

# Some Fibonacci congruences with square moduli

Anthony G. Shannon<sup>1</sup> , Peter J.-S. Shiue<sup>2</sup> ,  
and Christopher Saito<sup>3</sup> 

<sup>1</sup> Warrane College, University of New South Wales  
Kensington, NSW 2033, Australia

e-mails: tshannon@warrane.unsw.edu.au, tshannon38@gmail.com

<sup>2</sup> Department of Mathematical Sciences, University of Nevada, Las Vegas  
Las Vegas, Nevada, 89154-4020, United States  
e-mail: shiue@unlv.nevada.edu

<sup>3</sup> Department of Mathematical Sciences, University of Nevada, Las Vegas  
Las Vegas, Nevada, 89154-4020, United States  
e-mail: saitocl@unlv.nevada.edu

**Received:** 17 January 2026

**Accepted:** 20 March 2026

**Revised:** 19 March 2026

**Online First:** 21 March 2026

**Abstract:** Fibonacci congruence with prime moduli have been extensively studied. Square moduli are obviously not prime numbers, so why study such congruences? The answer is to investigate patterns for a special class of moduli, since there are many Fibonacci identities containing squares of Fibonacci numbers. Previous studies do not seem to have revealed anything of note that is notably distinct from other composite moduli.

**Keywords:** Basis vectors, Fibonacci numbers, Integral lattices, Jacobsthal sequences, Pythagorean triples, Sophie Germain.

**2020 Mathematics Subject Classification:** 11B39, 11B37, 11D09, 11H06.



# 1 Introduction

Congruences with square moduli have traditionally been studied with sieve methods [2, 20, 36] and as Farey sequences [1], and there have been numerous Fibonacci number congruences both with pure research [29] and for education [8]. Generally, these have been modulo an integer  $m$ , particularly when the modulus is prime, but also relatively prime conditions [16, 19, 35]. There have also been some which have explored aspects of square moduli [19, 27]. We propose to consider some aspects of these in this note. To do this, we extend some results from [34].

$$F_{2z} - F_{z-1}F_z + F_zF_{z+1} \equiv 0 \pmod{m^2} \quad (1.1)$$

and

$$F_{2z+1} - F_{z+1}^2 \equiv 0 \pmod{m^2} \quad (1.2)$$

For example, let  $z = 6$  in (1.1) and (1.2). Then

$$F_{12} - F_5F_6 + F_6F_7 = 208 \equiv 0 \pmod{4^2}$$

and

$$F_{13} - F_7^2 \equiv 0 \pmod{2^2}.$$

## 2 A partial investigation

The following results are well-known (see Koshy [24]).

**Lemma 1.** *Suppose  $m, n \in \mathbb{N}$ . Then  $n \mid m$  implies  $F_n \mid F_m$ .*

**Lemma 2.** *Let  $p$  be a prime number.*

1. *If  $p = 5$ , then  $p \mid F_5$ .*
2. *If  $p \equiv 1$  or  $4 \pmod{5}$ , then  $p \mid F_{p-1}$ .*
3. *If  $p \equiv 2$  or  $3 \pmod{5}$ , then  $p \mid F_{p+1}$ .*

Using Lemma 1, we have the following theorem.

**Theorem 1.** *Let  $a$  be the smallest positive integer such that  $F_a \equiv 0 \pmod{m^2}$ , where  $m \in \mathbb{N}$ . Then  $F_{ak} \equiv 0 \pmod{m^2}$  for  $k = 1, 2, 3, \dots$*

Using Lemma 2, we have the following examples.

**Example 1.**

- |                 |                     |                     |                     |
|-----------------|---------------------|---------------------|---------------------|
| A. $2 \mid F_3$ | D. $7 \mid F_8$     | G. $17 \mid F_9$    | J. $29 \mid F_{14}$ |
| B. $3 \mid F_4$ | E. $11 \mid F_{10}$ | H. $19 \mid F_{18}$ | K. $47 \mid F_{16}$ |
| C. $5 \mid F_5$ | F. $13 \mid F_7$    | I. $23 \mid F_{24}$ | L. $61 \mid F_{15}$ |

**Note.** Gandhi [23] has shown the above results in a different approach.

**Theorem 2.** Let  $p$  be a prime number. If  $p^i \mid F_n$ , then  $p^{i+1} \mid F_{np}$ , where  $i \in \mathbb{N}$ .

*Proof.* The proof follows from the identity of Halton [18]

$$F_{pn} = F_n \sum_{k=1}^p \frac{p}{k} F_n^{k-1} F_{n-1}^{p-k} F_k. \quad \square$$

By applying Theorem 2, we obtain

- |                      |                        |                        |                        |
|----------------------|------------------------|------------------------|------------------------|
| A. $2^2 \mid F_6$    | D. $7^2 \mid F_{56}$   | G. $17^2 \mid F_{153}$ | J. $29^2 \mid F_{406}$ |
| B. $3^2 \mid F_{12}$ | E. $11^2 \mid F_{110}$ | H. $19^2 \mid F_{342}$ | K. $47^2 \mid F_{752}$ |
| C. $5^2 \mid F_{25}$ | F. $13^2 \mid F_{91}$  | I. $23^2 \mid F_{552}$ | L. $61^2 \mid F_{915}$ |

Therefore, we have the following table.

