

Two new arithmetic operations

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Abstract: Two arithmetic operations are introduced and some of their properties are studied. It is proved that they can be operations of semi-groups, but not of monoids. It is shown that their reverse operations are not one-valued. Some connections between the new operations and the well-known arithmetic functions φ, ψ, σ are shown.

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1 Introduction

Over the set of the natural numbers \mathbb{N} different operations are introduced and their properties are studied. Here, two new arithmetic operations are introduced and some of their properties are studied.



2 Main results

Let everywhere in the paper, the natural numbers m, n have the following canonical representations:

$$m = \prod_{i=1}^{k+l} p_i^{\alpha_i} \quad \text{and} \quad n = \prod_{i=k+1}^{k+l+r} p_i^{\beta_i}.$$

Let us define the following two arithmetic operations:

$$m *_1 n = \prod_{i=1}^k p_i^{\alpha_i} \prod_{i=k+l+1}^{k+l+r} p_i^{\beta_i} \quad (1)$$

and

$$m *_2 n = \prod_{i=1}^k p_i^{\alpha_i} \prod_{i=k+1}^{k+l} p_i^{\min(\alpha_i, \beta_i)} \prod_{i=k+l+1}^{k+l+r} p_i^{\beta_i}. \quad (2)$$

Obviously,

$$m *_1 m = 1,$$

$$m *_2 m = m.$$

The well-known function “greatest common divisor” is defined by:

$$(m, n) = \prod_{i=k+1}^{k+l} p_i^{\min(\alpha_i, \beta_i)}.$$

Therefore,

$$m *_2 n = (m *_1 n) \cdot (m, n).$$

Theorem 1. For every two natural numbers m, n such that $(m, n) = 1$,

$$m *_1 n = m *_2 n. \quad (3)$$

Proof. From $(m, n) = 1$ it follows that the above m and n have the reduced form:

$$m = \prod_{i=1}^k p_i^{\alpha_i}$$

and

$$n = \prod_{i=k+l+1}^{k+l+r} p_i^{\beta_i}.$$

Then, from (1) and (2) it follows (3). □

Theorem 2. $\langle \mathbb{N}, *_1 \rangle$ is a commutative semi-group, but not a monoid and for each $m \in \mathbb{N}$ there are infinitely many $n \in \mathbb{N}$ such that $m *_1 n = n *_1 m = 1$.

Proof. It is obvious that for every two $m, n \in \mathbb{N} : m *_1 n \in \mathbb{N}$. Let $a, b, c, d, e, f, g \in \mathbb{N}$, let every two of them are co-prime, and

$$\begin{aligned} m &= adeg, \\ n &= bdfg, \\ s &= cefg. \end{aligned}$$

Then

$$\begin{aligned} (m *_1 n) *_1 s &= (adeg *_1 bdfg) *_1 cefg \\ &= abef *_1 cefg \\ &= abcg \\ &= adeg *_1 bcde \\ &= adeg *_1 (bdfg *_1 cefg) \\ &= m *_1 (n *_1 s). \end{aligned}$$

Let $m \in \mathbb{N}$. Then

$$m *_1 1 = 1 *_1 m = 1 \neq m,$$

i.e. $\langle \mathbb{N}, *_1 \rangle$ cannot be a monoid, but for

$$m = \prod_{i=1}^k p_i^{\alpha_i}$$

and for arbitrary $\beta_1, \dots, \beta_k \geq 1$:

$$m *_1 \prod_{i=1}^k p_i^{\beta_i} = 1.$$

Therefore, there are infinitely many reverse elements. The equality

$$m *_1 n = n *_1 m$$

is obvious. □

Theorem 3. $\langle \mathbb{N}, *_2 \rangle$ is a commutative semi-group, but not a monoid.

