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Matrix representations of an extended family of Horadam polynomials

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Abstract: In this article, we describe the concept of d-Horadam polynomials, which is a generalization of the classical Horadam polynomials $W_n(0, W_1(x); p(x), q(x))$. We derive essential characteristics of these newly defined polynomials, such as their generating function, a Binet-type expression, several combinatorial relations, and summation identities. Subsequently, we construct the infinite matrix of d-Horadam polynomials, which can be represented as a Riordan array. Moreover, by employing the Riordan approach, we establish two distinct decompositions of the infinite Pascal matrix involving the d-Horadam polynomials.

Keywords: Horadam polynomials, d-Horadam polynomials, Pascal matrix, Riordan matrix.
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1 Introduction

Over the years, integer sequences and their polynomial counterparts have attracted considerable attention due to their natural appearances and broad range of applications in different areas. Among them, the Fibonacci sequence is one of the most extensively studied examples. It is determined by the recurrence relation

$$F_{n+1} = F_n + F_{n-1}, \quad n \geq 1,$$

subject to the initial conditions $F_0 = 0$ and $F_1 = 1$. In 1883, Catalan introduced the Fibonacci polynomials, which satisfy the recurrence relation

$$F_{n+1}(x) = xF_n(x) + F_{n-1}(x), \quad n \geq 1,$$

with the starting values $F_0(x) = 0$ and $F_1(x) = 1$.

For integers $p, q, r, s \in \mathbb{Z}$, the Horadam sequence $\{W_n\}$, denoted by $W_n = W_n(r, s; p, q)$, is defined through the linear recurrence relation

$$W_{n+1} = pW_n + qW_{n-1}, \quad n \geq 1,$$

together with the initial conditions $W_0 = r$ and $W_1 = s$ (see, e.g., [6, 7]). Extending this framework, the Horadam polynomials are introduced as $W_n(x) = W_n\big(W_0(x), W_1(x); p(x), q(x)\big)$ and are governed by the recurrence relation

$$W_{n+1}(x) = p(x)W_n(x) + q(x)W_{n-1}(x), \quad n \geq 1,$$

with prescribed initial polynomials $W_0(x)$ and $W_1(x)$ (see, e.g., [21]).

Nalli and Haukkanen [17] introduced the $h(x)$-Fibonacci polynomials, which are specified by the recurrence relation

$$F_{h,n+1}(x) = h(x)F_{h,n}(x) + F_{h,n-1}(x), \quad n \geq 1,$$

with the initial conditions $F_{h,0}(x) = 0$ and $F_{h,1}(x) = 1$. Subsequently, Lee and Asci [15] defined the (p, q)-Fibonacci polynomials via

$$F_{p,q,n+1}(x) = p(x)F_{p,q,n}(x) + q(x)F_{p,q,n-1}(x), \quad n \geq 1,$$

subject to $F_{p,q,0}(x) = 0$ and $F_{p,q,1}(x) = 1$. Let $d \in \mathbb{Z}^+$ and let $p_i(x)$ $(i = 1, 2, \dots, d+1)$ be real polynomials. In [19], Sadaoui and Krelifa extended the notion of (p, q)-Fibonacci polynomials to the class of d-Fibonacci polynomials, defined through

$$F_{n+1}(x) = p_1(x)F_n(x) + p_2(x)F_{n-1}(x) + \cdots + p_{d+1}(x)F_{n-d}(x), \quad n \geq 1,$$

with initial conditions $F_n(x) = 0$ for $n \leq 0$ and $F_1(x) = 1$. Kuloğlu and Özkan [12] introduced the d-Tribonacci polynomials, regarded as a higher-order generalization of the classical Tribonacci polynomials, and investigated their properties. Furthermore, Özimamoğlu [18] defined the d-Oresme polynomials and derived some of their properties.

Lawden [14] proposed the $n \times n$ lower-triangular Pascal matrix denoted by $P = (p_{i,j})$, which is defined entrywise as

$$p_{i,j} = \begin{cases} 0, & \text{for } i < j, \\ \binom{i-1}{j-1}, & \text{for } i \geq j, \end{cases}$$

where the indices satisfy $i, j = 1, 2, \dots, n$ (see also [2, 4] for related discussions). The infinite dimensional Pascal matrix, denoted by P, is expressed as

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{1}$$

Pascal matrices find numerous applications across probability theory, numerical computation, surface modeling, and combinatorial analysis. In [1, 3, 25], the researchers examined the linear algebraic structures associated with the generalized Pascal functional matrix, the classical Pascal matrix, and the extended Pascal matrix, respectively. In [16] and [24], two distinct factorizations of the Pascal matrix incorporating the Fibonacci matrix were established. Moreover, numerous works on Pascal matrices exist in the literature [11, 22, 23].

In [20], Shapiro and collaborators presented the Riordan group in the following manner.