Table 1.  $F_{nk} \equiv 0 \pmod{p^2}$ ,  $2 \leq p \leq 61$ .

$F_{nk}$	$n$	$\equiv 0 \pmod{p^2}$
$F_{6k}$	$3 \times 2$	$\equiv 0 \pmod{2^2}$
$F_{12k}$	$4 \times 3$	$\equiv 0 \pmod{3^2}$
$F_{25k}$	$5 \times 5$	$\equiv 0 \pmod{5^2}$
$F_{56k}$	$8 \times 7$	$\equiv 0 \pmod{7^2}$
$F_{110k}$	$10 \times 11$	$\equiv 0 \pmod{11^2}$
$F_{91k}$	$7 \times 13$	$\equiv 0 \pmod{13^2}$
$F_{153k}$	$9 \times 17$	$\equiv 0 \pmod{17^2}$
$F_{342k}$	$18 \times 19$	$\equiv 0 \pmod{19^2}$
$F_{552k}$	$24 \times 23$	$\equiv 0 \pmod{23^2}$
$F_{406k}$	$14 \times 29$	$\equiv 0 \pmod{29^2}$
$F_{752k}$	$16 \times 47$	$\equiv 0 \pmod{47^2}$
$F_{915k}$	$15 \times 61$	$\equiv 0 \pmod{61^2}$

**Theorem 3.** Let  $k$  be a positive integer and  $p$  be a prime number.

$$F_{12 \cdot 2^{2t-4} k} \equiv 0 \pmod{(2^2)^t}, \quad t \geq 2 \quad (2.1)$$

$F_{r \cdot p^{2t-2} k} \equiv 0 \pmod{(p^2)^t}$ ,  $p \geq 3$ ,  $r$  is the smallest positive integer such that

$$F_r \equiv 0 \pmod{p^2}, \quad t \geq 1 \quad (2.2)$$

*Proof.* Use mathematical induction on  $t$ . □

Using Theorem 3, we have the following algorithm to solve the following problem.

**Problem.** Given  $m^2 \in \mathbb{N}$ , find all possible  $n \in \mathbb{N}$ , such that  $F_{nk} \equiv 0 \pmod{m^2}$ , where  $k \in \mathbb{N}$ .

**Algorithm**

*Step 1.* Find the prime factorization of  $m^2 = (p_1^2)^{m_1} (p_2^2)^{m_2} \cdots (p_v^2)^{m_v}$ .

*Step 2.* Find  $n_i$  such that  $F_{n_i} \equiv 0 \pmod{(p_i^2)^{m_i}}$  by Theorem 3.

*Step 3.* Find  $\text{LCM}(n_1, n_2, \dots, n_i) = n$ . Then  $F_{nk} \equiv 0 \pmod{m^2}$ .

**Example 2.** For example, let us find all  $n \in \mathbb{N}$  such that  $F_{nk} \equiv 0 \pmod{20^2}$ . Following the algorithm:

*Step 1.*  $m^2 = 20^2 = (2^2)^2 5^2$ .

*Step 2.* We use Theorem 3. For  $\text{mod}(2^2)^2$ , we apply (2.1) with the identification  $t = 2$ . Then  $F_{12k} \equiv 0 \pmod{(2^2)^2}$ . Hence,  $n_1 = 12$ . For  $\text{mod } 5^2$ , we apply (2.2) with the identification  $p = 5$  and  $t = 1$ . Looking at Table 2, we find  $r = 25$ . Then  $F_{25k} \equiv 0 \pmod{5^2}$ . Hence,  $n_2 = 25$ .

*Step 3.*  $\text{LCM}(n_2) = \text{LCM}(12, 25) = 300$ . Thus,  $F_{300k} \equiv 0 \pmod{20^2}$ .

As another example, let us find all  $n \in \mathbb{N}$  such that  $F_{nk} \equiv 0 \pmod{24^2}$ . Following the algorithm:

*Step 1.*  $m^2 = 24^2 = (2^2)^3 3^2$ .

*Step 2.* We use Theorem 3. For  $\text{mod}(2^2)^3$ , we apply (2.1) with the identification  $t = 3$ . Then  $F_{48k} \equiv 0 \pmod{(2^2)^3}$ . Hence,  $n_1 = 48$ . For  $\text{mod } 3^2$ , we apply (2.2) with the identification  $p = 3$  and  $t = 1$ . Looking at Table 2, we find  $r = 12$ . Then  $F_{12k} \equiv 0 \pmod{3^2}$ . Hence,  $n_2 = 12$ .

