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On ψ-quadratic k-tuples and their generalizations

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Abstract: In this paper, we introduce the notion of ψ-quadratic k-tuples. We establish the non-existence of ψ-quadratic pairs and provide a partial analysis of ψ-quadratic triples. We also give examples and propose generalizations of these new concepts.
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1 Notations

The letter p, with or without a subscript, will always denote prime number. Let $n > 1$ be a positive integer with prime factorization

$$n = p_1^{a_1} \cdots p_r^{a_r} .$$

We define the Dedekind function $\psi(n)$ by the formula

$$\psi(n) = n \prod_{i=1}^{r} \left(1 + \frac{1}{p_i}\right) \quad \text{and} \quad \psi(1) = 1 . \tag{1}$$

2 Introduction

The Dedekind function $\psi(n)$ has been studied extensively in connection with various arithmetic properties of integers. Its role in generalizations of classical notions such as perfect and amicable numbers has attracted particular attention in recent years. Motivated by these developments, in the present paper we introduce and investigate ψ-quadratic k-tuples and their generalizations, defined through polynomial identities involving $\psi(n)$ and sums of powers of integers. We establish the non-existence of ψ-quadratic pairs and provide a partial analysis of ψ-quadratic triples. Moreover, we construct explicit families of ψ-quadratic quadruples and extend the study to ψ-cubic, ψ-quartic and ψ-quintic tuples, where several infinite classes and open questions naturally arise. The results presented here extend previous works of the author [6,7] on σ-quadratic k-tuples and ψ-amicable numbers, respectively. They contribute to the systematic study of Diophantine equations involving the Dedekind function $\psi(n)$ and can be regarded as power generalizations of ψ-amicable numbers. Many articles have addressed related problems. We point out the papers [1–5, 8–19, 21]. Many other similar results can be found in the literature.

3 ψ-Quadratic pairs

The positive integers x and y form a ψ-quadratic pair if

$$\psi^2(x) = x^2 + y^2 \,. \tag{2}$$

With the following theorem, we establish that no ψ-quadratic pairs exist.

Theorem 3.1. *The equation* (2) *has no solutions in positive integers.*

Proof. Put

$$u = \psi(x) - x \,, \qquad v = \psi(x) + x \,, \qquad d = \gcd(u, v) \,. \tag{3}$$

Now (2) and (3) imply

$$uv = y^2 \,.$$

Bearing in mind that

$$u = du_1 \,, \qquad v = dv_1 \,, \qquad \gcd(u_1, v_1) = 1$$

we deduce

$$uv = d^2 u_1 v_1 = y^2 \,.$$

Consequently, both u_1 and v_1 must be a perfect squares. Thus

$$u = da^2 \,, \qquad v = db^2 \,, \qquad \gcd(a, b) = 1 \tag{4}$$

for some $a, b \in \mathbb{Z}$. Therefore

$$v - u = d(b^2 - a^2)$$

which, together with (3), yields

$$d(b^2 - a^2) = 2x \,. \tag{5}$$

We consider five cases.

- **Case 1.**

$$x = 2^k\,, \quad k \geq 1\,. \tag{6}$$

From (1), (3), (4) and (6), we obtain

$$u = 2^{k-1}\,, \quad v = 5 \cdot 2^{k-1}\,, \quad d = 2^{k-1}\,, \quad a^2 = 1\,, \quad b^2 = 5\,,$$

which is impossible.

- **Case 2.**

$$x = p_1^{a_1} \cdots p_r^{a_r}\,, \quad r \geq 1\,, \tag{7}$$

where $p_1, \ldots, p_r$ are odd prime numbers. Using (1), (3) and (7), we get

$$u = p_1^{a_1-1} \cdots p_r^{a_r-1}(p_1+1) \cdots (p_r+1) - p_1^{a_1} \cdots p_r^{a_r}\,,$$

$$v = p_1^{a_1-1} \cdots p_r^{a_r-1}(p_1+1) \cdots (p_r+1) + p_1^{a_1} \cdots p_r^{a_r}\,.$$

Since u and v are odd, it follows from (4) that a and b are odd. Then

$$b^2 - a^2 \equiv 0 \pmod{8}\,. \tag{8}$$

It is clear that (7) and (8) contradict (5).