Let $i, j \in \mathbb{N}$, and consider an infinite matrix $A = (a_{i,j})$ with complex entries. For a fixed $k \in \mathbb{N}$, denote the generating function of the k-th column of A by

$$c_k(z) = \sum_{m=0}^{\infty} a_{m,k} z^m.$$

The matrix A is referred to as a Riordan matrix, denoted by $A = (g(z), f(z))$, if its column generating functions satisfy

$$c_k(z) = g(z)\,[f(z)]^k,$$

where

$$g(z) = \sum_{m=0}^{\infty} g_m z^m \quad \text{and} \quad f(z) = \sum_{m=0}^{\infty} f_m z^m,\ f(0) = 0.$$

Let $\mathcal{R}$ denote the collection of all Riordan matrices. The set $\mathcal{R}$ forms a group under matrix multiplication, denoted by $*$, and is referred to as the Riordan group. The following properties of the Riordan group hold:

- For a column vector $C(z)$, the action of a Riordan matrix is given by

$$(g(z), f(z)) * C(z) = g(z)\, C(f(z)).$$

- The product of two Riordan matrices satisfies

$$(g(z), f(z)) * (h(z), l(z)) = \big(g(z)\, h(f(z)),\, l(f(z))\big).$$

- The identity element of $\mathcal{R}$ is

$$i_{\mathcal{R}} = (1, z).$$

- The inverse of a Riordan matrix $(g(z), f(z))$ is given by

$$(g(z), f(z))^{-1} = \left(\frac{1}{g(\bar{f}(z))}, \bar{f}(z) \right),$$

  where $\bar{f}(z)$ denotes the compositional inverse of $f(z)$.

The Riordan group possesses a wide range of applications. As discussed in [20], three notable applications include Euler's problem concerning the king's walks, various binomial and inverse identities, and expansions of the Bessel–Neumann type. In addition, Cheon *et al.* [5] introduced a generalization of the Lucas polynomial sequence derived from a Riordan array constructed using weighted Delannoy numbers.

The organization of this study is outlined as follows: in Section 2, we introduce the d-Horadam polynomials, which constitute a novel extension of the classical Horadam polynomials $W_n(0, W_1(x); p(x), q(x))$. Various properties of the d-Horadam polynomials are established, including their generating functions, Binet-type formulas, combinatorial identities, and summation expressions. We also define the matrix W_d and demonstrate that its powers generate the d-Horadam polynomials. In Section 3, the infinite d-Horadam polynomial matrix is constructed, which forms a Riordan matrix. Subsequently, two distinct factorizations of the infinite Pascal matrix incorporating the d-Horadam polynomials are derived.

2 d-Horadam polynomials

In this part, we describe a novel extension of the classical Horadam polynomials

$$W_n(0, W_1(x); p(x), q(x)).$$

Definition 2.1. *Let $d \in \mathbb{Z}^+$ and let $s_i(x)$ be real polynomials for $i = 1, 2, \dots, d+1$. The d-Horadam polynomials $W_n^{(d)}(x)$ are defined recursively by*

$$W_{n+1}^{(d)}(x) = s_1(x)W_n^{(d)}(x) + s_2(x)W_{n-1}^{(d)}(x) + \cdots + s_{d+1}(x)W_{n-d}^{(d)}(x), \quad n \geq 1, \tag{2}$$

with initial conditions

$$W_n^{(d)}(x) = 0 \textit{ for } n \leq 0, \quad \textit{and } W_1^{(d)}(x).$$

For the sake of simplicity, we denote $W_n^{(d)}$ and s_i in place of $W_n^{(d)}(x)$ and $s_i(x)$, respectively, in (2). Table 1 displays several initial terms of the d-Horadam polynomials.

Table 1. Selected values of the d-Horadam polynomials.

n	$W_n^{(d)}$
2	$s_1 W_1^{(d)}$
3	$(s_1^2 + s_2) W_1^{(d)}$
4	$(s_1^3 + 2s_1s_2 + s_3) W_1^{(d)}$
5	$(s_1^4 + 3s_1^2s_2 + 2s_1s_3 + s_2^2 + s_4) W_1^{(d)}$
6	$(s_1^5 + 4s_1^3s_2 + 3s_1^2s_3 + 3s_1s_2^2 + 2s_1s_4 + 2s_2s_3 + s_5) W_1^{(d)}$

In (2), by setting $s_1 = p(x)$, $s_2 = q(x)$, and $s_i = 0$ for $i = 3, 4, \dots, d+1$, we recover the classical Horadam polynomials $W_n^{(d)} = W_n(x)$. Hence, the d-Horadam polynomials represent a natural generalization of the traditional Horadam polynomials. Furthermore, Table 2 lists specific instances of d-Horadam polynomials corresponding to particular choices of d.