*Step 3.*  $\text{LCM}(n_1, n_2) = \text{LCM}(48, 12) = 48$ . Thus,  $F_{48k} \equiv 0 \pmod{24^2}$ .

Continuing this process, we investigate specific Fibonacci numbers modulo  $m^2$  for  $1 < m \leq 30$ , see Table 2.

Table 2.  $F_n \equiv 0 \pmod{m^2}$ ,  $2 < m \leq 30$

$F_n$	$n$	Value [12]	$\equiv 0 \pmod{p^2}$
$F_6$	$3 \times 2$	8	$\equiv 0 \pmod{2^2}$
$F_{12}$	$4 \times 3$	144	$\equiv 0 \pmod{3^2}$
$F_{12}$	$3 \times 4$	144	$\equiv 0 \pmod{4^2}$
$F_{25}$	$5 \times 5$	75,025	$\equiv 0 \pmod{5^2}$
$F_{12}$	$2 \times 6$	144	$\equiv 0 \pmod{6^2}$
$F_{56}$	$8 \times 7$	2,258,851,433,717	$\equiv 0 \pmod{7^2}$
$F_{48}$	$6 \times 8$	4,807,526,976	$\equiv 0 \pmod{8^2}$
$F_{108}$	$12 \times 9$	16,641,027,750,620,563,662,096	$\equiv 0 \pmod{9^2}$
$F_{150}$	$15 \times 10$	9,969,216,677,189,303,386,214,405,760,200	$\equiv 0 \pmod{10^2}$
$F_{110}$	$10 \times 11$	43,566,776,258,854,844,738,105	$\equiv 0 \pmod{11^2}$
$F_{12}$	$1 \times 12$	144	$\equiv 0 \pmod{12^2}$
$F_{91}$	$7 \times 13$	4,660,046,610,375,530,309	$\equiv 0 \pmod{13^2}$
$F_{168}$	$12 \times 14$	57,602,132,235,424,755,886,206,198,685,365,216	$\equiv 0 \pmod{14^2}$
$F_{150}$	$10 \times 15$	9,969,216,677,189,303,386,214,405,760,200	$\equiv 0 \pmod{15^2}$
$F_{144}$	$9 \times 16$	555,565,404,224,292,694,404,015,791,808	$\equiv 0 \pmod{16^2}$
$F_{153}$	$9 \times 17$	$2 \times 17^2 \times 1597 \times 6376021 \times 7175323114950564593$	$\equiv 0 \pmod{17^2}$
$F_{108}$	$6 \times 18$	16,641,027,750,620,563,662,096	$\equiv 0 \pmod{18^2}$
$F_{342}$	$18 \times 19$	See <a href="https://r-knott.surrey.ac.uk/fibonacci/fibtable301.html">https://r-knott.surrey.ac.uk/fibonacci/fibtable301.html</a>	$\equiv 0 \pmod{19^2}$
$F_{300}$	$15 \times 20$	See <a href="https://r-knott.surrey.ac.uk/fibonacci/fibtable301.html">https://r-knott.surrey.ac.uk/fibonacci/fibtable301.html</a>	$\equiv 0 \pmod{20^2}$
$F_{168}$	$8 \times 21$	57,602,132,235,424,755,886,206,198,685,365,216	$\equiv 0 \pmod{21^2}$
$F_{330}$	$15 \times 22$	See <a href="https://r-knott.surrey.ac.uk/fibonacci/fibtable301.html">https://r-knott.surrey.ac.uk/fibonacci/fibtable301.html</a>	$\equiv 0 \pmod{22^2}$
$F_{552}$	$24 \times 23$	See <a href="https://mersennus.net/fibonacci/">https://mersennus.net/fibonacci/</a>	$\equiv 0 \pmod{23^2}$
$F_{48}$	$2 \times 24$	4,807,526,976	$\equiv 0 \pmod{24^2}$
$F_{625}$	$25 \times 25$	See <a href="https://mersennus.net/fibonacci/">https://mersennus.net/fibonacci/</a>	$\equiv 0 \pmod{25^2}$
$F_{546}$	$21 \times 26$	See <a href="https://mersennus.net/fibonacci/">https://mersennus.net/fibonacci/</a>	$\equiv 0 \pmod{26^2}$
$F_{972}$	$36 \times 27$	See <a href="https://mersennus.net/fibonacci/">https://mersennus.net/fibonacci/</a>	$\equiv 0 \pmod{27^2}$
$F_{168}$	$6 \times 28$	57,602,132,235,424,755,886,206,198,685,365,216	$\equiv 0 \pmod{28^2}$
$F_{406}$	$14 \times 29$	See <a href="https://r-knott.surrey.ac.uk/fibonacci/fibtable301.html">https://r-knott.surrey.ac.uk/fibonacci/fibtable301.html</a>	$\equiv 0 \pmod{29^2}$
$F_{300}$	$10 \times 30$	$2^4 \times 3^2 \times 5^2 \times 11 \times 31 \times 41 \times 61 \times 101 \times 151 \times 401$ $\times 601 \times 2521 \times 3001 \times 12301 \times 18451 \times 570601 \times$ $230686501 \times 87129547172401$	$\equiv 0 \pmod{30^2}$

