^{n+1}. \end{aligned}$$

Then we obtain

$$\begin{aligned} \sum_{k=0}^n (x + q^{k+1} - 1) \mathcal{L}_{e_k}(x, q) &= \mathcal{L}_{e_{n+1}}(x, q) + q^{n+1} \mathcal{L}_{e_n}(x, q) - \mathcal{L}_{e_0}(x, q) + x\mathcal{L}_{e_0}(x, q) \\ &\quad - \mathcal{L}_{e_1}(x, q) + qn(1-x) - q^2 \sum_{k=0}^{n-1} q^k \\ &= \mathcal{L}_{e_{n+1}}(x, q) + q^{n+1} \mathcal{L}_{e_n}(x, q) - q([n+1] + (n+1)(x-1) - 2) - 2. \quad \square \end{aligned}$$

- If we choose $q \rightarrow 1^-$ in (25), then we obtain

$$\sum_{k=0}^n \mathcal{L}_{e_k}(x) = \frac{\mathcal{L}_{e_{n+1}}(x) + \mathcal{L}_{e_n}(x)}{x} - (n+1).$$

- If we choose $x = 1$ in (25), then we obtain

$$\sum_{k=0}^n q^{k+1} \mathcal{L}_{e_k}(q) = \mathcal{L}_{e_{n+1}}(q) + q^{n+1} \mathcal{L}_{e_n}(q) + q - q^2 [n] - 2.$$

- If we choose $x = 1$ and $q \rightarrow 1^-$ in (25), then we obtain

$$\sum_{k=0}^n L_{e_k} = L_{e_{n+1}} + L_{e_n} - (n+1) = L_{e_{n+2}} - (n+2).$$

Theorem 3.14. *The following identity holds.*

$$\sum_{k=0}^n (x + q^{k+1} - 1) \mathfrak{L}_{u_k}(x, q) = \mathfrak{L}_{u_{n+1}}(x, q) + q^n \mathfrak{L}_{u_n}(x, q) \quad (26)$$

$$+ q [n] + (x - 1) (q(n + 1) + 2) - 2 - q.$$

where $[n]$ the q -integer of the number n .

Proof. Using the recurrence relation for q -Leonardo Lucas polynomials of the first kind, the identity is proved similarly to Theorem 3.13. \square

Theorem 3.15. *The following identity holds:*

$$\sum_{k=1}^n x^{n-k} q^{k+1} \mathcal{L}_{e_k}(x, q) = \mathcal{L}_{e_{n+2}}(x, q) - (2x^{n+2} + 2qx^n - q) - q^3 \frac{x^n - q^n}{x - q}.$$

Proof. Using the recurrence relation of q -Leonardo Pisano polynomials, we obtain

$$\frac{q^n}{x^n} \mathcal{L}_{e_n}(x, q) = \frac{1}{qx^n} \mathcal{L}_{e_{n+2}}(x, q) - \frac{1}{qx^{n-1}} \mathcal{L}_{e_{n+1}}(x, q) + \left(\frac{1}{x^n} - \frac{1}{x^{n-1}} - \frac{q^{n+1}}{x^n} \right)$$

for $n \geq 0$. Then we obtain

$$\begin{aligned} \sum_{k=1}^n \left(\frac{q}{x} \right)^k \mathcal{L}_{e_k}(x, q) &= \frac{1}{qx^n} \mathcal{L}_{e_{n+2}}(x, q) - \frac{1}{q} \mathcal{L}_{e_2}(x, q) - 1 + \frac{1}{x^n} - \frac{q^2}{x} \sum_{k=0}^{n-1} \left(\frac{q}{x} \right)^k \\ &= \frac{1}{qx^n} \mathcal{L}_{e_{n+2}}(x, q) - \frac{1}{q} \left(2x^2 + 2q - \frac{q}{x^n} \right) - \frac{q^2}{x} \frac{1 - \left(\frac{q}{x} \right)^n}{1 - \frac{q}{x}}. \end{aligned}$$

Finally,

$$\sum_{k=1}^n x^{n-k} q^{k+1} \mathcal{L}_{e_k}(x, q) = \mathcal{L}_{e_{n+2}}(x, q) - (2x^{n+2} + 2qx^n - q) - q^3 \frac{x^n - q^n}{x - q}. \quad \square$$

Theorem 3.16. *The following identity holds:*

$$\sum_{k=1}^n x^{n-k} q^k \mathfrak{L}_{u_n}(x, q) = \mathfrak{L}_{u_{n+2}}(x, q) - (2x^{n+2} + 4x^n + q) + q^2 \frac{x^n - q^n}{x - q}, \quad (27)$$

where $\mathfrak{L}_{u_n}(x, q)$ is the n -th q -Leonardo Lucas polynomials of the first kind.

Proof. Using the recurrence relation of the q -Leonardo Lucas polynomials, we obtain

$$\frac{q^n}{x^n} \mathfrak{L}_{u_n}(x, q) = \frac{1}{x^n} \mathfrak{L}_{u_{n+2}}(x, q) - \frac{1}{x^{n-1}} \mathfrak{L}_{u_{n+1}}(x, q) + q \left(\frac{1}{x^{n-1}} - \frac{1}{x^n} + q^n \right)$$

for $n \geq 0$. Then,

$$\begin{aligned} \sum_{k=1}^n \left(\frac{q}{x} \right)^k \mathfrak{L}_{u_n}(x, q) &= \frac{1}{x^n} \mathfrak{L}_{u_{n+2}}(x, q) - \mathfrak{L}_{u_2}(x, q) + q - \frac{q}{x^n} + q \sum_{k=1}^n \frac{q^k}{x^k} \\ &= \frac{1}{x^n} \mathfrak{L}_{u_{n+2}}(x, q) - \left(2x^2 + 4 + \frac{q}{x^n} \right) + \frac{q^2}{x} \frac{1 - \left(\frac{q}{x} \right)^n}{1 - \frac{q}{x}}. \end{aligned}$$