Table 2. Special cases of $W_n^{(d)}$ such that $W_1^{(d)} = 1$ and $s_i = 0$ for $i = 3, 4, \ldots, d+1$.

s_1	s_2	d-Horadam polynomials $W_n^{(d)}$
x	1	Fibonacci polynomials $F_n(x)$
x	-1	Vieta–Fibonacci polynomials $V_n(x)$ [8]
$2x$	1	Pell polynomials $P_n(x)$ [10]
1	$2x$	Jacobsthal polynomials $J_n(x)$ [9]
$3x$	-2	Mersenne polynomials $M_n(x)$ [13]

From (2), the characteristic equation associated with the d-Horadam polynomials is expressed as

$$v^{d+1} - s_1 v^d - \cdots - s_{d+1} = 0. \tag{3}$$

Theorem 2.1. *For $n \geq d$, it follows that*

$$\begin{aligned} v^n = W_{n-d+1}^{(d)} v^d &+ \left(s_2 W_{n-d}^{(d)} + \cdots + s_{d+1} W_{n-2d+1}^{(d)}\right) v^{d-1} \\ &+ \left(s_3 W_{n-d}^{(d)} + \cdots + s_{d+1} W_{n-2d+2}^{(d)}\right) v^{d-2} + \cdots + s_{d+1} W_{n-d}^{(d)}. \end{aligned} \tag{4}$$

Proof. The proof of the theorem proceeds by mathematical induction on n. For the base case $n = 1$, the validity of equation (4) is immediate. Suppose that equation (4) holds for $n = i$. We then demonstrate that it also holds for $n = i + 1$. Combining (2) with the characteristic equation (3), we derive

$$\begin{aligned} v^{i+1} &= v^i v \\ &= W_{i-d+1}^{(d)} \left(s_1 v^d + \cdots + s_{d+1}\right) + \left(s_2 W_{i-d}^{(d)} + \cdots + s_{d+1} W_{i-2d+1}^{(d)}\right) v^d \\ &+ \left(s_3 W_{i-d}^{(d)} + \cdots + s_{d+1} W_{i-2d+2}^{(d)}\right) v^{d-1} + \cdots + s_{d+1} W_{i-d}^{(d)} v \\ &= \left(s_1 W_{i-d+1}^{(d)} + \cdots + s_{d+1} W_{i-2d+1}^{(d)}\right) v^d + \left(s_2 W_{i-d+1}^{(d)} + \cdots + s_{d+1} W_{i-2d+2}^{(d)}\right) v^{d-1} \\ &+ \cdots + \left(s_d W_{i-d+1}^{(d)} + s_{d+1} W_{i-d}^{(d)}\right) v + s_{d+1} W_{i-d+1}^{(d)}. \end{aligned}$$

□

Theorem 2.2. *The generating function corresponding to the d-Horadam polynomials is expressed as*

$$g^{(d)}(v) = \frac{W_1^{(d)} v}{1 - s_1 v - s_2 v^2 - \cdots - s_{d+1} v^{d+1}}.$$

Proof. We have

$$\begin{aligned} g^{(d)}(v) &= \sum_{i=0}^{\infty} W_i^{(d)} v^i \\ &= W_0^{(d)} + W_1^{(d)} v + W_2^{(d)} v^2 + \cdots + W_n^{(d)} v^n + \cdots. \end{aligned} \tag{5}$$

If we multiply the equation (5) by $s_1 v, s_2 v^2, \ldots, s_{d+1} v^{d+1}$, respectively, then we obtain the following equations.

$$\begin{aligned} s_1 v g^{(d)}(v) &= s_1 W_0^{(d)} v + s_1 W_1^{(d)} v^2 + s_1 W_2^{(d)} v^3 + \cdots \\ s_2 v^2 g^{(d)}(v) &= s_2 W_0^{(d)} v^2 + s_2 W_1^{(d)} v^3 + s_2 W_2^{(d)} v^4 + \cdots \\ &\vdots \\ s_{d+1} v^{d+1} g^{(d)}(v) &= s_{d+1} W_0^{(d)} v^{d+1} + s_{d+1} W_1^{(d)} v^{d+2} + \cdots . \end{aligned}$$

Upon performing the required computations and applying (2), we obtain

$$g^{(d)}(v)\left(1 - s_1 v - s_2 v^2 - \cdots - s_{d+1} v^{d+1}\right) = W_0^{(d)} + \left(W_1^{(d)} - s_1 W_0^{(d)}\right) v$$

and so

$$g^{(d)}(v) = \frac{W_1^{(d)} v}{1 - s_1 v - s_2 v^2 - \cdots - s_{d+1} v^{d+1}}. \qquad \square$$

Let the roots of (3) be $\{\gamma_1, \gamma_2, \ldots, \gamma_{d+1}\}$. That is, we have

$$\frac{W_1^{(d)} v}{1 - s_1 v - s_2 v^2 - \cdots - s_{d+1} v^{d+1}} = \sum_{i=1}^{d+1} \frac{c_i}{1 - \gamma_i v}.$$

From Theorem 2.2, we obtain

$$\sum_{i=0}^{\infty} W_i^{(d)} v^i = \sum_{i=1}^{d+1} c_i \sum_{k=0}^{\infty} \gamma_i^k v^k.$$

Consequently, the Binet-type formula for $W_n^{(d)}$ can be stated in the following corollary.

Corollary 2.1. *The Binet-type expression for the d-Horadam polynomials is given by*

$$W_n^{(d)} = \sum_{k=1}^{d+1} c_k \gamma_k^n.$$

In particular, the multinomial coefficients allow for an explicit representation of the d-Horadam polynomials.