Table 3.  $F_{nk} \equiv 0 \pmod{m^2}, \forall k \in \mathbb{N}, 1 < m \leq 30$

$F_{nk}$	$n$	$\equiv 0 \pmod{p^2}$	$F_{nk}$	$n$	$\equiv 0 \pmod{p^2}$
$F_{6k}$	6	$\equiv 0 \pmod{2^2}$	$F_{153k}$	153	$\equiv 0 \pmod{17^2}$
$F_{12k}$	12	$\equiv 0 \pmod{3^2}$	$F_{108k}$	108	$\equiv 0 \pmod{18^2}$
$F_{12k}$	12	$\equiv 0 \pmod{4^2}$	$F_{342k}$	342	$\equiv 0 \pmod{19^2}$
$F_{25k}$	25	$\equiv 0 \pmod{5^2}$	$F_{300k}$	300	$\equiv 0 \pmod{20^2}$
$F_{12k}$	12	$\equiv 0 \pmod{6^2}$	$F_{168k}$	168	$\equiv 0 \pmod{21^2}$
$F_{56k}$	56	$\equiv 0 \pmod{7^2}$	$F_{330k}$	330	$\equiv 0 \pmod{22^2}$
$F_{48k}$	48	$\equiv 0 \pmod{8^2}$	$F_{552k}$	552	$\equiv 0 \pmod{23^2}$
$F_{108k}$	108	$\equiv 0 \pmod{9^2}$	$F_{48k}$	48	$\equiv 0 \pmod{24^2}$
$F_{150k}$	150	$\equiv 0 \pmod{10^2}$	$F_{625k}$	625	$\equiv 0 \pmod{25^2}$
$F_{110k}$	110	$\equiv 0 \pmod{11^2}$	$F_{546k}$	546	$\equiv 0 \pmod{26^2}$
$F_{12k}$	12	$\equiv 0 \pmod{12^2}$	$F_{972k}$	972	$\equiv 0 \pmod{27^2}$
$F_{91k}$	91	$\equiv 0 \pmod{13^2}$	$F_{168k}$	168	$\equiv 0 \pmod{28^2}$
$F_{168k}$	168	$\equiv 0 \pmod{14^2}$	$F_{406k}$	406	$\equiv 0 \pmod{29^2}$
$F_{150k}$	300	$\equiv 0 \pmod{15^2}$	$F_{300k}$	300	$\equiv 0 \pmod{30^2}$
$F_{144k}$	144	$\equiv 0 \pmod{16^2}$			

Table 4. Divisibility symbols

Symbol	Meaning	Comment
$C(m^2)$	The smallest $k \in \mathbb{Z}_+ : F_{k+n} \equiv F_n \pmod{m^2}$	Period of cycle $\pmod{m^2}$
$D(m^2)$	Smallest $z \in \mathbb{Z}_+ : F_z \equiv 0 \pmod{m^2}$	Period of divisibility <sup>†</sup> $\pmod{m^2}$
$T(m^2)$	Smallest $l \in \mathbb{Z}_+ : F_{z+1}^l \equiv 1 \pmod{m^2}$	$z \times l = k$ [25]

<sup>†</sup> Also called *entry point* of  $m^2$  or *rank of apparition* of  $m^2$ .

For example, from Table 2,  $C(m^2)$  for  $m = 7, 11, 13$ :

$$F_{9+5} \equiv F_9 \pmod{7^2} \text{ and } F_{9+8} \equiv F_9 \pmod{11^2},$$

$$F_{10+5} \equiv F_5 \pmod{11^2} \text{ and } F_{15+5} \equiv F_5 \pmod{13^2};$$

$D(m^2)$  for  $m = 2, F_6 \equiv 0 \pmod{2^2}, z = 6$ , and  $T(m^2)$  for  $m = 2, F_5 \equiv 1 \pmod{2^2}, z = 1$ .

Table 5.  $F_n \pmod{n^2}$ ,  $2 < n \leq 20$  from [27]

