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The generalized Horadam model and the fundamental sequence of a nonhomogeneous linear recursive relation

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Abstract: This study concerns some properties of the generalized nonhomogeneous Horadam model and its application to four types of known sequences of numbers. Regarding the four sequences of this model, we provide results on combinatorial and linear expressions, and also on the analytical representation of Binet. Our approach is based on results we have developed regarding the general setting of nonhomogeneous linear recursive sequences, especially the fundamental sequence related to their homogeneous part.

Keywords: Generalized nonhomogeneous Horadam model, Nonhomogeneous linear recurrence relation, Fundamental Fibonacci system, Fundamental sequence.
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1 Introduction

Let p, q with $(q \neq 0)$ be two scalars of $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), and consider the sequence $\{GH_n\}_{n\geq 0}$ defined by

$$GH_{n+1} = pGH_n + qGH_{n-1} + HC_{n+1}, \text{for } n \geq 1, \tag{1}$$

where GH_0, GH_1 are the initial data and $\{HC_n\}_{n\geq 0}$ is a given sequence of real or complex numbers. For $HC_n = 0$, for every $n \geq 0$ in (1), we get the Horadam's model (see [19]). Thus, we will call the sequence $\{GH_n\}_{n\geq 0}$ defined by (1) *the generalized nonhomogeneous Horadam model.* In [18], some generalized sequences of Leonardo numbers have been studied, among them we can cite the following three sequences $\{L_n\}_{n\geq 0}$, $\{I_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ defined by

$$L_n = L_{n-1} + L_{n-2} + n, \tag{2}$$

$$I_n = I_{n-1} + I_{n-2} + (-1)^n, \tag{3}$$

$$H_n = H_{n-1} + H_{n-2} + F_n, \tag{4}$$

for every $n \geq 2$, where $L_0 = L_1 = 1,$; $I_0 = I_1 = 1$ and $H_0 = H_1 = -1$, and $\{F_n\}_{n\geq 0}$ is the sequence of usual Fibonacci numbers. Moreover, there is also $\{W_n\}_{n\geq 0}$ the sequence defined by $W_0 = t_0, W_1 = t_1$ and

$$W_n = sW_{n-1} - fW_{n-2} + (s - f - 1)\sum_{j=0}^{k} \alpha_j n^j, \text{ for every } n \geq 2, \tag{5}$$

We can easily show that the expressions (2)–(5) represent special cases of the nonhomogeneous Horadam model (1). On the other side, the nonhomogeneous generalized Horadam model represents a special case of the nonhomogeneous linear recursive sequence defined as follows. Let $a_0, \ldots, a_{r-1}$ $(r \geq 2)$, with $a_{r-1} \neq 0$, be in $\mathbb{R}$ or $\mathbb{C}$, and $\{D_n\}_{n\geq 0}$ a given sequence of $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Consider $\{R_n\}_{n\geq 0}$ the sequence of $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ defined by the nonhomogeneous linear recursive relation of order r

$$R_{n+1} = a_0R_n + a_1R_{n-1} + \cdots + a_{r-1}R_{n-r+1} + D_{n+1}, \text{ for every } n \geq r - 1, \tag{6}$$

where $R_0, \ldots, R_{r-1}$ represent the given initial data. In the following, we refer to $\{R_n\}_{n\geq 0}$ as the sequence (6) and R_n as its general term. Considered as a nonhomogeneous difference equation, Expression (6) represents a fundamental tools in several fields of mathematics (see, for instance, [2,3,8–12,14,16]), as well in applied science and engineering (see, for example, [1,6,15]). In the context of our study, we rely on the concept of the fundamental Fibonacci system (see [4,7,10]), related to the homogeneous part of (6) defined as follows. Let $\{v_n\}_{n\geq 0}$ be the sequence associated to the homogeneous part of (6), namely

$$v_{n+1} = a_0v_n + \cdots + a_{r-1}v_{n-r+1}, \text{ for every } n \geq r - 1, \tag{7}$$

with $v_n = \alpha_n$ for $0 \leq n \leq r - 1$ being arbitrary initial conditions. We associate the family of sequences $\{v_n^{(d)}\}_{n\geq 0}$, indexed by d, with $1 \leq d \leq r$, defined as follows

$$v_{n+1}^{(d)} = a_0v_n^{(d)} + \cdots + a_{r-1}v_{n-r+1}^{(d)}, \tag{8}$$

for $n \geq r - 1$, where $v_n^{(d)} = \delta_{d-1,n}$, for $0 \leq n \leq r - 1$, and $\delta_{s-1,j}$ is the Kronecker symbol.

The set $\{\{v_n^{(d)}\}_{n\geq 0}\ ;\ 1 \leq d \leq r\}$ is called the *fundamental Fibonacci system*. Subsequently, the sequence $\{v_n^{(r)}\}_{n\geq 0}$ will play an important role in the rest of this study and will be called *fundamental sequence*. It is worth noting that since its introduction in the seminal work [10] (see also [4, 17]), the fundamental Fibonacci system has been used recently for establishing several results in various fields, especially in number theory (see, for instance, [4, 7, 17]).