Proof. As above, it is obvious that for every two $m, n \in \mathbb{N} : m *_2 n \in \mathbb{N}$. Let $a, b, c, d_m, d_n, e_m, e_s, f_n, f_s, g_m, g_n, g_s \in \mathbb{N}$, let every two of them be co-prime, and

$$\begin{aligned} m &= ad_m e_m g_m, \\ n &= bd_n f_n g_n, \\ s &= ce_s f_s g_s. \end{aligned}$$

Then

$$\begin{aligned} (m *_2 n) *_2 s &= (ad_m e_m g_m *_2 bd_n f_n g_n) *_2 ce_s f_s g_s \\ &= ab \min(d_m, d_n) e_m f_n \min(g_m, g_n) *_1 ce_s f_s g_s \\ &= abc \min(d_m, d_n) \min(e_m, e_s) \min(f_n, f_s) \min(g_m, g_n, g_s) \\ &= ad_m e_m g_m *_2 bcd_n e_s \min(f_n, f_s) \min(g_n, g_s) \\ &= ad_m e_m g_m *_2 (bd_n f_n g_n *_2 ce_s f_s g_s) \\ &= m *_2 (n *_2 s). \end{aligned}$$

Similarly to the proof of Theorem 2, $m *_2 1 = 1 *_2 m = 1 \neq m$, but now for each m there is no n such that $m *_2 n = 1$. Finally, the equality

$$m *_2 n = n *_2 m$$

is obvious. □

Theorem 4. For every three natural numbers m, n, r :

$$\begin{aligned} m^r *_1 n^r &= (m *_1 n)^r, \\ m^r *_2 n^r &= (m *_2 n)^r. \end{aligned}$$

Proof. In the first case, we have:

$$\begin{aligned} m^r *_1 n^r &= \left(\prod_{i=1}^{k+l} p_i^{\alpha_i} \right)^r *_1 \left(\prod_{i=k+1}^{k+l+r} p_i^{\beta_i} \right)^r \\ &= \prod_{i=1}^{k+l} p_i^{r\alpha_i} *_1 \prod_{i=k+1}^{k+l+r} p_i^{r\beta_i} \\ &= \prod_{i=1}^k p_i^{r\alpha_i} \prod_{i=k+l+1}^{k+l+r} p_i^{r\beta_i} \\ &= \left(\prod_{i=1}^k p_i^{\alpha_i} \prod_{i=k+l+1}^{k+l+r} p_i^{\beta_i} \right)^r \\ &= (m *_1 n)^r. \end{aligned}$$

In the second case, we have:

$$\begin{aligned} m^r *_2 n^r &= \left(\prod_{i=1}^{k+l} p_i^{\alpha_i} \right)^r *_2 \left(\prod_{i=k+1}^{k+l+r} p_i^{\beta_i} \right)^r \\ &= \prod_{i=1}^{k+l} p_i^{r\alpha_i} *_2 \prod_{i=k+1}^{k+l+r} p_i^{r\beta_i} \\ &= \prod_{i=1}^k p_i^{r\alpha_i} \prod_{i=k+1}^{k+l} p_i^{r \min(\alpha_i, \beta_i)} \prod_{i=k+1}^{k+l+r} p_i^{r\beta_i} \\ &= \left(\prod_{i=1}^k p_i^{\alpha_i} \prod_{i=k+1}^{k+l} p_i^{\min(\alpha_i, \beta_i)} \prod_{i=k+1}^{k+l+r} p_i^{\beta_i} \right)^r \\ &= (m *_2 n)^r. \end{aligned}$$

This proves the theorem. □

If we would like to define the reverse operation \circ_j of the operation $*_j$ for $j = 1, 2$, i.e., from $m *_j n = t$ to follow that $m = t \circ_j n$, then the operation \circ_j must not be one-valued. Really, for

$$t = \prod_{i=1}^k p_i^{\alpha_i} \prod_{i=k+l+1}^{k+l+r} p_i^{\beta_i}$$

in the case $j = 1$, m can have the form

$$m = \prod_{i=1}^k p_i^{\alpha_i} \prod_{i=k+1}^{k+l} p_i^{\gamma_i},$$

for every $\gamma_1, \dots, \gamma_k \geq 1$, and in the case $j = 2$, m can have the form

$$m = \prod_{i=1}^k p_i^{\alpha_i} \prod_{i=k+1}^{k+l} p_i^{\beta_i + \gamma_i},$$

for every $\gamma_1, \dots, \gamma_k \geq 0$.