	$p$	<b>2</b>	<b>3</b>	<b>5</b>	<b>7</b>	<b>11</b>	<b>13</b>	<b>17</b>	<b>19</b>	<b>Comments</b>	
$n$	$F_n$	$p^2$	<b>4</b>	<b>9</b>	<b>25</b>	<b>49</b>	<b>121</b>	<b>169</b>	<b>289</b>		<b>361</b>
<b>3</b>	<b>2</b>		2	2	2	2	2	2	2	2	$F_3 \equiv 2 \pmod{2^2}$
<b>4</b>	<b>3</b>		3	3	3	3	3	3	3	3	$F_4 \equiv 3 \pmod{2^2}$
<b>5</b>	<b>5</b>		1	5	5	5	5	5	5	5	$F_5 \equiv 1 \pmod{2^2}$
<b>6</b>	<b>8</b>		0	8	8	8	8	8	8	8	$F_6 \equiv 0 \pmod{2^2}$
<b>7</b>	<b>13</b>		1	1	13	13	13	13	13	13	
<b>8</b>	<b>21</b>		1	3	21	21	21	21	21	21	
<b>9</b>	<b>34</b>		2	7	9	34	34	34	34	34	
<b>10</b>	<b>55</b>		3	11	15	6	55	55	55	55	
<b>11</b>	<b>89</b>		1	8	14	40	89	89	89	89	
<b>12</b>	<b>144</b>		0	0	19	46	13	144	144	144	$F_{12} \equiv 0 \pmod{2^2}$ $F_{12} \equiv 0 \pmod{3^2}$
<b>13</b>	<b>233</b>		1	2	8	37	112	64	233	233	
<b>14</b>	<b>377</b>		1	8	2	34	14	39	88	16	$F_{14} \equiv F_9 \pmod{7^2}$
<b>15</b>	<b>610</b>		2	7	10	22	5	103	32	249	$F_{15} \equiv F_5 \pmod{11^2}$
<b>16</b>	<b>987</b>		3	6	12	7	19	142	120	265	
<b>17</b>	<b>1597</b>		1	4	22	29	34	76	152	153	$F_{17} \equiv F_9 \pmod{11^2}$
<b>18</b>	<b>2584</b>		0	1	9	36	43	49	272	57	$F_{18} \equiv 0 \pmod{2^2}$
<b>19</b>	<b>4181</b>		1	5	31	16	67	125	135	210	
<b>20</b>	<b>6765</b>		1	6	15	3	110	5	118	267	$F_{20} \equiv F_5 \pmod{13^2}$

This table can also lead to, for instance, [26]

$$\begin{aligned}
 F_{72} &= 498,454,011,879,264 \\
 &= 12 \times 41,537,834,323,272 \\
 &= 12 \times 12 \times 3,461,486,195,606 \\
 &= 0 \pmod{12^2}
 \end{aligned}$$

and also

$$F_{2z} = F_{z-1}F_z + F_zF_{z+1}.$$

So that we can readily develop Table 1. Furthermore, each of the rows in Table 1, can be further generalized with conjectures; *e.g.*,

$F_6$	$3 \times 2$	$8 \equiv 0 \pmod{2^2}$
$F_{12}$	$6 \times 2$	$144 \equiv 0 \pmod{2^2}$
$F_{18}$	$9 \times 2$	$2584 \equiv 0 \pmod{2^2}$
$F_{24}$	$12 \times 2$	$4638 \equiv 0 \pmod{2^2}$
$F_{6k}$	$k \in \mathbb{N}$	$F_{6k} \equiv 0 \pmod{2^2}$

$F_{12}$	$3 \times 4$	$144 \equiv 0 \pmod{3^2}$
$F_{24}$	$6 \times 4$	$4638 \equiv 0 \pmod{3^2}$
$F_{36}$	$9 \times 4$	$14,930,352 \equiv 0 \pmod{3^2}$
$F_{48}$	$12 \times 4$	$4,807,526,976 \equiv 0 \pmod{3^2}$
$F_{12k}$	$k \in \mathbb{N}$	$F_{12k} \equiv 0 \pmod{3^2}$

Also of relevance to this section is Table 6 which includes part of Singmaster [30], and which warrants further investigation in relation to the aims of this paper, too.  $x_n$  is the least non-negative residue of  $u_n \pmod{m}$  where  $\{u_n\}$  is the second order Jacobsthal sequence [22]. Obviously,  $x_n = u_n, n < m$ , when the relevant recurrence equation holds, and after which the relevant recurrence congruence holds.

Table 6. Some square moduli congruence emerging patterns with Jacobsthal numbers

$n$	0	1	2	3	4	5	6	7	8	9	10
$u_n \cdot$	1	1	3	5	11	21	43	85	171	341	683
$x_n \pmod{2^2}$	1	1	3	1	3	1	3	1	3	1	3
$x_n \pmod{2 \cdot 2^2}$	1	1	3	5	3	5	3	5	3	1	3
$x_n \pmod{4^2}$	1	1	3	5	11	5	11	5	11	5	11
$x_n \pmod{5^2}$	1	1	3	5	11	21	18	10	21	16	8

### 3 s-Lattices

Somewhat similar conjectures could be made with what are called  $s$ -lattice recurrence relations developed by the late John Crothers [10, 11]. His aim was to test some unsolved number theory problems, particularly in the context of phyllotaxis, on a Fibonacci spiral intersected by concentric circles rather than the usual Cartesian rectangular grids. This has become more fashionable with the recent success of Douady *et al.* [13] and the spiral examples. This was partly built on the work of the Bravais brothers [5] almost two centuries ago. This also followed on, in parts, some relevant aspects of the work of Alan Turing himself [32], Donald Coxeter [9], Heiko Harboth [21], Marcellus Waddill [34], Colin Paul Spears *et al.* [31], Turner and Shannon [33] and James Franklin [15].