In the present paper, we aim to study the nonhomogeneous generalized Horadam model (1). More precisely, we establish the linear expression and the combinatorial formulas of the general term GH_n, as well as its analytic Binet representations. Since Expressions (2)–(5) are special cases of (1), similar results for the general terms L_n, I_n, H_n and W_n are provided. Our approach for studying the nonhomogeneous generalized Horadam model is based on some new results established for the generalized linear nonhomogeneous difference equations (6). That is, in the general setting (6), we exhibit an explicit formula of R_n only in terms of the fundamental sequence derived from the homogeneous part of Expression (6), and also the sequence $\{D_n\}_{n\geq 0}$. In addition, the combinatorial and the analytic Binet representations for R_n are established. We take advantage of the results of this approach to study the generalized nonhomogeneous Horadam model and its special cases specified previously.

The content of this paper is as follows. Section 2 is devoted to the linear expression of the general term R_n of (6), in terms of the fundamental sequence. In addition, the combinatorial and analytic Binet expressions of R_n are provided. The results from Section 2 are applied in Section 3 to the study of the generalized nonhomogeneous Horadam model (1) and its related special cases (2)–(5). Conclusion and perspectives are presented in Section 4.

For convenience and clarity, in the remainder of this study, the order r is always assumed to be ≥ 2.

2 Linear, combinatorial and analytic Binet expressions of (6)

In this section, we establish that the general term R_n (6) is expressed linearly in terms of the fundamental sequence $\{v_n^{(r)}\}_{n\geq 0}$ and the sequence $\{D_n\}_{n\geq 0}$. Moreover, the combinatorial formula and the analytic Binet representation for R_n are provided through those of the fundamental sequence.

2.1 Linear expression of R_n in terms $\{v_n^{(r)}\}_{n\geq 0}$

It was shown in [4, 17] (see also [7] and the references therein) that each sequence $\{v_n^{(d)}\}_{n\geq 0}$ of the set $\{\{v_n^{(d)}\}_{n\geq 0}\ ;\ 1 \leq d \leq r\}$, namely, a fundamental Fibonacci system, can be expressed in terms of the sequence $\{v_n^{(r)}\}_{n\geq 0}$. Indeed, this statement is made explicit in the following lemma.

Lemma 2.1. [7] *Let $\{v_n\}_{n\geq 0}$ be the sequence* (7) *and $\{\{v^{(d)}\}_{n\geq 0}\ ;\ 1 \leq d \leq r\}$ be its related fundamental Fibonacci system. Then, for every d ($1 \leq d \leq r-1$), we have*

$$v_n^{(d)} = \sum_{t=1}^{d} a_{r-t} v_{n-d+t-1}^{(r)}, \tag{9}$$

for every $n \geq 0$. Especially, we have $v_n^{(1)} = a_{r-1} v_{n-1}^{(r)}$, for every $n \geq 1$.

Let us consider the vector column $\mathbb{W}_n = (v_n, v_{n-1}, \dots, v_{n-r+1})^T$ where Z^T means the transpose of Z, and the $r \times r$ companion matrix is defined by

$$\mathbb{A} = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{r-1} \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}. \tag{10}$$

The recursive relation (7) implies that the vector column $\mathbb{W}_n$ satisfies the matrix equation $\mathbb{W}_{n+1} = \mathbb{A}\mathbb{W}_n$, for every $n \geq r-1$. Therefore, an iterative process implies that we have $\mathbb{W}_{n+r-1} = \mathbb{A}^n \mathbb{W}_{r-1}$, for all $n \geq 0$. Following the results of [10] (see also [7] and the references therein), the entries $a_{js}^{(n)}$ of the matrix $\mathbb{A}^n = (a_{ij}^{(n)})_{0 \leq i, j \leq r-1}$ are stated, using the fundamental Fibonacci system, as follows

$$a_{ij}^{(n)} = v_{n+r-i}^{(r-j+1)}. \tag{11}$$

Consider the two vectors $X_n = (R_n, \dots, R_{n-r+1})^T$ and $Y_n = (D_n, 0, \dots, 0)^T$ for $n \geq r-1$. We can easily verify that Expression (6) leads to the matrix equation $X_{n+1} = \mathbb{A}X_n + Y_{n+1}$, for all $n \geq r-1$, where $\mathbb{A}$ is the former companion matrix given by (10). Therefore, an iterative process permits us to obtain

$$X_n = \mathbb{A}^{n-r+1} X_{r-1} + \sum_{k=r}^{n} \mathbb{A}^{n-k} Y_k, \ \text{ for every } \ n \geq r. \tag{12}$$

The second step consists of providing explicit formula of R_n the general term of the sequence defined by Expression (6) only with help of the two sequences $\{v_n^{(r)}\}_{n \geq 0}$ and $\{D_n\}_{n \geq 0}$. To reach our goal, we combine Expressions (11) and (12). First, Expression (11) allows us to get

$$\mathbb{A}^{n-r+1} X_{r-1} = \begin{pmatrix} \sum_{s=1}^{r} v_n^{(r-s+1)} R_{r-s} \\ \vdots \\ \sum_{s=1}^{r} v_{n-r+1}^{(r-s+1)} R_{r-s} \end{pmatrix} \text{ and } \mathbb{A}^{n-k} Y_k = \begin{pmatrix} v_{n-k+r-1}^{(r-s+1)} D_k \\ \vdots \\ v_{n-k}^{(r-s+1)} D_k \end{pmatrix}.$$

Therefore, using Expression (12) we derive

$$R_n = \sum_{s=1}^{r} v_n^{(r-s+1)} R_{r-s} + \sum_{k=r}^{n} v_{n-k+r-1}^{(r)} D_k, \text{ for every } n \geq r. \tag{13}$$

Expression (13) has been provided under another form in [10, Proposition 1]. Expression (9) shows that we have $v_n^{(r-s+1)} = \sum_{t=1}^{r-s+1} a_{r-t} v_{n-r+s+t-2}^{(r)}$, and by substituting the former formula of $v_n^{(r-s+1)}$ in (13) we arrive to the following main result of this section.

