

On the inverse of a certain triangular matrix and its connection to the largest odd divisor

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Abstract: The inverse R of a certain infinite triangular matrix A is shown to be directly related to the largest odd divisor function, thus proving a conjecture of Barry. We also provide a proof of a formula for R given by Yin and obtain bivariate generating functions for A and R .

Keywords: Bivariate generating function, Largest odd divisor, Lower triangular matrix.

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1 Introduction

For an integer $n \geq 1$, let $\ell(n)$ be the largest odd divisor of n (e.g., [2]). This arithmetic function appears as [A000265](#) in the On-Line Encyclopedia of Integer Sequences (OEIS) [5] and statements regarding sums involving it appeared, for example, in [1] and [4, Problem B-6]. Define an infinite lower triangular matrix $A = (a_{n,k})_{n,k \geq 1}$ by

$$a_{n,k} = \begin{cases} \frac{1}{\lfloor \frac{n+1}{2} \rfloor}, & \text{if } k \leq n \leq 2k - 1, \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

Let $R = (r_{n,k})_{n,k \geq 1}$ be the inverse of A (registered as sequence [A127793](#) in the OEIS). Barry [5, [A000265](#)] conjectured the following sum representation of $\ell(n)$:



$$\ell(n) = \sum_{k \geq 1} r_{n,k} \left\lfloor \frac{k+1}{2} \right\rfloor.$$

The main purpose of this work is to prove Barry's conjecture (Theorem 2.1). After accomplishing this, we devote our attention to the matrices A and R . Yin [5, A127793] stated that

$$R = e_{1,1} + \sum_{i \geq 2} \sum_{p \geq 0} \left(\left\lfloor \frac{i}{2} \right\rfloor e_{2^p i, i} - \left\lfloor \frac{i}{2} \right\rfloor e_{2^p(i+1), i} \right), \quad (2)$$

where $e_{i,j}$ is the infinite matrix with a 1 at entry (i, j) and 0 elsewhere. To the best of our knowledge, Yin did not provide a proof of (2). In Theorem 2.2 we prove Yin's formula. We also obtain a neat closed form formula for the entries of R (Theorem 2.3), and find the bivariate generating functions for A and R (Theorems 2.4 and 2.5).

Before we begin, let us set some notation. Let $\mathbb{N} = \{1, 2, \dots\}$ denote the set of natural numbers. For a real number x , we denote by $\lfloor x \rfloor$ and $\lceil x \rceil$ the floor and ceiling of x , respectively. For a condition P , write $\llbracket P \rrbracket$ for the Iverson bracket, i.e., $\llbracket P \rrbracket = 1$ if P is true and 0 otherwise (e.g., [3, p. 24]). For $n \in \mathbb{N}$, set $I_n = \left\{ \left\lceil \frac{n+1}{2} \right\rceil, \dots, n \right\}$. Let $\mathbf{1}$ be the infinite all-1 column vector.

2 Main results

We begin by proving Barry's conjecture.

Theorem 2.1. *Let $n \in \mathbb{N}$. Then,*

$$\ell(n) = \sum_{k \geq 1} r_{n,k} \left\lfloor \frac{k+1}{2} \right\rfloor.$$

Proof. Let ℓ be the infinite column vector defined by $\ell = (\ell(n))_{n \geq 1}$ and let $v = (v_k)_{k \geq 1}$ be the infinite column vector defined by $v_k = \left\lfloor \frac{k+1}{2} \right\rfloor$. Thus, we wish to prove that $\ell = Rv$. Since $R = A^{-1}$, this is equivalent to proving $A\ell = v$. First, notice that (1) may be written as

$$a_{n,k} = \begin{cases} \frac{1}{\left\lfloor \frac{n+1}{2} \right\rfloor}, & \text{if } \left\lceil \frac{n+1}{2} \right\rceil \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

Thus, for each $n \in \mathbb{N}$,

$$(A\ell)_n = \frac{1}{\left\lfloor \frac{n+1}{2} \right\rfloor} \sum_{i=\left\lceil \frac{n+1}{2} \right\rceil}^n \ell(i).$$