Let us define for the above m :

$$RF(m) = \prod_{i=1}^k p_i^{\alpha_i - 1}.$$

In [1–5], this function was called *restrictive factor*.

Theorem 5. For every two natural numbers m, n :

$$RF(m *_1 n) = RF(m) *_1 RF(n), \quad (4)$$

$$RF(m *_2 n) = RF(m) *_2 RF(n). \quad (5)$$

Proof. Let the natural numbers m and n be given. Then:

$$\begin{aligned} RF(m *_1 n) &= RF\left(\prod_{i=1}^k p_i^{\alpha_i} \prod_{i=k+1}^{k+l} p_i^{\alpha_i} *_1 \prod_{i=k+1}^{k+l} p_i^{\beta_i} \prod_{i=k+l+1}^{k+l+r} p_i^{\beta_i}\right) \\ &= RF\left(\prod_{i=1}^k p_i^{\alpha_i} \prod_{i=k+l+1}^{k+l+r} p_i^{\beta_i}\right) \\ &= \prod_{i=1}^k p_i^{\alpha_i - 1} \prod_{i=k+l+1}^{k+l+r} p_i^{\beta_i - 1} \\ &= \prod_{i=1}^k p_i^{\alpha_i - 1} \prod_{i=k+1}^{k+l} p_i^{\alpha_i - 1} *_1 \prod_{i=k+1}^{k+l} p_i^{\beta_i - 1} \prod_{i=k+l+1}^{k+l+r} p_i^{\beta_i - 1} \\ &= RF(m) *_1 RF(n) \end{aligned}$$

and

$$\begin{aligned} RF(m *_2 n) &= RF\left(\prod_{i=1}^k p_i^{\alpha_i} \prod_{i=k+1}^{k+l} p_i^{\alpha_i} *_2 \prod_{i=k+1}^{k+l} p_i^{\beta_i} \prod_{i=k+l+1}^{k+l+r} p_i^{\beta_i}\right) \\ &= RF\left(\prod_{i=1}^k p_i^{\alpha_i} \prod_{i=k+1}^{k+l} p_i^{\min(\alpha_i, \beta_i)} \prod_{i=k+l+1}^{k+l+r} p_i^{\beta_i}\right) \\ &= \prod_{i=1}^k p_i^{\alpha_i - 1} \prod_{i=k+1}^{k+l} p_i^{\min(\alpha_i, \beta_i) - 1} \prod_{i=k+l+1}^{k+l+r} p_i^{\beta_i - 1} \\ &= \prod_{i=1}^k p_i^{\alpha_i - 1} \prod_{i=k+1}^{k+l} p_i^{\min(\alpha_i - 1, \beta_i - 1)} \prod_{i=k+l+1}^{k+l+r} p_i^{\beta_i - 1} \\ &= \prod_{i=1}^k p_i^{\alpha_i - 1} \prod_{i=k+1}^{k+l} p_i^{\alpha_i - 1} *_2 \prod_{i=k+1}^{k+l} p_i^{\beta_i - 1} \prod_{i=k+l+1}^{k+l+r} p_i^{\beta_i - 1} \\ &= RF(m) *_2 RF(n). \end{aligned}$$

Therefore (4) and (5) are valid. □

Let $\gamma(n)$ denote the product of distinct prime divisors of n (“core of n ”). From (1) it follows that

$$\gamma((m, n)) = \prod_{i=k+1}^{k+l} p_i$$

and

$$\gamma(m *_1 n) = \prod_{i=1}^k p_i \prod_{i=k+l+1}^{k+l+r} p_i = \frac{\prod_{i=1}^{k+l+r} p_i}{\prod_{i=k+1}^{k+l} p_i} = \frac{\gamma(mn)}{\gamma((m, n))} = \frac{\gamma(m) \cdot \gamma(n)}{(\gamma((m, n)))^2}. \quad (6)$$

Thus, one has

$$\gamma(m *_1 n) \leq \gamma(m) \cdot \gamma(n) \leq \gamma(mn),$$

with equality for $(m, n) = 1$.