Theorem 2.1. *Let $\{R_n\}_{n \geq 0}$ be the sequence* (6)*, with initial conditions $R_0, \dots, R_{r-1}$, with $r \geq 2$ and $a_{r-1} \neq 0$. Then, the linear form of R_n is exhibited under the form*

$$R_n = \sum_{s=1}^{r} \sum_{t=1}^{r-s+1} a_{r-t} v_{n-r+s+t-2}^{(r)} R_{r-s} + \sum_{k=r}^{n} v_{n-k+r-1}^{(r)} D_k, \tag{14}$$

for any $n \geq r$, where $\{v_n^{(r)}\}_{n \geq 0}$ is the fundamental sequence.

To the best of our knowledge, Expression (14) is not current in the literature. On the other hand, when Expression (6) is seen as a differential equation, it is well known that the general term R_n represents its solution. Furthermore, R_n can be written under the form $R_n = R_n^{\langle hs\rangle} + R_n^{\langle ps\rangle}$, for every $n \geq 0$, where $\{R_n^{\langle hs\rangle}\}_{n\geq 0}$ is the solution of the homogeneous part (7) of (6), and $\{R_n^{\langle ps\rangle}\}_{n\geq 0}$ is known in the current literature as the particular solution of (6), when this expression is considered as a difference equation. Therefore, Expression (14) permits us to formulate the following property.

Proposition 2.1. *Let $\{R_n\}_{n\geq 0}$ be the sequence* (6)*. Then, for every $n \geq r$, we have $R_n^{\langle hs\rangle} + R_n^{\langle ps\rangle}$, where $R_n^{\langle hs\rangle} = \sum_{s=1}^{r}\sum_{t=1}^{r-s+1} a_{r-t}v_{n-r+s+t-2}^{(r)}R_{r-s}$ and $R_n^{\langle ps\rangle} = \sum_{k=r}^{n} v_{n-k+r-1}^{(r)}D_k$, for $n \geq r$, with initial data $R_0, \ldots, R_{r-1}$, and $\{v_n^{(r)}\}_{n\geq 0}$ is the fundamental sequence.*

Expression (14) will allow us to provide the combinatorial and the analytic Binet representations of the general term R_n.

2.2 Combinatorial expression of R_n

In the aim to exhibit the combinatorial expression of the sequence (6), we consider the expression

$$\Omega_r(n) = \sum_{k_0+2k_1+\cdots+rk_{r-1}=n-r} \frac{(k_0+\cdots+k_{r-1})!}{k_0!k_1!\cdots k_{r-1}!} a_0^{k_0}a_1^{k_1}\cdots a_{r-1}^{k_{r-1}}, \tag{15}$$

where the k_i $(0 \leq i \leq r-1)$ are in $\mathbb{N}$, with $\Omega_r(n) = 1$ and $\Omega_r(n) = 0$ for every $n \leq r-1$. It was shown in [13, Theorem 2.5 and Subsection 2.4] that the term v_n of the sequence defined by (7) is formulated in terms of $\Omega_r(n)$ in the following form

$$v_n = u_0\Omega_r(n) + u_1\Omega_r(n-1) + \cdots + u_{r-1}\Omega_r(n-r+1),$$

for any $n \geq r$, where $u_k = a_{r-1}\alpha_k + \cdots + a_k\alpha_{r-1}$ $(0 \leq k \leq r-1)$ and the $\Omega_r(n)$ are as in (15). In addition, we shown that the sequence $\{W_n\}_{n\geq 0}$, with $W_n = \Omega_r(n+1)$, satisfies (7). Furthermore, a direct computation shows that

$$W_n = v_n^{(r)} = \Omega_r(n+1). \tag{16}$$

The substitution of Formula (16) in (14) permits us to obtain the theorem.

Theorem 2.2. *Let $\{R_n\}_{n\geq 0}$ be the sequence given by Expression* (6)*, whose initial conditions are $R_0, \ldots, R_{r-1}$. Then, the combinatorial expression of R_n is formulated as follows*

$$R_n = \sum_{s=1}^{r}\sum_{t=1}^{r-s+1} a_{r-t}\Omega_r(n-r+s+t-1)R_{r-s} + \sum_{k=r}^{n}\Omega_r(n-k+r)D_k, \tag{17}$$

for $n \geq r$, where $\Omega_r(n)$ is given as in (15)*.*

On the other hand, Expression (17) represents a combinatorial solution of (6), viewed as a difference equation. Hence, the sequences $\{R_n^{\langle hs\rangle}\}_{n\geq 0}$ and $\{R_n^{\langle ps\rangle}\}_{n\geq 0}$ related to the homogeneous part (7) of (6), and the particular sequence satisfying (6), respectively. Hence, we have

$$R_n^{\langle hs\rangle} = \sum_{s=1}^{r}\sum_{t=1}^{r-s+1} a_{r-t}\Omega_r(n-r+s+t-1)R_{r-s} \quad \text{and} \quad R_n^{\langle ps\rangle} = \sum_{k=r}^{n}\Omega_r(n-k+r)D_k,$$

for every $n \geq r$, where $\Omega_r(n)$ is given by (15).