Thus, it suffices to prove that

$$\sum_{i=\left\lceil \frac{n+1}{2} \right\rceil}^n \ell(i) = \left\lfloor \frac{n+1}{2} \right\rfloor^2. \quad (4)$$

To this end, for $p \in \mathbb{N}$, set

$$F(p) = \sum_{i=p+1}^{2p} \ell(i).$$

We prove by induction on p that $F(p) = p^2$. For the base case $p = 1$, we have $F(1) = \ell(2) = 1 = 1^2$. Assume that $p \geq 2$ and that the assertion holds for $p - 1$. Then,

$$\begin{aligned} F(p) - F(p-1) &= \sum_{i=p+1}^{2p} \ell(i) - \sum_{i=p}^{2p-2} \ell(i) \\ &= \ell(2p-1) + \ell(2p) - \ell(p) \\ &= 2p-1 + \ell(p) - \ell(p) \\ &= 2p-1. \end{aligned}$$

With the induction hypothesis, it follows that

$$F(p) = F(p-1) + 2p-1 = (p-1)^2 + 2p-1 = p^2,$$

completing the induction.

We now distinguish between two cases of n .

1. $n = 2m$ is even. Then (4) may be written as $F(m) = m^2$ and this identity holds as we have just proved.
2. $n = 2m + 1$ is odd. Then (4) may be written as

$$\sum_{i=m+1}^{2m+1} \ell(i) = (m+1)^2.$$

But

$$\begin{aligned} \sum_{i=m+1}^{2m+1} \ell(i) &= \sum_{i=m+1}^{2m} \ell(i) + \ell(2m+1) \\ &= F(m) + 2m+1 \\ &= m^2 + 2m+1 \\ &= (m+1)^2. \end{aligned} \quad \square$$

In order to prove Yin's formula, notice that by extracting the coefficient of $e_{n,k}$ in (2), Yin's formula is immediately seen to be equivalent to the following entry-wise description of R :

$$r_{n,k} = \left\lfloor \frac{k}{2} \right\rfloor \sum_{p \geq 0} ([n = 2^p k] - [n = 2^p(k+1)]). \quad (5)$$

Our proof of Yin's formula relies on the following result.

Lemma 2.1. *For every $n, k \in \mathbb{N}$ we have*

$$\sum_{p \geq 0} ([2^p k \in I_n] - [2^p(k+1) \in I_n]) = [k = n]. \quad (6)$$

Proof. Denote by $S(n, k)$ the left-hand side of (6). First, assume that $n = 1$. Then $I_1 = \{1\}$ and we distinguish between two cases of k .

1. $k = 1$. Then $2^p k = 1 \iff p = 0$, and there exists no $p \geq 0$ with $2^p(k+1) = 1$. Thus, $S(1, 1) = 1 - 0 = 1$.
2. $k > 1$. Then there exists no $p \geq 0$ with $2^p k = 1$ or $2^p(k+1) = 1$. Thus $S(1, k) = 0$.

In any case, $S(1, k) = [[k = 1]]$. Assume now that $n \geq 2$ and set $m = \lfloor \frac{n}{2} \rfloor$. We prove the following statement. Let $x \in \mathbb{N}$. Then

$$2x \in I_n \iff x \in I_m. \quad (7)$$

We distinguish between two cases of n .

1. $n = 2m$ is even. Then $I_n = \{m + 1, \dots, 2m\}$ and, using that x is an integer,

$$2x \in I_n \iff m + 1 \leq 2x \leq 2m \iff \frac{m + 1}{2} \leq x \leq m \iff x \in I_m.$$

2. $n = 2m + 1$ is odd. Then, $I_n = \{m + 1, \dots, 2m + 1\}$ and, using that x is an integer,

$$2x \in I_n \iff m + 1 \leq 2x \leq 2m + 1 \iff \frac{m + 1}{2} \leq x \leq m + \frac{1}{2} \iff x \in I_m.$$