On the other hand, one has

$$m *_1 n = \frac{mn}{\prod_{i=k+1}^{k+l} p_i^{\alpha_i + \beta_i}}, \quad (7)$$

and, as $\alpha_i + \beta_i \geq 2$ ($i = k + 1, \dots, k + l$), we get from (6) that

$$m *_1 n \leq \frac{m \cdot n}{(\gamma((m, n)))^2},$$

with equality for $(m, n) = 1$.

Let $\varphi(n)$ be the Euler totient function. Then,

$$\begin{aligned} \varphi(m) &= m \cdot \prod_{i=1}^k \left(1 - \frac{1}{p_i}\right) \prod_{i=k+1}^{k+l} \left(1 - \frac{1}{p_i}\right), \\ \varphi(n) &= n \cdot \prod_{i=k+1}^{k+l} \left(1 - \frac{1}{p_i}\right) \prod_{i=k+l+1}^{k+l+r} \left(1 - \frac{1}{p_i}\right). \end{aligned}$$

So, using (6) and (7) we obtain:

$$\varphi(m *_1 n) = (m *_1 n) \prod_{i=1}^k \left(1 - \frac{1}{p_i}\right) \prod_{i=k+l+1}^{k+l+r} \left(1 - \frac{1}{p_i}\right)$$

and

$$\begin{aligned} \varphi(m)\varphi(n) &= mn \cdot \prod_{i=1}^k \left(1 - \frac{1}{p_i}\right) \prod_{i=k+1}^{k+l} \left(1 - \frac{1}{p_i}\right)^2 \prod_{i=k+l+1}^{k+l+r} \left(1 - \frac{1}{p_i}\right) \\ &= (m *_1 n) \prod_{i=k+1}^{k+l} p_i^{\alpha_i + \beta_i} \prod_{i=1}^k \left(1 - \frac{1}{p_i}\right) \prod_{i=k+1}^{k+l} \left(1 - \frac{1}{p_i}\right)^2 \prod_{i=k+l+1}^{k+l+r} \left(1 - \frac{1}{p_i}\right) \\ &= \varphi(m *_1 n) \prod_{i=k+1}^{k+l} p_i^{\alpha_i + \beta_i} \prod_{i=k+1}^{k+l} \left(1 - \frac{1}{p_i}\right)^2 \\ &\geq \varphi(m *_1 n) (m, n)^2 \prod_{i=k+1}^{k+l} \left(1 - \frac{1}{p_i}\right)^2 \\ &= \varphi(m *_1 n) (\varphi((m, n)^2))^2 \\ &= \varphi(m *_1 n) (\varphi(\gamma((m, n))^2))^2. \end{aligned}$$

This becomes an equality when $m = apq, n = bpq$, where a, b are natural numbers such that $(a, b) = 1$ and p, q are prime numbers. With this we have proven the following theorem.

Theorem 6. For every two natural numbers m, n :

$$\varphi(m)\varphi(n) \geq \varphi(m *_1 n)(\varphi((m, n)))^2. \quad (8)$$

In the same manner, we can check that

$$\psi(m)\varphi(n) \geq \psi(m *_1 n)(\psi((m, n)))^2,$$

where ψ is Dedekind's function.

From the well-known inequality

$$\sigma(ab) > a\sigma(b),$$

where $a, b > 1$ are natural numbers and σ is the well-known arithmetic function, it follows that

$$\sigma(mn) > \sigma(m *_2 n) \prod_{i=k+1}^{k+l} p_i^{\alpha_i + \beta_i} \geq \sigma(m *_2 n)(\gamma((m, n)))^2.$$

3 Conclusion

In future, other properties of the newly defined operations will be investigated. A promising direction of research is to identify which other arithmetic functions satisfy equalities similar to (4), (5) and inequalities similar to (8).

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