It seems to us that Theorem 2.2 is not current in the literature in this form.

2.3 Analytical Binet formula of R_n

The analytic Binet formula of R_n can also be derived with the aid of the analytic Binet representation of the fundamental sequence $\{v_n^{(r)}\}_{n\geq 0}$. Indeed, let λ_s ($1 \leq s \leq h$) be the (characteristic) roots of the polynomial

$$P(z) = z^r - a_0 z^{r-1} - a_1 z^{r-2} - \cdots - a_{r-1} = (z-\lambda_1)^{m_1}(z-\lambda_2)^{m_2}\cdots(z-\lambda_h)^{m_h}$$

of the sequence (7). Then, we have $v_n^{(r)} = \sum_{s=1}^{h}\left[\sum_{j=0}^{m_h-1} A_{sj}^{(r)} n^j\right]\lambda_s^n$, for every $n \geq 0$, where the scalars $A_{sj}^{(r)}$ ($0 \leq s \leq h$ and $0 \leq j \leq m_s - 1$) are the scalars obtained by solving the linear system $v_n^{(r)} = \sum_{s=1}^{h}\left[\sum_{j=0}^{m_h-1} A_{sj}^{(r)} n^j\right]\lambda_s^n = \delta_{r-1,n}$ for $n = 0, 1, \ldots, r-1$, which is a generalized Vandermonde system. With the aid of Expression (14), the analytic Binet formula for R_n is obtained from the analytic Binet formulas of the general terms $v_{n-r+s+t-2}^{(r)}$ and $v_{n-k+r-1}^{(r)}$. A long, simple calculation allows us to obtain

$$v_{n+l}^{(r)} = \sum_{s=1}^{h}\left(\sum_{j=0}^{m_j-1} A_{sj}^{(r)}(n+l)^j\right)\lambda_s^{n+l} = \sum_{s=1}^{h}\left(\sum_{j=0}^{m_j-1}\sum_{w=0}^{j} b_w A_{sj}^{(r)} n^w\right)\lambda_s^{n+l}, \tag{18}$$

for $n + l \geq 0$, where $b_w = \binom{j}{w} l^{j-w}$. Thus we can write the expression $\sum_{j=0}^{m_j-1}\sum_{w=0}^{j} b_w A_{sj}^{(r)} n^w$ in an adequate form. That is, a direct computation shows that

$$\sum_{j=0}^{m_j-1}\sum_{w=0}^{j} b_w A_{sj}^{(r)} n^w = \sum_{i=0}^{m_s-1}\left(\sum_{j=i}^{m_s-1} A_{sj}^{(r)}\right) b_i n^i.$$

Denote

$$\Delta_{m_s}(n) = \sum_{i=0}^{m_s-1}\left(\sum_{j=i}^{m_s-1} A_{sj}^{(r)}\right) b_i n^i.$$

Hence, the substitution of this former expression in (18) we obtain the analytic Binet formula for $v_{n+l}^{(r)}$, for a given $l \in \mathbb{Z}$ under the form $v_{n+l}^{(r)} = \sum_{s=1}^{h}\Delta_{m_s}(n)\lambda_s^{n+l}$, for $n + l \geq 0$. Thus, by making $l = -r + k + t - 2$ and $l = -k + r - 1$, we derive

$$v^{(r)}_{n-r+k+t-2} = \sum_{s=1}^{h} \Delta^{(1)}_{m_s}(n)\lambda_s^{n-r+k+t-2} \quad \text{and} \quad v^{(r)}_{n-k+r-1} = \sum_{s=1}^{h} \Delta^{(2)}_{m_s}(n)\lambda_s^{n-k+r-1},$$

where

$$\Delta^{(1)}_{m_s}(n) = \sum_{i=0}^{m_s-1} \left(\sum_{j=i}^{m_s-1} A^{(r)}_{sj} \right) b^{(1)}_i n^i \text{ and } \Delta^{(2)}_{m_s}(n) = \sum_{i=0}^{m_s-1} \left(\sum_{j=i}^{m_s-1} A^{(r)}_{sj} \right) b^{(2)}_i n^i, \tag{19}$$

with $b^{(1)}_w = \binom{j}{w}(-r+k+t-2)^{j-w}$ and $b^{(2)}_w = \binom{j}{w}(-k+r-1)^{j-w}$. In summary, we can exhibit the explicit analytical Binet formula of the sequence $\{R_n\}_{n\geq 0}$ in terms of the initial condition and the (characteristic) roots of the polynomial $P(z) = z^r - a_0 z^{r-1} - a_1 z^{r-2} - \cdots - a_{r-1}$, and their multiplicities. More precisely, we can formulate this assertion explicitly in the following theorem.