The term  $S$ -lattice means a simple integer coordinate lattice in two variables defined as

- a solution set to any irreducible linear homogeneous congruence where  $m > 1$ ,  $a, b, m$  are pairwise relatively prime in  $ax + by \equiv 0 \pmod{m}$ , and
- a subgroup of  $Z^2$  that is closed under coordinate addition and that is generated by both integer basis vector pairs  $\{(r, 1), ((m, 0))\}$  and

$$\{(1, 5), ((0, m))\}, 0 < r < m, 0 < s < m, rs \equiv 1 \pmod{m},$$

in which the reference to “simple” is derived from a unique correspondence between any such lattice and two finite simple continued fractions, and the definition appeals to the notion of a lattice with a more geometrical interpretation as a subgroup of coordinates of  $Z^2$  under coordinate addition, as distinct from simply being a solution set to a linear congruence.

Bases vector pairs of an  $s$ -lattice  $S$ , modulo 7 (for example) are defined in a somewhat similar pattern to that of Sophie Germain [7]. For example, a row basis vector pair of  $S$  is  $\{(r, 1), (m, 0)\} = \{(3, 1), (7, 0)\}$ . For possibilities of extending the continued fraction algorithm via the Jacobi Perron Algorithm [28]. An informal overview of some key aspects of this theory follows.

**Theorem 4.** If the second order recursive sequence  $\{x_i\}$  defined by the initial conditions  $x_{-1} = 0$ ,  $x_0 = 1$ , and for  $i = 1, 2, \dots, n$ , by

$$x_i = a_i x_{i-1} + x_{i-2} \quad (3.1)$$

and the second order recursive sequence  $\{y_i\}$  defined by the initial conditions  $y_n = 0$ ,  $y_{n-1} = 1$ , and for  $i = n-2, n-3, \dots, 0, -1$  by

$$y_i = a_{i+2} y_{i+1} + y_{i+2} \quad (3.2)$$

where  $a_2, a_3, \dots, a_{n-1} > 0, a_1, a_n > 1$ , then for  $I = 0, 1, \dots, n-1$ , the vector division transformations

$$E_i = a_i E_{i-1} + E_{i-2} \quad (3.3)$$

where

$$E_i = \{(x_k, (-1)^k y_k) \mid n \text{ odd}, (x_k, (-1)^{k+1} y_k) \mid n \text{ even}\}, \quad (3.4)$$

hold over the  $s$ -lattice in two variables  $S(x_{n-1}^+, x_n)$ , where

$$x_n = y_n \quad (3.5)$$

and where

$$S(x_{n-1}^+, x_n) = \left\{ S \left( \frac{y_0^+}{x_n} \right), n \text{ odd}, S \left( \frac{y_0^-}{x_n} \right), n \text{ even} \right\}, \quad (3.5)$$

in which the symbol  $S(x_{n-1}^+, x_n)$  refers to  $S$  being generated by the basis vector pair

$$\left\{ (x_{n-1}, 1), (x_n, 0) \right\}, 0 < x_{n-1} < \frac{x_n}{2}$$

and is referred to as the row generator form of  $S$ , and the expression  $S \left( \frac{y_0^+}{x_n} \right)$ , if  $n$  is odd, means that  $S$  is generated by basis vector pairs

$$\left\{ 1, y_0, (0, x_n) \right\}, 0 < y_0 < \frac{x_n}{2},$$

and is referred to as the column generator form of  $S$ , and the expression  $S \left( \frac{y_0^-}{x_n} \right)$ , if  $n$  is even, means that  $S$  is generated by basis vector pairs

$$\left\{ 1, y_0, (0^+, x_n) \right\}, \frac{-x_n}{2} < y_0 < 0.$$

**Example 3.** Equations (3.7) give the vector division transformation over a lattice  $S$  generated by the basis vector pair  $\{(17, 1), (65, 0)\}$ , which is denoted by  $S(17, 65)$ . The number of partial quotients is odd ( $= 5$ ). Hence, by Theorem 4,  $S$  has a positive column generator of

$y_0 = 23 : 0 < 23 < \left\lfloor \frac{65}{2} \right\rfloor$  akin to a Euclidean algorithm in this grid:

$$\begin{array}{r}
(65, 0) = 3(17, 1) + (14, -3) \\
(17, 1) = 1(14, -3) + (3, 4) \\
(14, -3) = 4(3, 4) + (2, -19) \\
(3, 4) = 1(2, -19) + (1, 23) \\
(2, -19) = 2(1, 23) + (0, -65).
\end{array}
\quad \left. \vphantom{\begin{array}{l} \\ \\ \\ \\ \\ \end{array}} \right| (3.7)$$