This proves (7). We proceed to prove the following recursion for $S(n, k)$:

$$S(n, k) = [[k = n]] - [[k = m]] + S(m, k). \quad (8)$$

We have

$$\begin{aligned} S(n, k) &= [[k \in I_n]] - [[k + 1 \in I_n]] + \sum_{p \geq 1} ([[2^p k \in I_n]] - [[2^p(k + 1) \in I_n]]) \\ &= [[k \in I_n]] - [[k + 1 \in I_n]] + \sum_{p \geq 0} ([[2 \cdot 2^p k \in I_n]] - [[2 \cdot 2^p(k + 1) \in I_n]]) \\ &= [[k \in I_n]] - [[k + 1 \in I_n]] + \sum_{p \geq 0} ([[2^p k \in I_m]] - [[2^p(k + 1) \in I_m]]) \\ &= [[k \in I_n]] - [[k + 1 \in I_n]] + S(m, k), \end{aligned}$$

where in the last identity we used (7). By considering each of the following five cases:

$$k < m, \quad k = m, \quad m + 1 \leq k \leq n - 1, \quad k = n, \quad k > n,$$

it is established that

$$[[k \in I_n]] - [[k + 1 \in I_n]] = [[k = n]] - [[k = m]].$$

This finishes the proof of (8).

We can now prove the statement of the lemma. We proceed by strong induction on n . Assume now that $n \geq 2$ and that the assertion holds for every natural number t with $t < n$, i.e., for every $k \in \mathbb{N}$, we have $S(t, k) = [[k = t]]$. By (8), we have

$$S(n, k) = [[k = n]] - [[k = m]] + S(m, k). \quad (9)$$

By the induction hypothesis applied to $t = m$, we have $S(m, k) = [[k = m]]$. Substituting this into (9) yields

$$S(n, k) = [[k = n]] - [[k = m]] + [[k = m]] = [[k = n]].$$

This completes the induction, and finishes the proof of the lemma. \square

We may now prove Yin's formula, in its entry-wise form (5).

Theorem 2.2. *We have $R = A^{-1}$.*

Proof. We prove that $AR = I$. Let $n, k \in \mathbb{N}$. Using (1),

$$(AR)_{n,k} = \sum_{i \geq 1} a_{n,i} r_{i,k} = \frac{1}{\lfloor \frac{n+1}{2} \rfloor} \sum_{i=\lfloor \frac{n+1}{2} \rfloor}^n r_{i,k}.$$

Thus, we need to prove that

$$\sum_{i=\lfloor \frac{n+1}{2} \rfloor}^n r_{i,k} = \begin{cases} \lfloor \frac{n+1}{2} \rfloor, & \text{if } n = k, \\ 0, & \text{otherwise.} \end{cases}$$

Using (5) and Lemma 2.1,

$$\begin{aligned} \sum_{i=\lfloor \frac{n+1}{2} \rfloor}^n r_{i,k} &= \left\lfloor \frac{k}{2} \right\rfloor \sum_{p \geq 0} \sum_{i=\lfloor \frac{n+1}{2} \rfloor}^n ([[i = 2^p k]] - [[i = 2^p(k+1)]]) \\ &= \left\lfloor \frac{k}{2} \right\rfloor \sum_{p \geq 0} ([[2^p k \in I_n]] - [[2^p(k+1) \in I_n]]) \\ &= \left\lfloor \frac{k}{2} \right\rfloor [[k = n]] \\ &= \begin{cases} \lfloor \frac{n}{2} \rfloor, & \text{if } n = k, \\ 0, & \text{otherwise,} \end{cases} \\ &= \begin{cases} \lfloor \frac{n+1}{2} \rfloor, & \text{if } n = k, \\ 0, & \text{otherwise.} \end{cases} \quad \square \end{aligned}$$

Remark 2.1. *When sequence [A127793](#) was registered in the OEIS, it was not clear that it qualifies as a sequence of integers. Yin's formula guarantees this. Furthermore, it was conjectured that each row of R sums to 1. This is not hard to prove. Indeed, first notice that the conjecture is true for A :*

$$\sum_{k \geq 1} a_{n,k} = \sum_{k=\lfloor \frac{n+1}{2} \rfloor}^n \frac{1}{\lfloor \frac{n+1}{2} \rfloor} = \frac{n - \lfloor \frac{n+1}{2} \rfloor + 1}{\lfloor \frac{n+1}{2} \rfloor} = 1.$$

In matrix form: $A\mathbf{1} = \mathbf{1}$. Multiplying both sides by $A^{-1} = R$ gives $R\mathbf{1} = \mathbf{1}$, proving that each row of R sums to 1 as well.