Theorem 2.3. *Let $\{R_n\}_{n\geq 0}$ be the sequence defined by the nonhomogeneous linear recursive expression* (6)*, whose initial conditions are $R_0, \ldots, R_{r-1}$. Then, the analytical Binet formula for R_n is imparted by*

$$R_n = \sum_{k=1}^{r} \sum_{t=1}^{r-k+1} \sum_{s=1}^{h} a_{r-t}\Delta^{(1)}_{m_s}(n)\lambda_s^{n-r+k+t-2} R_{r-k} + \sum_{k=r}^{n} \sum_{s=1}^{h} \Delta^{(2)}_{m_s}(n)\lambda_s^{n-k+r-1} D_k, \tag{20}$$

for $n \geq r$, where $\Delta^{(1)}_{m_s}(n)$ and $\Delta^{(2)}_{m_s}(n)$ are given as in (19)*, and λ_j ($1 \leq j \leq s$) are the (characteristic) roots of the polynomial $P(z) = z^r - a_0 z^{r-1} - a_1 z^{r-2} - \cdots - a_{r-1}$, of the multiplicities m_j ($1 \leq j \leq s$), respectively.*

In addition, the result of Theorem 2.3 shows that the analytic Binet representation of the two sequences $\{R_n^{\langle hs\rangle}\}_{n\geq 0}$ and $\{R_n^{\langle ps\rangle}\}_{n\geq 0}$ are given in the following form

$$R_n^{\langle hs\rangle} = \sum_{k=1}^{r} \sum_{t=1}^{r-k+1} \sum_{s=1}^{h} a_{r-t}\Delta^{(1)}_{m_s}(n)\lambda_s^{n-r+k+t-2} R_{r-k} \text{ and } R_n^{\langle ps\rangle} = \sum_{k=r}^{n} \sum_{s=1}^{h} \Delta^{(2)}_{m_s}(n)\lambda_s^{n-k+r-1} D_k,$$

When the roots of $P(z) = z^r - a_0 z^{r-1} - a_1 z^{r-2} - \cdots - a_{r-1}$ are all simple roots, the analytic Binet formula of $v_n^{(r)} = \Omega_r(r+1)$ can be expressed under an explicit form. To this aim, the following lemma was considered in [5] (see the references therein).

Lemma 2.2. [5] *Suppose that the (characteristic) roots $\lambda_1, \ldots, \lambda_r$ of*

$$P(z) = z^r - a_0 z^{r-1} - a_1 z^{r-2} - \cdots - a_{r-1}$$

are all simple. Then, we have

$$v_n^{(r)} = \Omega_r(n+1) = \sum_{i=1}^{r} \frac{1}{P'(\lambda_i)} \lambda_i^n = \sum_{i=1}^{r} \frac{1}{\prod\limits_{f\neq i} (\lambda_i - \lambda_f)} \lambda_i^n, \tag{21}$$

for every $n \geq r$, where $P'(z) = \frac{dP}{dz}(z)$ and the $\Omega_r(n)$ are as in (15)*. In other terms, we get $v_n^{(r)} = \sum\limits_{i=1}^{r} A_i^{(r)} \lambda_i^n$, where $A_i^{(r)} = \frac{1}{P'(\lambda_i)} = \sum\limits_{i=1}^{r} \frac{1}{\prod_{f\neq i}(\lambda_i - \lambda_f)}$.*

Consequently, when the (characteristic) roots of the polynomial $P(z)$ are all simple, Expression (14) allows us to furnish an explicit compact analytic Binet representation for the general term R_n, as shown in the following proposition.

Proposition 2.2. *Let $\{R_n\}_{n\geq 0}$ be the sequence donated by the nonhomogeneous linear recursive relation* (6)*, whose initial conditions are $R_0, \ldots, R_{r-1}$. Suppose that the (characteristic) roots of the polynomial $P(z)$ are all simple, namely, $P(z) = \prod_{i=1}^{r}(z - \lambda_i)$. Then, the analytic Binet formula for the general term R_n is given by*

$$R_n = \sum_{s=1}^{r} \sum_{t=1}^{r-s+1} \sum_{i=1}^{r} a_{r-t} \frac{\lambda_i^{-r+s+t-2}}{P'(\lambda_i)} \lambda_i^n R_{r-s} + \sum_{k=r}^{n} \sum_{i=1}^{r} \frac{\lambda_i^{r-k-1}}{P'(\lambda_i)} \lambda_i^n D_k,$$

for every $n \geq 0$.

It should be noted that the results in this section originate from the expression established in [10, Proposition 1]. However, the results of Theorems 2.1, 2.2, 2.3 and Proposition 2.2 are not known in the literature. Moreover, these results can be applied to various fields. In particular, as pointed out in the Introduction, Expression (6) represents the general setting of Expressions (1) and (2)–(5). Especially, the result of Proposition 2.2 occurs in several known sequences of usual numbers defined by the nonhomogeneous linear recurrence expression, such that the sequence of Leonardo numbers and some of its generalizations. The next section is devoted to some of these sequences related to the generalized nonhomogeneous Horadam model.