Equations (3.7a) and (3.7b) show the sets of division transformations corresponding to the  $x$  and  $y$  components respectively of the vector division transformations of Equations (3.7). The partial quotients in (3.7a) are in reverse order to those in (3.7b).

$$\begin{array}{r}
65 = 3 \times 17 + 14 \\
17 = 1 \times 14 + 3 \\
14 = 4 \times 3 + 2 \\
3 = 1 \times 2 + 1 \\
2 = 2 \times 1 + 0
\end{array}
\quad \left. \vphantom{\begin{array}{l} \\ \\ \\ \\ \\ \end{array}} \right| (3.7a)$$

and

$$\begin{array}{r}
65 = 2 \times 23 + 19 \\
23 = 1 \times 19 + 4 \\
19 = 4 \times 4 + 3 \\
4 = 1 \times 3 + 1 \\
3 = 3 \times 1 + 0
\end{array}
\quad \left. \vphantom{\begin{array}{l} \\ \\ \\ \\ \\ \end{array}} \right| (3.7b)$$

Here we see some conjectures illustrated, such as: a coset  $C$  of a simple integer coordinate lattice  $S \pmod{m^2}$  in two variables is a solution set to any irreducible linear non-homogeneous congruence:

$$ax + by \equiv c \pmod{m^2} \quad (3.8)$$

“Simple” here implies a unique correspondence between any such lattice and two finite simple continued fractions, one with respect to the forward sense of the partial quotients and the other with respect to the reverse sense of these partial quotients as in (3.7a) and (3.7b) above.

An abstract theoretical geometry is framed in terms of how our human internal modelling seems to work. In particular, by using a Fibonacci connection we can eventually perceive how all rational numbers may be represented by a single lattice through a division transformation array. Thus, it is now known that there are exceptions to the rule that atoms combine in simple proportions to form molecules; there are quasicrystals [6], which do not conform to the Bravais packing rules! In relation to phyllotaxis, bacterial communities seem to be elaborately interwoven by communication links akin to biological neural networks [4]. How and what we perceive are complicated by our notion of continuity as understood by Euclid and in which Bell [3] referred to time as a “flowing river” which seems rudimentary in comparison with the modern mathematical notion of a ‘dense space continuum’. Mathematical modelling of the environment is based upon a human perception of the environment, but we need to explore geometries which might model how a molecule ‘models’ its environment in order to function.

## 4 Concluding comments

An apparently open, but related question [22], seems to be: how many fundamental solutions does the Pell equation

$$u^2 - 5v^2 = -4r^2 \quad (4.1)$$

have if  $m \mid r$  where  $m = \prod_{i=1}^t p_i^{a_i}$  with  $a_i \geq 1$  and  $p_i \equiv \mp 1 \pmod{10}$  for each  $i$ ? A conjecture is that if  $r(m^2)$  is the number of divisors of  $m^2$ , then the number of fundamental solutions of (4.1) is given by

$$\mp r + r\sqrt{5} \text{ and } \mp 4r + 2r\sqrt{5}. \quad (4.2)$$

Similar studies have been carried out in great detail in Nagell [27], and da Fonseca *et al.* [14] also conclude with some related open problems.

## References

- [1] Alladi, K., & Shannon, A. G. (1977). On a property of Farey Fibonacci fractions. *The Fibonacci Quarterly*, 15(2), 153–155.
- [2] Baier, S., Lynch, S. B., & Zhao, L. (2019). A lower bound for the large sieve with square moduli. *Bulletin of the Australian Mathematical Society*, 100(2), 225–229.
- [3] Bell, E. T. (1937). *Men of Mathematics*. New York: Simon and Schuster.
- [4] Ben-Jacob, R., & Cohen, I. (1997). Cooperative formation of bacterial patterns. In: Shapiro, J. A., & Dworkin, M. (eds). *Bacteria as Multicellular Organisms*. New York: Oxford University Press, 394–416.
- [5] Bravais, L., & Bravais, A. (1837). Essai sur la disposition des feuilles curvisériées. *Annales des sciences naturelles. Botanique et biologie*. 7: 42–110, 193–221, 291–348; and 8: 11–42.
- [6] Cahn, J. W., Shechtman, D., & Gratias, D. (1986). Indexing of icosahedral crystals. *Journal of Materials Research*, 1(1), 13–26.
- [7] del Centina, A. (2007). Unpublished manuscripts of Sophie Germain and a reevaluation of her work on Fermat’s Last Theorem. *Archive for History of Exact Sciences*, 62(4), 349–392.
- [8] Collings, S. N. (1970). Fibonacci numbers. *Mathematics Teaching*, 52, 23.
- [9] Coxeter, H. S. M. (1953). The golden section, phyllotaxis, and Wythoff’s game. *Scripta Mathematica*, 19, 135–143.
- [10] Crothers, J. N. (2013). An introduction to simple modular lattices. *Advanced Studies in Contemporary Mathematics*, 23(3), 637–653.
- [11] Crothers, J. N., & Shannon, A. G. (2016). *Quantics: Diophantine Congruences in Special Fibonacci Lattices*. Ahmedabad: Sara Books.