Theorem 2.3. *For $n \geq 0$, it follows that*

$$W_{n+1}^{(d)} = W_1^{(d)} \sum_{\substack{k_1, k_2, \ldots, k_{d+1} \\ k_1 + 2k_2 + \cdots + (d+1)k_{d+1} = n}} \binom{k_1 + k_2 + \cdots + k_{d+1}}{k_1, k_2, \ldots, k_{d+1}} s_1^{k_1} s_2^{k_2} \cdots s_{d+1}^{k_{d+1}}.$$

Proof. By Theorem 2.2, we find that

$$\begin{aligned} &\sum_{k=0}^{\infty} W_{k+1}^{(d)} v^k \\ &= \frac{W_1^{(d)}}{1 - s_1 v - s_2 v^2 - \cdots - s_{d+1} v^{d+1}} \\ &= W_1^{(d)} \sum_{k=0}^{\infty} \left(s_1 v + s_2 v^2 + \cdots + s_{d+1} v^{d+1}\right)^k \end{aligned}$$

$$= W_1^{(d)} \sum_{k=0}^{\infty} \left[\sum_{k_1+k_2+\cdots+k_{d+1}=k} \binom{k}{k_1, k_2, \ldots, k_{d+1}} s_1^{k_1} \cdots s_{d+1}^{k_{d+1}} \right] v^{k_1+2k_2+\cdots+(d+1)k_{d+1}}$$

$$= W_1^{(d)} \sum_{k=0}^{\infty} \left[\sum_{\substack{k_1,k_2,\ldots,k_{d+1} \\ k_1+2k_2+\cdots+(d+1)k_{d+1}=k}} \binom{k_1+k_2+\cdots+k_{d+1}}{k_1, k_2, \ldots, k_{d+1}} s_1^{k_1} \cdots s_{d+1}^{k_{d+1}} \right] v^k. \qquad \square$$

Theorem 2.4. *The summation formula of the d-Horadam polynomials is expressed as*

$$\sum_{i=0}^{\infty} W_i^{(d)} = \frac{W_1^{(d)}}{1 - s_1 - s_2 - \cdots - s_{d+1}}.$$

Proof. We have

$$\sum_{i=0}^{\infty} W_i^{(d)} = W_0^{(d)} + W_1^{(d)} + \cdots + W_n^{(d)} + \cdots . \tag{6}$$

Multiplying (6) by $s_1, s_2, \ldots, s_{d+1}$, respectively, hence we get

$$\begin{aligned} s_1 \sum_{i=0}^{\infty} W_i^{(d)} &= s_1 W_0^{(d)} + s_1 W_1^{(d)} + \cdots + s_1 W_n^{(d)} + \cdots \\ s_2 \sum_{i=0}^{\infty} W_i &= s_2 W_0^{(d)} + s_2 W_1^{(d)} + \cdots + s_2 W_n^{(d)} + \cdots \\ &\vdots \\ s_{d+1} \sum_{i=0}^{\infty} W_i^{(d)} &= s_{d+1} W_0^{(d)} + s_{d+1} W_1^{(d)} + \cdots + s_{d+1} W_n^{(d)} + \cdots . \end{aligned}$$

By performing the required calculations and applying (2), one can readily obtain

$$\sum_{i=0}^{\infty} W_i^{(d)} (1 - s_1 - s_2 - \cdots - s_{d+1}) = W_0^{(d)} + \left(W_1^{(d)} - s_1 W_0^{(d)} \right)$$

and so

$$\sum_{i=0}^{\infty} W_i^{(d)} = \frac{W_1^{(d)}}{1 - s_1 - s_2 - \cdots - s_{d+1}}. \qquad \square$$

Nalli and Haukkanen [17] proposed the matrix

$$Q_h(x) = \begin{bmatrix} h(x) & 1 \\ 1 & 0 \end{bmatrix},$$

and Lee and Asci [15] introduced the matrix

$$Q_{p,q}(x) = \begin{bmatrix} p(x) & q(x) \\ 1 & 0 \end{bmatrix},$$

which serves as the Fibonacci matrix

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Subsequently, Sadaoui and Krelifa [19] introduced the matrix

$$Q_d = \begin{bmatrix} p_1(x) & p_2(x) & \cdots & p_d(x) & p_{d+1}(x) \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}_{(d+1)\times(d+1)}. \tag{7}$$

Now we define the matrix W_d which is a generalization of the Q_d in (7) as follows:

$$W_d = \begin{bmatrix} W_1^{(d)} s_1 & W_1^{(d)} s_2 & \cdots & W_1^{(d)} s_d & W_1^{(d)} s_{d+1} \\ W_1^{(d)} & 0 & 0 & \cdots & 0 \\ 0 & W_1^{(d)} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & W_1^{(d)} & 0 \end{bmatrix}_{(d+1)\times(d+1)}. \tag{8}$$

This explicitly implies that the determinant of W_d is given by the polynomial $\left(W_1^{(d)}\right)^{d+1}(-1)^d s_{d+1}$. The matrix representation of $W_n^{(d)}$ is given in the next theorem.