3 Generalized nonhomogeneous Horadam model

This section concerns the application of results of Section 2 to the nonhomogeneous generalized Horadam model, and provides its applications in the study of some known nonhomogeneous sequences, such as the Leonardo sequence and its generalization.

3.1 Linear formula for model (1)

We devote this subsection to establishing the linear formula for the general term GH_n of the generalized Horadam model (1). Furthermore, we describe explicit formulas for the sequences of generalized Leonardo numbers considered in [18], with the aid of its related fundamental Fibonacci system $\{\{v_n^{(1)}\}_{n\geq 0}, \{v_n^{(2)}\}_{n\geq 0}\}$, given by

$$v_{n+1}^{(1)} = pv_n^{(1)} + qv_{n-1}^{(1)}, \text{ and } v_{n+1}^{(2)} = pv_n^{(2)} + qv_{n-1}^{(2)}, \tag{22}$$

where $v_0^{(1)} = 1,\ v_1^{(1)} = 0$ and $v_0^{(2)} = 0,\ v_1^{(2)} = 1$. Therefore, as a consequence of Theorem 2.1, the following result can be stated.

Theorem 3.1. *Let $\{GH_n\}_{n\geq 0}$ be the sequence defined by* (1)*, whose initial conditions are GH_0 and GH_1. Then, the linear formula of GH_n is given as follows*

$$GH_n = v_n^{(2)} GH_1 + qv_{n-1}^{(2)} GH_0 + \sum_{k=2}^{n} v_{n-k+1}^{(2)} HC_k, \tag{23}$$

for all $n \geq 2$, where $\{v_n^{(2)}\}_{n\geq 0}$ is the fundamental sequence given by (22).

Theorem 3.1 permits us to exhibit the formulas for the generalized Leonardo models considered in [18], namely, (2)–(5), by using generalized the nonhomogeneous Horadam model (1). That is, we formulate here the corollary.

Corollary 3.1. *Let* $\{L_n\}_{n\geq 0}$, $\{I_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ *and* $\{W_n\}_{n\geq 0}$ *be the sequences conceived as in* (2)–(5). *Therefore, for every* $n \geq 2$, *the linear expressions of the general terms* L_n, I_n, H_n *and* W_n *are in the form*

$$\begin{cases} L_n = v^{(2)}_{L,n} + v^{(2)}_{L,n-1} + \sum_{k=2}^{n} v^{(2)}_{L,n-k+1} k, \\ I_n = v^{(2)}_{I,n} + v^{(2)}_{I,n-1} + \sum_{k=2}^{n} v^{(2)}_{I,n-k+1} (-1)^k, \\ H_n = -v^{(2)}_{H,n} - v^{(2)}_{H,n-1} + \sum_{k=2}^{n} v^{(2)}_{H,n-k+1} F_k, \end{cases}$$

where $L_0 = L_1 = 1$, $I_0 = I_1 = 1$, $H_0 = H_1 = -1$, *and* $\{v^{(2)}_{L,n}\}_{n\geq 0}$, $\{v^{(2)}_{I,n}\}_{n\geq 0}$, $\{v^{(2)}_{H,n}\}_{n\geq 0}$ *are the fundamental sequences related to* $\{L_n\}_{n\geq 0}$, $\{I_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ *(respectively), which are given by* (22) *such that* $p = q = 1$. *Finally, for the general term* W_n *we have*

$$W_n = v^{(2)}_{W,n} - f\, t_1 v^{(2)}_{W,n-1} + \sum_{l=2}^{n} \sum_{j=0}^{k} (s - f - 1) v^{(2)}_{W,n-l+1} \alpha_j n^j,$$

for any $n \geq 2$, *where* $\{v^{(2)}_{W,n}\}_{n\geq 0}$ *is nothing else but the fundamental sequence given as in* (22) *defined by setting* $p = s$ *and* $q = -f$.

3.2 Combinatorial formulas for model (1)

As in the general case, the properties of Subsection 2.2 allow us to exhibit the explicit combinatorial expressions for the model (1). Let $\{v^{(2)}_n\}_{n\geq 0}$ be the fundamental sequence given by (22). Since $r = 2$, the application of formulas (15) and (16) permits us to get $v^{(2)}_n = \Omega_2(n+1)$, where

$$\Omega_2(n) = \sum_{k_0 + 2k_1 = n-2} \frac{(k_0 + k_1)!}{k_0! k_1!} a_0^{k_0} a_1^{k_1} = \sum_{k=0}^{[\frac{n-2}{2}]} \binom{n-k-2}{k} a_0^{n-2k-2} a_1^k, \tag{24}$$

for every $n \geq 2$. Here $[x]$ means the integer part of x. Therefore, the application of Theorem 2.2 (or equivalently replacing (24) in (23)) allows us to show the following result.

Theorem 3.2. *Let* $\{GH_n\}_{n\geq 0}$ *be the sequence defined by* (1), *whose initial conditions are* GH_0 *and* GH_1. *Then, the combinatorial formula of* GH_n *is as follows*

$$GH_n = \Omega_2(n+1,)GH_0 + q\Omega_2(n)GH_1 + \sum_{k=2}^{n} \Omega_2(n-k+2) HC_k,$$

for every $n \geq 2$, *where* $\Omega_2(n)$ *is given as in* (24).