Hence we obtain

$$\sum_{k=1}^n x^{n-k} q^k \mathfrak{L}_{u_n}(x, q) = \mathfrak{L}_{u_{n+2}}(x, q) - (2x^{n+2} + 4x^n + q) + q^2 \frac{x^n - q^n}{x - q}. \quad \square$$

4 Conclusion

In this paper, q -Leonardo Pisano and q -Leonardo Lucas polynomials of the first and second kind are introduced and their various properties are mentioned. In particular, the relation of polynomials q -Fibonacci polynomials and q -Lucas polynomials are shown.

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References

- [1] Alp, Y., & Koçer, E. G. (2021). Hybrid Leonardo numbers. *Chaos, Solitons and Fractals*, 150, 111–128.
- [2] Alp, Y., & Koçer, E. G. (2021). Some properties of Leonardo numbers. *Konuralp Journal of Mathematics*, 9(1), 183–189.
- [3] Alp, Y., & Koçer, E. G. (2025). Bivariate Leonardo polynomials and Riordan arrays. *Notes on Number Theory and Discrete Mathematics*, 31(2), 236–250.
- [4] Andrews, G. E. (2004). Fibonacci numbers and the Rogers–Ramanujan identities. *The Fibonacci Quarterly*, 42, 3–19.
- [5] Bednarz, U., & Wołowiec-Musiał M. (2023). Generalized Fibonacci–Leonardo numbers. *Journal of Difference Equations and Applications*, 30(1), 111–121.
- [6] Belbachir, H., & Benmezai, A. (2014). An alternative approach to Cigler’s q -Lucas polynomials. *Applied Mathematics and Computation*, 226, 691–698.
- [7] Carlitz, L. (1975). Fibonacci notes 4: q -Fibonacci polynomials. *The Fibonacci Quarterly*, 13(2), 97–102.
- [8] Catarino, P., & Borges, A. (2019). On Leonardo numbers. *Acta Mathematica Universitatis Comenianae*, 89(1), 75–86.
- [9] Catarino P., & Borges, A. (2020). A note on incomplete Leonardo numbers. *Integers*, 20, Article ID A43.
- [10] Çelemoğlu, Ç. (2024). Pell Leonardo numbers and their matrix representations. *Journal of New Results in Science*, 13(2), 101–108.
- [11] Esen, S. (2018). *q -Fibonacci and q -Lucas Polynomials*. [Master’s Thesis. Graduate School of Natural and Applied Sciences of Gazi University].

- [12] da Fonseca, C. M., Kızılateş, C., Saraiva, P., & Shannon, A. G. (2025). Generalised Leonardo numbers. *Logic Journal of the IGPL*, 33(6), DOI: 10.1093/jigpal/jzaf005
- [13] Kac, V., & Cheung, P. (2001). *Quantum Calculus*. New York: Springer, pp. 5–20.
- [14] Kara, N., & Yilmaz, F. (2023). On hybrid numbers with Gaussian Leonardo coefficients. *Mathematics*, 11(6), Article ID 1551.
- [15] Karataş, A. (2022). On complex Leonardo numbers. *Notes on Number Theory and Discrete Mathematics*, 28(3), 458–465.
- [16] Kim, D. S., & Kim, T. (2020). On sums of finite products of balancing polynomials. *Journal of Computational and Applied Mathematics*, 377, Article ID 112913.
- [17] Kim, T., & Kim, D. S. (2025). Spivey-type recurrence relations for degenerate Bell and Dowling polynomials. *Russian Journal of Mathematical Physics*, 32(2), 288–296.
- [18] Kim, T., Kim, D. S., Dolgy, D. V., & Kwon, J. (2020). A note on sums of finite products of Lucas-balancing polynomials. *Proceedings of the Jangjeon Mathematical Society*, 23(1), 1–22.
- [19] Kuhapatanakul, K., & Ruankong, P. (2024). On generalized Leonardo p -numbers. *Journal of Integer Sequences*, 27, Article ID 24.4.6.
- [20] Özkan, E., & Akkuş, H. (2024). Generalized bronze Leonardo sequence. *Notes on Number Theory and Discrete Mathematics*, 30(4), 811–824.
- [21] Prasad, K., & Kumari, M. (2025). The Leonardo polynomials and their algebraic properties. *Proceedings of the Indian National Science Academy*, 91, 662–671.
- [22] Shannon, A. G., & Deveci, Ö. (2022). A note on generalized and extended Leonardo sequences. *Notes on Number Theory and Discrete Mathematics*, 28(1), 109–114.
- [23] Shannon, A. G., Shiue, P. J.-S., & Huang, S. C. (2023). Notes on generalized and extended Leonardo numbers. *Notes on Number Theory and Discrete Mathematics*, 29(4), 752–773.
- [24] Shattuck, M. (2022). Combinatorial proofs of identities for the generalized Leonardo numbers. *Notes on Number Theory and Discrete Mathematics*, 28(4), 778–790.
- [25] Tan, E., & Ho-Hon, L. (2023). On Leonardo p -numbers. *Integers*, 23, Article ID A7.
- [26] Tasci, D. (2023). On Gaussian Leonardo numbers. *Contributions to Mathematics*, 7, 34–40.