Theorem 2.5. *For $n \geq 1$, it follows that*

$$W_d^n = \begin{bmatrix} W_{n+1}^{(d)} & s_2 W_n^{(d)} + \cdots + s_{d+1} W_{n-d+1}^{(d)} & \cdots & s_{d+1} W_n^{(d)} \\ W_n^{(d)} & s_2 W_{n-1}^{(d)} + \cdots + s_{d+1} W_{n-d}^{(d)} & \cdots & s_{d+1} W_{n-1}^{(d)} \\ \vdots & \vdots & \ddots & \vdots \\ W_{n-d+1}^{(d)} & s_2 W_{n-d}^{(d)} + \cdots + s_{d+1} W_{n-2d+1}^{(d)} & \cdots & s_{d+1} W_{n-d}^{(d)} \end{bmatrix}.$$

Proof. The proof of the theorem is carried out by induction on n. For the base case $n = 1$, using (2) and (8), we obtain

$$W_d = \begin{bmatrix} W_2^{(d)} & s_2 W_1^{(d)} + \cdots + s_{d+1} W_{2-d}^{(d)} & \cdots & s_{d+1} W_1^{(d)} \\ W_1^{(d)} & s_2 W_0^{(d)} + \cdots + s_{d+1} W_{1-d}^{(d)} & \cdots & s_{d+1} W_0^{(d)} \\ \vdots & \vdots & \ddots & \vdots \\ W_{2-d}^{(d)} & s_2 W_{1-d}^{(d)} + \cdots + s_{d+1} W_{2-2d}^{(d)} & \cdots & s_{d+1} W_{1-d}^{(d)} \end{bmatrix}.$$

Assume that the induction hypothesis holds for $n = t$. That is,

$$W_d^t = \begin{bmatrix} W_{t+1}^{(d)} & s_2 W_t^{(d)} + \cdots + s_{d+1} W_{t-d+1}^{(d)} & \cdots & s_{d+1} W_t^{(d)} \\ W_t^{(d)} & s_2 W_{t-1}^{(d)} + \cdots + s_{d+1} W_{t-d}^{(d)} & \cdots & s_{d+1} W_{t-1}^{(d)} \\ \vdots & \vdots & \ddots & \vdots \\ W_{t-d+1}^{(d)} & s_2 W_{t-d}^{(d)} + \cdots + s_{d+1} W_{t-2d+1}^{(d)} & \cdots & s_{d+1} W_{t-d}^{(d)} \end{bmatrix}.$$

We show that it is true for $n = t + 1$. Hence we have

$$W_d^{t+1} = W_d^t W_d = \begin{bmatrix} W_{t+2}^{(d)} & s_2 W_{t+1}^{(d)} + \cdots + s_{d+1} W_{t-d+2}^{(d)} & \cdots & s_{d+1} W_{t+1}^{(d)} \\ W_{t+1}^{(d)} & s_2 W_t^{(d)} + \cdots + s_{d+1} W_{t-d+1}^{(d)} & \cdots & s_{d+1} W_t^{(d)} \\ \vdots & \vdots & \ddots & \vdots \\ W_{t-d+2}^{(d)} & s_2 W_{t-d+1}^{(d)} + \cdots + s_{d+1} W_{t-2d+2}^{(d)} & \cdots & s_{d+1} W_{t-d+1}^{(d)} \end{bmatrix}.$$

This completes the proof. □

For $n,k\geq 0$, we get $W_d^n W_d^k = W_d^{n+k}$. Furthermore, it is observed that the $(1,1)$-entry of the matrix W_d^{n+k} is obtained as the product of the first row of W_d^n with the first column of W_d^k. Consequently, by Theorem 2.5, we arrive at the following corollary.

Corollary 2.2. *Let $n,t\geq 0$. It then follows that*

$$\begin{aligned} W_1^{(d)}W_{n+t+1}^{(d)} &= W_{n+1}^{(d)}W_{t+1}^{(d)} + s_2 W_n^{(d)}W_t^{(d)} \\ &+s_3\left(W_{n-1}^{(d)}W_t^{(d)} + W_n^{(d)}W_{t-1}^{(d)}\right) \\ &+s_4\left(W_{n-2}^{(d)}W_t^{(d)} + W_{n-1}^{(d)}W_{t-1}^{(d)} + W_n^{(d)}W_{t-2}^{(d)}\right) \\ &\vdots \\ &+s_{d+1}\left(W_{n-d+1}^{(d)}W_t^{(d)} + \cdots + W_n^{(d)}W_{t-d+1}^{(d)}\right). \end{aligned}$$

Theorem 2.6. *For $n\geq 0$, then we get*

$$W_{(d+1)n}^{(d)} = \sum_{\substack{k_1,k_2,\ldots,k_{d+1}\\ (d+1)k_1+dk_2+\cdots+k_{d+1}=n}} \binom{k_1+k_2+\cdots+k_{d+1}}{k_1,k_2,\ldots,k_{d+1}} s_1^{k_1}s_2^{k_2}\cdots s_{d+1}^{k_{d+1}} W_{n-(k_1+k_2+\cdots+k_{d+1})}^{(d)}. \quad (9)$$