We now provide a neat closed form formula for $r_{n,k}$.

Theorem 2.3. *For $n \in \mathbb{N}$, write $n = 2^t u$ with u odd and nonnegative integer t . For each $j \in \{0, 1, \dots, t\}$, set $\theta_j = 2^{t-j} u$. Then, for every $k \in \mathbb{N}$, we have*

$$r_{n,k} = \sum_{p=0}^t \left(\left\lfloor \frac{\theta_p}{2} \right\rfloor [[k = \theta_p]] - \left\lfloor \frac{\theta_p}{2} \right\rfloor [[k = \theta_p - 1]] \right).$$

Proof. Let $p \geq 0$. The term $\left\lceil \frac{k}{2} \right\rceil \llbracket n = 2^p k \rrbracket$ in (5) is nonzero if and only if $p \leq t$ and $k = \theta_p$. Thus,

$$\sum_{p \geq 0} \left\lceil \frac{k}{2} \right\rceil \llbracket n = 2^p k \rrbracket = \sum_{p=0}^t \left\lceil \frac{\theta_p}{2} \right\rceil \llbracket k = \theta_p \rrbracket. \quad (10)$$

Similarly, the term $\left\lceil \frac{k}{2} \right\rceil \llbracket n = 2^p(k+1) \rrbracket$ is nonzero if and only if $p \leq t$ and $k = \theta_p - 1$. In addition, $\left\lceil \frac{\theta_p - 1}{2} \right\rceil = \left\lceil \frac{\theta_p}{2} \right\rceil$. Thus,

$$\sum_{p \geq 0} \left\lceil \frac{k}{2} \right\rceil \llbracket n = 2^p(k+1) \rrbracket = \sum_{p=0}^t \left\lceil \frac{\theta_p}{2} \right\rceil \llbracket k = \theta_p - 1 \rrbracket. \quad (11)$$

Combining (10) and (11) proves the assertion. \square

Theorem 2.4. Let $A(x, y)$ be the bivariate generating function for the matrix A , i.e.,

$$A(x, y) = \sum_{n, k \geq 1} a_{n, k} x^n y^k.$$

Then

$$A(x, y) = \frac{1 + xy}{x(1 - y)} \log \left(\frac{1 - x^2 y^2}{1 - x^2 y} \right).$$

Proof. Split $A(x, y)$ into its even and odd parts, i.e., write $A(x, y) = E(x, y) + O(x, y)$, where

$$E(x, y) = \sum_{m, k \geq 1} a_{2m, k} x^{2m} y^k, \quad O(x, y) = \sum_{m, k \geq 1} a_{2m-1, k} x^{2m-1} y^k.$$

Start with $E(x, y)$. For $n = 2m$ we have, $\left\lfloor \frac{n+1}{2} \right\rfloor = m$ and $\left\lceil \frac{n+1}{2} \right\rceil = m + 1$. Thus, by (3),

$$\begin{aligned} E(x, y) &= \sum_{m \geq 1} \left(\frac{1}{m} \sum_{k=m+1}^{2m} y^k \right) x^{2m} \\ &= \sum_{m \geq 1} \frac{y^{m+1}(1 - y^m)}{m(1 - y)} x^{2m} \\ &= \frac{1}{1 - y} \sum_{m \geq 1} \frac{1}{m} (x^{2m} y^{m+1} - x^{2m} y^{2m+1}) \\ &= \frac{y}{1 - y} \sum_{m \geq 1} \frac{(x^2 y)^m}{m} - \frac{y}{1 - y} \sum_{m \geq 1} \frac{(x^2 y^2)^m}{m}. \end{aligned}$$