Theorem 3.2 permits us to provide the combinatorial formulas for Leonardo's generalized models, given in the next result.

Corollary 3.2. *Let $\{L_n\}_{n\geq 0}$, $\{I_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{W_n\}_{n\geq 0}$ be the sequences defined by* (2)–(5). *Then, for every $n \geq 2$, the combinatorial expression of the general terms L_n, I_n, H_n and W_n as follows*

$$\begin{cases} L_n = \Omega_2(n+1) + \Omega_2(n) + \sum_{k=2}^{n} \Omega_2(n-k+2)k, \\ I_n = \Omega_2(n+1) + \Omega_2(n) + \displaystyle\sum_{k=2}^{n} \Omega_2(n-k+2)(-1)^k, \\ H_n = -\Omega_2(n+1) - \Omega_2(n) + \displaystyle\sum_{k=2}^{n} \Omega_2(n-k+2)F_k, \end{cases}$$

where $\Omega_2(n)$ is given as in (24) *with $a_0 = a_1 = 1$. Finally, for the general term W_n we have*

$$W_n = \Omega_2(n+1) - f\, t_1\Omega_2(n) + \sum_{l=2}^{n}\sum_{j=0}^{k} \alpha_j(s-f-1)\Omega_2(n-l+2)n^j,$$

for every $n \geq 2$, where $\Omega_2(n)$ is given as in (24) *with $a_0 = s$ and $a_1 = -f$.*

3.3 Analytical Binet formulas for model (1)

This subsection concerns the analytical Binet formula for the general term GH_n of the model (1), whose initial conditions are GH_0 and GH_1. To this aim, let us consider the characteristic polynomial $P(z) = z^2 - pz - q$, of the homogeneous part $v_{n+1} = pv_n + qv_{n-1}$ of (1), whose discriminant is $p^2 + 4q$.

Suppose that $p^2 + 4q \neq 0$. In this case, the roots of the polynomial $P(z)$ are simple and given by $\lambda_1 = \frac{p+\sqrt{p^2+4q}}{2}$, $\lambda_2 = \frac{p-\sqrt{p^2+4q}}{2}$. In addition, since $P^{'}(z) = 2z - p$, we have $P^{'}(\lambda_j) = \pm\sqrt{p^2+4q} \neq 0$, where $j = 1, 2$. Hence, Proposition 2.2 implies that

$$GH_n = (a_1GH_1 + a_0GH_1\lambda_1 + a_1GH_0\lambda_1)\frac{\lambda_1^{n-2}}{P'(\lambda_1)} + (a_1GH_1 + a_0GH_1\lambda_2 + a_1GH_0\lambda_2)\frac{\lambda_2^{n-2}}{P'(\lambda_2)} + \sum_{k=2}^{n}\left(\frac{\lambda_1^{n+1-k}}{P'(\lambda_1)} + \frac{\lambda_2^{n+1-k}}{P'(\lambda_2)}\right)HC_k,$$

for every $n \geq 2$. In summary, we can deduce the following theorem.

Theorem 3.3. *Let $\{GH_n\}_{n\geq 0}$ be a sequence defined by* (1), *whose initial conditions are $GH_0 = \alpha_0$ and $GH_1 = \alpha_1$. Suppose that the roots $\lambda_1 = \frac{p+\sqrt{p^2+4q}}{2}$, $\lambda_2 = \frac{p-\sqrt{p^2+4q}}{2}$ of the characteristic polynomial $P(z) = z^2 - pz - q$ are distinct (equivalently, $p^2 + 4q \neq 0$). Then, the analytical Binet formula for GH_n is in the form*

$$GH_n = \frac{A_{p,q}(\alpha_0,\alpha_1,\lambda_1)}{P'(\lambda_1)}\lambda_1^{n-2} + \frac{B_{p,q}(\alpha_0,\alpha_1,\lambda_2)}{P'(\lambda_2)}\lambda_2^{n-2} + \sum_{k=2}^{n}\left(\frac{\lambda_1^{n+1-k}}{P'(\lambda_1)} + \frac{\lambda_2^{n+1-k}}{P'(\lambda_2)}\right)HC_k,$$

for every $n \geq 2$, where $p^{'}(\lambda_j) = \pm\sqrt{p^2+4q}$ ($j = 1, 2$), $A_{p,q}(\alpha_0,\alpha_1,\lambda_1) = q\alpha_1 + (p\alpha_1 + q\alpha_0)\lambda_1$, and $B_{p,q}(\alpha_0,\alpha_1,\lambda_2) = q\alpha_1 + (p\alpha_1 + q\alpha_0)\lambda_2$.

The result of Theorem 3.3 can also be established with the aid of a long straightforward computation using Theorem 2.3.The following corollary follows from an application of Theorem 3.3 for the sequence of of the generalized Leonardo (2)–(5).