Proof. We represent T as the expression appearing on the right-hand side of (9). Utilizing the Binet-type expression provided in Corollary 2.1 along with the characteristic equation given in (3), we deduce that for all $n\geq 2$,

$$\begin{aligned} T &= \sum_{\substack{k_1,k_2,\ldots,k_{d+1}\\ (d+1)k_1+dk_2+\cdots+k_{d+1}=n}} \binom{k_1+k_2+\cdots+k_{d+1}}{k_1,k_2,\ldots,k_{d+1}} s_1^{k_1}s_2^{k_2}\cdots s_{d+1}^{k_{d+1}} \sum_{i=1}^{d+1} c_i\gamma_i^{n-(k_1+k_2+\cdots+k_{d+1})} \\ &= \sum_{\substack{k_1,k_2,\ldots,k_{d+1}\\ (d+1)k_1+dk_2+\cdots+k_{d+1}=n}} \binom{k_1+k_2+\cdots+k_{d+1}}{k_1,k_2,\ldots,k_{d+1}} s_1^{k_1}s_2^{k_2}\cdots s_{d+1}^{k_{d+1}} \sum_{i=1}^{d+1} c_i\gamma_i^{dk_1+(d-1)k_2+\cdots+k_d} \\ &= c_1 \sum_{\substack{k_1,k_2,\ldots,k_{d+1}\\ (d+1)k_1+dk_2+\cdots+k_{d+1}=n}} \binom{k_1+k_2+\cdots+k_{d+1}}{k_1,k_2,\ldots,k_{d+1}} \left(s_1\gamma_1^d\right)^{k_1}\left(s_2\gamma_1^{d-1}\right)^{k_2}\cdots\left(s_{d+1}\right)^{k_{d+1}} \\ &\vdots \\ &+ c_{d+1} \sum_{\substack{k_1,k_2,\ldots,k_{d+1}\\ (d+1)k_1+dk_2+\cdots+k_{d+1}=n}} \binom{k_1+k_2+\cdots+k_{d+1}}{k_1,k_2,\ldots,k_{d+1}} \left(s_1\gamma_{d+1}^d\right)^{k_1}\left(s_2\gamma_{d+1}^{d-1}\right)^{k_2}\cdots\left(s_{d+1}\right)^{k_{d+1}} \\ &= c_1\left(s_1\gamma_1^d + s_2\gamma_1^{d-1}+\cdots+s_{d+1}\right)^n + \cdots + c_{d+1}\left(s_1\gamma_{d+1}^d + s_2\gamma_{d+1}^{d-1}+\cdots+s_{d+1}\right)^n \\ &= \sum_{i=1}^{d+1} c_i\gamma_i^{(d+1)n} \\ &= W_{(d+1)n}^{(d)}. \end{aligned}$$

□

Theorem 2.7. *For all $n \geq 0$, it follows that*

$$\sum_{t=0}^{n} \binom{n}{t} (-2s_{d+1})^t W^{(d)}_{(d+1)(n-t)}$$
$$= \sum_{\substack{k_1,k_2,\dots,k_{d+1} \\ (d+1)k_1+dk_2+\dots+k_{d+1}=n}} \binom{k_1+\dots+k_{d+1}}{k_1,\dots,k_{d+1}} s_1^{k_1} s_2^{k_2} \cdots \left(-s_{d+1}^{k_{d+1}}\right) W^{(d)}_{n-(k_1+k_2+\dots+k_{d+1})}. \tag{10}$$

Proof. Let R denote the expression on the right-hand side of (10). By following the reasoning presented in the proof of Theorem 2.6, we derive

$$\begin{aligned}
R &= c_1 \left(s_1\gamma_1^d + \dots + s_d\gamma_1 - s_{d+1}\right)^n + \dots + c_{d+1} \left(s_1\gamma_{d+1}^d + \dots + s_d\gamma_{d+1} - s_{d+1}\right)^n \\
&= \sum_{j=1}^{d+1} c_j \left(\gamma_j^{d+1} - 2s_{d+1}\right)^n \\
&= \sum_{j=1}^{d+1} c_j \sum_{t=0}^{n} \binom{n}{t} \gamma_j^{(d+1)(n-t)} (-2s_{d+1})^t \\
&= \sum_{t=0}^{n} \binom{n}{t} (-2s_{d+1})^t W^{(d)}_{(d+1)(n-t)}.
\end{aligned}$$

□

3 The infinite matrix of d-Horadam polynomials

In this part, we present a novel infinite matrix, referred to as the infinite matrix of d-Horadam polynomials. Subsequently, we give two distinct factorizations of the infinite Pascal matrix.

Definition 3.1. *The infinite matrix of d-Horadam polynomials is defined as*

$$\mathcal{H}_d = \begin{bmatrix} W_1^{(d)} & 0 & 0 & \cdots \\ s_1 W_1^{(d)} & W_1^{(d)} & 0 & \cdots \\ (s_1^2+s_2) W_1^{(d)} & s_1 W_1^{(d)} & \ddots & \ddots \\ (s_1^3+2s_1s_2+s_3) W_1^{(d)} & (s_1^2+s_2) W_1^{(d)} & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix}, \tag{11}$$

where the first column entries of $\mathcal{H}_d$ are given by $(\mathcal{H}_d)_{i,1} = W_i^{(d)}$ for all $i \in \mathbb{Z}^+$.