Using $\sum_{m \geq 1} \frac{z^m}{m} = \log\left(\frac{1}{1-z}\right)$ (e.g., [6, (2.5.2)]), we obtain

$$E(x, y) = \frac{y}{1 - y} \log \left(\frac{1 - x^2 y^2}{1 - x^2 y} \right).$$

Now we consider $O(x, y)$. For $n = 2m - 1$ we have $\lfloor \frac{n+1}{2} \rfloor = \lceil \frac{n+1}{2} \rceil = m$. Thus, by (3),

$$\begin{aligned} O(x, y) &= \sum_{m \geq 1} \left(\frac{1}{m} \sum_{k=m}^{2m-1} y^k \right) x^{2m-1} \\ &= \sum_{m \geq 1} \frac{y^m(1-y^m)}{m(1-y)} x^{2m-1} \\ &= \frac{1}{1-y} \sum_{m \geq 1} \frac{1}{m} (x^{2m-1}y^m - x^{2m-1}y^{2m}) \\ &= \frac{1}{x(1-y)} \sum_{m \geq 1} \frac{(x^2y)^m}{m} - \frac{1}{x(1-y)} \sum_{m \geq 1} \frac{(x^2y^2)^m}{m} \\ &= \frac{1}{x(1-y)} \log \left(\frac{1-x^2y^2}{1-x^2y} \right). \end{aligned}$$

Adding the expressions for $E(x, y)$ and $O(x, y)$ finishes the proof. \square

Theorem 2.5. Let $R(x, y)$ be the bivariate generating function for the matrix R , i.e.,

$$R(x, y) = \sum_{n, k \geq 1} r_{n, k} x^n y^k.$$

Then

$$R(x, y) = \sum_{p \geq 0} \frac{yx^{2^p}(1-x^{2^p})}{(1-yx^{2^p})^2(1+yx^{2^p})}.$$

Proof. By (5),

$$\begin{aligned} R(x, y) &= \sum_{n, k \geq 1} \left(\left\lfloor \frac{k}{2} \right\rfloor \sum_{p \geq 0} ([n = 2^p k] - [n = 2^p(k+1)]) \right) x^n y^k \\ &= \sum_{k \geq 1} \left\lfloor \frac{k}{2} \right\rfloor \sum_{p \geq 0} (x^{2^p k} - x^{2^p(k+1)}) y^k \\ &= \sum_{p \geq 0} (1-x^{2^p}) \sum_{k \geq 1} \left\lfloor \frac{k}{2} \right\rfloor (yx^{2^p})^k. \end{aligned}$$

Set

$$S(z) = \sum_{k \geq 1} \left\lfloor \frac{k}{2} \right\rfloor z^k.$$

We claim that

$$S(z) = \frac{z}{(1-z)^2(1+z)}.$$

Indeed, splitting into $k = 2m$ and $k = 2m - 1$, we have

$$S(z) = \sum_{m \geq 1} mz^{2m} + \sum_{m \geq 1} mz^{2m-1}.$$

With $\sum_{m \geq 1} mt^m = \frac{t}{(1-t)^2}$, we obtain

$$S(z) = \frac{z^2}{(1-z^2)^2} + \frac{z}{(1-z^2)^2} = \frac{z}{(1-z)^2(1+z)},$$

as asserted. It follows that

$$R(x, y) = \sum_{p \geq 0} (1 - x^{2p}) S(yx^{2p}) = \sum_{p \geq 0} \frac{(1 - x^{2p})yx^{2p}}{(1 - yx^{2p})^2(1 + yx^{2p})},$$

and the proof is complete. \square

Remark 2.2. *The matrices A and R are indexed by $n, k \geq 1$. If one prefers indexing by $n, k \geq 0$, then the corresponding generating functions $\tilde{A}(x, y)$ and $\tilde{R}(x, y)$ are given by*

$$\begin{aligned} \tilde{A}(x, y) &= \sum_{n, k \geq 0} a_{n+1, k+1} x^n y^k = \frac{A(x, y)}{xy} = \frac{1 + xy}{x^2 y (1 - y)} \log \left(\frac{1 - x^2 y^2}{1 - x^2 y} \right), \\ \tilde{R}(x, y) &= \sum_{n, k \geq 0} r_{n+1, k+1} x^n y^k = \frac{R(x, y)}{xy} = \sum_{p \geq 0} \frac{x^{2p-1} (1 - x^{2p})}{(1 - yx^{2p})^2 (1 + yx^{2p})}. \end{aligned}$$

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