- [12] Daykin, D. E., & Dresel, L. A. G. (1970). Factorization of Fibonacci numbers. *The Fibonacci Quarterly*, 8(1), 23–30.
- [13] Douady, S., Dumais, J., Golé, C., & Pick, N. (2024). *Do Plants Know Math? Unwinding the Story of Plant Spirals, from Leonardo da Vinci to Now*. Princeton and Oxford: Princeton University Press.
- [14] da Fonseca, C. M., Kizilates, C., Saraiva, P., & Shannon, A. G. (2025). Generalized Leonardo numbers. *Logic Journal of the IGPL*, 33(6), DOI: 10.1093/jigpal/jzaf005.
- [15] Franklin, J. (2014). *An Aristotelian Realist Philosophy of Mathematics: Mathematics as the Science of Quantity and Structure*. New York: Palgrave Macmillan.
- [16] Gica, A. (2022). Congruences modulo the square of a prime for sums containing Fibonacci numbers. *The Fibonacci Quarterly*, 60(3), 243–255.
- [17] Gillespie, F. S. (1992). A generalization of Kummer’s and related results. *The Fibonacci Quarterly*, 30(4), 349–367.
- [18] Halton, J. H. (1966). On the divisibility properties of Fibonacci numbers. *The Fibonacci Quarterly*, 4(3), 217–240.
- [19] Halton, J. H. (1967). Some properties associated with square Fibonacci numbers. *The Fibonacci Quarterly*, 5(4), 347–355.
- [20] Halupczok, K. (2012). A new bound for the large sieve inequality with power moduli. *International Journal of Number Theory*, 8(3), 689–695.
- [21] Harboth, H. (1991). Fibonacci representations of graphs. In: Bergum, G. E., Philippou, A. N., & Horadam, A. F. (eds.). *Applications of Fibonacci Numbers, Volume 4*. Dordrecht: Kluwer, 133–138.
- [22] Jacobsthal, E. E. (1919–1920). Fibonaccische Polynome und Kreisteilungs-gleichungen, Sitzungsber. *Berliner Mathematische Gesellschaft*, 17, 43–57.
- [23] Gandhi, K. R. R. (2011). Divisibility properties of Fibonacci numbers. *South Asian Journal of Mathematics*, 1(3), 140–144.
- [24] Koshy, T. (2011). *Fibonacci and Lucas Numbers with Applications*. John Wiley and Sons.
- [25] Long, C., Cohen, G. L., & Langtry, T., & Shannon, A. G. (1993). Arithmetic sequences and second order recurrences. In: Bergum, G. E., Philippou, A. N., & Horadam, A. F. (eds.). *Applications of Fibonacci Numbers, Volume 5*. Dordrecht: Kluwer, 449–457.
- [26] Matomäki, K. (2009). A note on primes of the form  $p = aq^2 + 1$ . *Acta Arithmetica*, 137(2), 133–137.
- [27] Nagell, T. (1981). *Introduction to Number Theory*. New York: Chelsea, Ch. 6.
- [28] Shannon, A. G., & Bernstein, L. (1973). The Jacobi–Perron algorithm and the algebra of recursive sequences. *Bulletin of the Australian Mathematical Society*, 8(2), 261–277.

- [29] Shannon, A. G., Horadam, A. F., & Collings, S. N. (1974). Some Fibonacci congruences. *The Fibonacci Quarterly*, 12(4), 351–354.
- [30] Singmaster, D. (1970). Some counterexamples and problems on linear recurrence relations. *The Fibonacci Quarterly*, 8(3), 264–267, 279.
- [31] Spears, C. P., Bickell-Johnson, M., & Yan, J. J. (2010). Fibonacci phyllotaxis by asymmetric cell division: Zeckendorf and Wythoff trees. *Congressus Numerantium*, 201, 257–271.
- [32] Turing, A. (1992). Morphogenesis. In: Saunders, P. T. (ed.). *The Collected Works of A. M. Turing*. Amsterdam: North Holland.
- [33] Turner, J. C., & Shannon, A. G. (1998). Introduction to a Fibonacci geometry. In: Bergum, G. E., Philippou, A. N., & Horadam, A. F. (eds.). *Applications of Fibonacci Numbers, Volume 7*. Dordrecht: Kluwer, 435–448.
- [34] Waddill, M. E. (1996). Properties of a  $k$ -order linear recursive sequence modulo  $m$ . In: Bergum, G. E., Philippou, A. N., & Horadam, A. F. (eds.). *Applications of Fibonacci Numbers, Volume 6*. Dordrecht: Kluwer, 505–519.
- [35] Wall, D. D. (1960). Fibonacci series modulo  $m$ . *The American Mathematical Monthly*, 67(6), 525–532.
- [36] Zhao, L. (2004). Large sieve inequality for characters to square moduli. *Acta Arithmetica*, 112(3), 297–308.