According to (11), the infinite matrix of d-Horadam polynomials can be represented as

$$\mathcal{H}_d = \begin{bmatrix} W_1^{(d)} & 0 & 0 & \cdots \\ W_2^{(d)} & W_1^{(d)} & 0 & \cdots \\ W_3^{(d)} & W_2^{(d)} & \ddots & \ddots \\ W_4^{(d)} & W_3^{(d)} & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

Consequently, the matrix $\mathcal{H}_d$ constitutes a Riordan matrix, as its first column is

$$\left(W_1^{(d)}, s_1W_1^{(d)}, \left(s_1^2+s_2\right) W_1^{(d)}, \left(s_1^3+2s_1s_2+s_3\right) W_1^{(d)}, \dots\right)^T,$$

we derive the following corollary from Theorem 2.2.

Corollary 3.1. *The generating function corresponding to the first column of the matrix $\mathcal{H}_d$ is*

$$g_{\mathcal{H}_d}(v)=\frac{W_1^{(d)}}{1-s_1v-s_2v^2-\cdots-s_{d+1}v^{d+1}}.$$

Within the matrix $\mathcal{H}_d$, for all $n\geq 1$ and $j\in\mathbb{Z}^+$, we obtain

$$(\mathcal{H}_d)_{n+1,j} \;=\; s_1(\mathcal{H}_d)_{n,j}+s_2(\mathcal{H}_d)_{n-1,j}+\cdots+s_{d+1}(\mathcal{H}_d)_{n-d,j}$$

according to Definition (2). Consequently, by setting $f_{\mathcal{H}_d}(v)=v$, we obtain the following corollary for the matrix $\mathcal{H}_d$.

Corollary 3.2. *The infinite matrix of d-Horadam polynomials, denoted by $\mathcal{H}_d$, is*

$$\begin{aligned}\mathcal{H}_d &= (g_{\mathcal{H}_d}(v),f_{\mathcal{H}_d}(v))\\ &= \left(\frac{W_1^{(d)}}{1-s_1v-s_2v^2-\cdots-s_{d+1}v^{d+1}},v\right).\end{aligned}$$

For all $i,j\in\mathbb{Z}^+$, the infinite matrix Φ_d can be defined as

$$(\Phi_d)_{i,j}=\frac{1}{W_1^{(d)}}\sum_{t=0}^{d+1}-s_t\binom{i-t-1}{j-1},$$

where $s_0=-1$. Hence we have

$$\Phi_d=\begin{bmatrix}\frac{1}{W_1^{(d)}} & 0 & 0 & 0 & \cdots\\ \frac{1-s_1}{W_1^{(d)}} & \frac{1}{W_1^{(d)}} & 0 & 0 & \cdots\\ \frac{1-s_1-s_2}{W_1^{(d)}} & \frac{2-s_1}{W_1^{(d)}} & \frac{1}{W_1^{(d)}} & \ddots & \ddots\\ \frac{1-s_1-s_2-s_3}{W_1^{(d)}} & \frac{3-2s_1-s_2}{W_1^{(d)}} & \frac{3-s_1}{W_1^{(d)}} & \ddots & \ddots\\ \vdots & \vdots & & \ddots & \ddots\end{bmatrix}. \tag{12}$$

We now provide the first factorization of the infinite Pascal matrix, as stated in the next theorem.

Theorem 3.1. *Let $\mathcal{H}_d$ denote the infinite matrix of d-Horadam polynomials, and let Φ_d be the infinite matrix defined in* (12)*. Then, we obtain*

$$P=\mathcal{H}_d*\Phi_d,$$

where P denotes the infinite Pascal matrix as defined in (1).

Proof. Based on the definition of the infinite Pascal matrix, it follows that

$$P=\left(\frac{1}{1-v},\frac{v}{1-v}\right). \tag{13}$$

The generating function corresponding to the first column of the matrix Φ_d is

$$\begin{aligned}
g_{\Phi_d}(v) &= \frac{1}{W_1^{(d)}} + \left(\frac{1-s_1}{W_1^{(d)}}\right)v + \left(\frac{1-s_1-s_2}{W_1^{(d)}}\right)v^2 + \left(\frac{1-s_1-s_2-s_3}{W_1^{(d)}}\right)v^3 + \cdots \\
&= \frac{1}{W_1^{(d)}}(1+v+v^2+v^3+\cdots) - \frac{s_1}{W_1^{(d)}}\left(v+v^2+v^3+\cdots\right) \\
&\quad -\frac{s_2}{W_1^{(d)}}\left(v^2+v^3+v^4+\cdots\right) - \cdots - \frac{s_{d+1}}{W_1^{(d)}}\left(v^{d+1}+v^{d+2}+v^{d+3}+\cdots\right) \\
&= \frac{1}{W_1^{(d)}}\left(\frac{1}{1-v} - \frac{s_1 v}{1-v} - \frac{s_2 v^2}{1-v} - \cdots - \frac{s_{d+1}v^{d+1}}{1-v}\right) \\
&= \frac{1}{W_1^{(d)}}\left(\frac{1-s_1v-s_2v^2-\cdots-s_{d+1}v^{d+1}}{1-v}\right). \qquad (14)
\end{aligned}$$