Corollary 3.3. *Let* $\lambda_1 = \frac{1+\sqrt{5}}{2}$ *and* $\lambda_2 = \frac{1-\sqrt{5}}{2}$*. Then, the analytical Binet representations of the general terms of the sequences* $\{L_n\}_{n\geq 0}$, $\{I_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ *and* $\{W_n\}_{n\geq 0}$ *defined by* (2)–(5)*, are given in the form*

$$\begin{cases} L_n = \dfrac{2\left(1+2\lambda_1\right)}{\sqrt{5}}\lambda_1^{n-2} - \dfrac{2\left(1+2\lambda_2\right)}{\sqrt{5}}\lambda_2^{n-2} + \sum\limits_{k=2}^{n} \dfrac{2}{\sqrt{5}}\left(\lambda_1^{n-k+1} - \lambda_2^{n-k+1}\right) k, \\ I_n = \dfrac{2\left(1+2\lambda_1\right)}{\sqrt{5}}\lambda_1^{n-2} - \dfrac{2\left(1+2\lambda_2\right)}{\sqrt{5}}\lambda_2^{n-2} + \sum\limits_{k=2}^{n} \dfrac{2}{\sqrt{5}}\left(\lambda_1^{n-k+1} - \lambda_2^{n-k+1}\right)(-1)^k, \\ H_n = \dfrac{2\left(1+2\lambda_1\right)}{\sqrt{5}}\lambda_1^{n-2} - \dfrac{2\left(1+2\lambda_2\right)}{\sqrt{5}}\lambda_2^{n-2} + \sum\limits_{k=2}^{n} \dfrac{2}{\sqrt{5}}\left(\lambda_1^{n-k+1} - \lambda_2^{n-k+1}\right) F_k, \end{cases}$$

for every $n \geq 0$*. Finally, for the general term* W_n*, suppose that* $s^2 - 4f \neq 0$*. Then, we have*

$$W_n = \Gamma_{s,f}(t_1,t_0)\eta_1^{n-2} - \Delta_{s,f}(t_1,t_0)\eta_2^{n-2} + \sum_{l=2}^{n}\sum_{j=0}^{k} \frac{2\alpha_j}{\sqrt{s^2-4f}}\left(\eta_1^{n-l+1} - \eta_2^{n-l+1}\right) k^j,$$

for any $n \geq 0$*, where*

$$\Gamma_{s,f}(t_1,t_0) = \frac{2\left(-ft_1 + st_1\eta_1 - ft_0\eta_1\right)}{\sqrt{s^2-4f}} \quad \textit{and} \quad \Delta_{s,f}(t_1,t_0) = \frac{2\left(-ft_1 + st_1\eta_2 - ft_0\eta_2\right)}{\sqrt{s^2-4f}},$$

such that $\eta_1 = \frac{s+\sqrt{s^2-4f}}{2}$*, and* $\eta_2 = \frac{s-\sqrt{s^2-4f}}{2}$.

For the the generalized nonhomogeneous Horadam model (1), suppose that $p^2+4q=0$. Thus, the characteristic polynomial of the homogeneous part is $P(z) = (z-\lambda)^2$, where $\lambda = \lambda_1 = \lambda_2 = \frac{p}{2}$. Then, the analytic Binet representation of the general term $v_n^{(2)}$ is imparted in the form $v_n^{(2)} = n\lambda^{n-1}$, for every $n \geq 0$. Hence, a direct application of Theorem 2.3 shows that the analytic Binet expression of the general term GH_n is

$$GH_n = n\lambda^{n-1}GH_1 + q(n-1)\lambda^{n-2}GH_0 + \sum_{k=2}^{n}(n-k+1)\lambda^{n-k}HC_k,$$

for every $n \geq 2$. In summary, the following proposition can be formulated.

Proposition 3.1. *Let* $\{GH_n\}_{n\geq 0}$ *be the sequence defined by the relation* (1)*, whose initial conditions are* GH_0 *and* GH_1*. Suppose that* $p^2+4q=0$*. Then, the analytical Binet formula for* GH_n *is*

$$GH_n = \left(\left(\frac{2}{p}GH_1 - GH_0\right)n + GH_0\right)\left(\frac{p}{2}\right)^n + \sum_{l=2}^{n}(n-l+1)\left(\frac{p}{2}\right)^{n-l}HC_l,$$

for every $n \geq 2$.

For the sequence $\{W_n\}_{n\geq 0}$ the following corollary can be stated.

Corollary 3.4. *Let* $\{W_n\}_{n\geq 0}$ *be the recursive sequence defined by* (5)*, where* $W_0 = t_0$ *and* $W_1 = t_1$*, and suppose that* $s^2 - 4f = 0$*. Then, the analytical Binet representation of the general term* W_n *is*

$$W_n = \left(\left(\frac{2}{s}t_1 - t_0\right)n + t_0\right)\left(\frac{s}{2}\right)^n + \sum_{l=2}^{n}\sum_{j=0}^{k}\alpha_j(n-l+1)n^j\left(\frac{s}{2}\right)^{n-l},$$

for every $n \geq 2$.

4 Conclusion

In this study, our main results of Section 2 concerned explicit formulas for sequences defined by a generalized nonhomogeneous linear recursive relation of finite order (6), whose initial conditions are arbitrary. The results of this section provided us with an interesting approach to address the linear representation and combinatorial aspect, as well as the analytic Binet representations for the general term of the generalized nonhomogeneous Horadam model (1). As a consequence, new formulas were provided for the sequences of generalized Leonardo numbers (2)–(5).

To our knowledge, it appears that our approach and results are not current in this form in the literature. Finally, our approach can be used for studying other models of sequences defined by nonhomogeneous recursive relations similar to (6).

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