Conversely, the generating function corresponding to the second column of the matrix Φ_d is

$$\begin{aligned}
g_{\Phi_d}(v)f_{\Phi_d}(v) &= \frac{1}{W_1^{(d)}}v + \left(\frac{2-s_1}{W_1^{(d)}}\right)v^2 + \left(\frac{3-2s_1-s_2}{W_1^{(d)}}\right)v^3 + \cdots \\
&= \frac{1}{W_1^{(d)}}(v+2v^2+3v^3+\cdots) - \frac{s_1 v}{W_1^{(d)}}(v+2v^2+3v^3+\cdots) \\
&\quad -\frac{s_2v^2}{W_1^{(d)}}(v+2v^2+3v^3+\cdots) - \cdots - \frac{s_{d+1}v^{d+1}}{W_1^{(d)}}(v+2v^2+3v^3+\cdots) \\
&= \frac{1}{W_1^{(d)}}(1-s_1v-s_2v^2-\cdots-s_{d+1}v^{d+1})(v+2v^2+3v^3+\cdots) \\
&= \frac{1}{W_1^{(d)}}\left(\frac{1-s_1v-s_2v^2-\cdots-s_{d+1}v^{d+1}}{1-v}\right)\left(\frac{v}{1-v}\right),
\end{aligned}$$

and so by the equation (14) we derive

$$f_{\Phi_d}(v) = \frac{v}{1-v}. \qquad (15)$$

Therefore by using the equations (14) and (15), we can obtain

$$\begin{aligned}
\Phi_d &= (g_{\Phi_d}(v), f_{\Phi_d}(v)) \\
&= \left(\frac{1}{W_1^{(d)}}\left(\frac{1-s_1v-s_2v^2-\cdots-s_{d+1}v^{d+1}}{1-v}\right), \frac{v}{1-v}\right). \qquad (16)
\end{aligned}$$

Finally from the definition of the matrix multiplication for the Riordan group, if we multiply the matrices $\mathcal{H}_d$ in Corollary 3.2, (13) and Φ_d in (16), respectively, the proof is completed. □

For all $i, j \in \mathbb{Z}^+$, the infinite matrix Ψ_d can be defined as

$$(\Psi_d)_{i,j} = \frac{1}{W_1^{(d)}}\sum_{t=0}^{d+1} -s_t\binom{i-1}{j+t-1},$$

where $s_0 = -1$. Hence we have

$$\Psi_d = \begin{bmatrix} \frac{1}{W_1^{(d)}} & 0 & 0 & 0 & \cdots \\ \frac{1-s_1}{W_1^{(d)}} & \frac{1}{W_1^{(d)}} & 0 & 0 & \cdots \\ \frac{1-2s_1-s_2}{W_1^{(d)}} & \frac{2-s_1}{W_1^{(d)}} & \frac{1}{W_1^{(d)}} & \ddots & \ddots \\ \frac{1-3s_1-3s_2-s_3}{W_1^{(d)}} & \frac{3-3s_1-s_2}{W_1^{(d)}} & \frac{3-s_1}{W_1^{(d)}} & \ddots & \ddots \\ \vdots & \vdots & & \ddots & \ddots \end{bmatrix}. \tag{17}$$

We now present the second factorization of the infinite Pascal matrix, as established in the next corollary.

Corollary 3.3. *Let $\mathcal{H}_d$ denote the infinite matrix of d-Horadam polynomials, and let Ψ_d be the infinite matrix defined in* (17)*. Then, we obtain*

$$P = \Psi_d * \mathcal{H}_d,$$

where P represents the infinite Pascal matrix as defined in (1)*.*

The inverse of $\mathcal{H}_d$ in Corollary 3.2 can be readily derived using the definition of the inverse element in the Riordan group, as stated in the following corollary.

Corollary 3.4. *The inverse of the infinite matrix of d-Horadam polynomials, denoted by $\mathcal{H}_d$, is*

$$\mathcal{H}_d^{-1} = \left(\frac{1 - s_1 v - s_2 v^2 - \cdots - s_{d+1} v^{d+1}}{W_1^{(d)}}, v \right).$$

4 Conclusions

In this study, we extend the known Horadam polynomials $W_n(0, W_1(x); p(x), q(x))$ by introducing a generalized form, referred to as the d-Horadam polynomials $W_n^{(d)}$. We derive their generating function, Binet-type formula, combinatorial identities, and summation expressions. A new matrix W_d is defined, whose powers generate the polynomials $W_n^{(d)}$. Furthermore, we describe the infinite matrix of d-Horadam polynomials, $\mathcal{H}_d$, which constitutes a Riordan matrix. Employing the Riordan method, we factorize the infinite Pascal matrix P and thereby obtain two distinct factorizations involving $\mathcal{H}_d$. Finally, we establish the Riordan characterization for the matrix inverse $\mathcal{H}_d^{-1}$.

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