- **Case 3.**

$$x = 2^k 3^r\,, \quad k \geq 1\,, \quad r \geq 1\,. \tag{9}$$

From (1), (3), (4) and (9), we derive

$$u = 2^k 3^r\,, \quad v = 2^k 3^{r+1}\,, \quad d = 2^k 3^r\,, \quad a^2 = 1\,, \quad b^2 = 3\,,$$

which is impossible.

- **Case 4.**

$$x = 2^k p^r\,, \quad p \neq 3\,, \quad k \geq 1\,, \quad r \geq 1\,. \tag{10}$$

Now (1), (3) and (10) give us

$$u = 2^{k-1} p^{r-1}(p+3)\,, \quad v = 2^{k-1} p^{r-1}(5p+3)\,. \tag{11}$$

  - **Case 4.1.**

$$p = 4l + 3\,, \quad l \geq 1\,. \tag{12}$$

By (11) and (12), we have

$$u = 2^k (4l+3)^{r-1}(2l+3)\,, \qquad v = 2^k (4l+3)^{r-1}(10l+9)\,. \tag{13}$$

Taking into account (4) and (13), we conclude that

$$d \equiv 0 \pmod{2^k}\,, \tag{14}$$

and that a and b are odd, which in turn implies (8). It is easy to see that (8), (10) and (14) contradict (5).

– **Case 4.2.**

$$p = 4l + 1\,, \quad l \geq 1\,. \tag{15}$$

From (11) and (15), we write

$$u = 2^{k+1}(4l+1)^{r-1}(l+1)\,, \qquad v = 2^{k+1}(4l+1)^{r-1}(5l+2)\,. \tag{16}$$

∗ <u>Case 4.2.1.</u>

$$(l+1, 5l+2) = 1\,. \tag{17}$$

Now (4), (16) and (17) lead to

$$d = 2^{k+1}(4l+1)^{r-1}\,, \quad a^2 = l+1\,, \quad b^2 = 5l+2\,,$$

which yields an impossible congruence

$$b^2 \equiv 2 \pmod 5\,,$$

because the squares are congruent only to 0, 1 or $4 \pmod 5$.

∗ <u>Case 4.2.2.</u>

$$(l+1, 5l+2) = 3\,. \tag{18}$$

Using (4), (16) and (18), we obtain

$$d = 3 \cdot 2^{k+1}(4l+1)^{r-1}\,, \quad a^2 = \frac{l+1}{3}\,, \quad b^2 = \frac{5l+2}{3}\,. \tag{19}$$

Apparently, (10) and (19) contradict (5).

- **<u>Case 5.</u>**

$$x = 2^k p_1^{a_1} \cdots p_r^{a_r}\,, \quad k \geq 1\,, \quad r \geq 2\,. \tag{20}$$

By (1), (3) and (20), we derive

$$u = 3 \cdot 2^{k-1} p_1^{a_1-1} \cdots p_r^{a_r-1}(p_1+1)\cdots(p_r+1) - 2^k p_1^{a_1}\cdots p_r^{a_r}\,, \tag{21}$$

$$v = 3 \cdot 2^{k-1} p_1^{a_1-1} \cdots p_r^{a_r-1}(p_1+1)\cdots(p_r+1) + 2^k p_1^{a_1}\cdots p_r^{a_r}\,. \tag{22}$$

Bearing in mind (4), (21) and (22), we establish (14) and that a and b are odd, which in turn implies (8). It is clear that (8), (14) and (20) contradict (5).

This completes the proof of Theorem 3.1. □

4 ψ-Quadratic triples

The positive integers a, b and c with $a \leq b$ form a ψ-quadratic triple if

$$\psi^2(a) = \psi^2(b) = a^2 + b^2 + c^2\,. \tag{23}$$

Theorem 4.1. *For each $k \geq 1$, the triples of the form $(2^k, 2^k, 2^{k-1})$ are ψ-quadratic triples. Moreover, if (a, b, c) is a ψ-quadratic triple with $a = b$, then*

$$a = b = 2^k \,, \quad c = 2^{k-1} \,, \quad k \geq 1 \,.$$

From Theorem 4.1 it follows that there exist infinitely many ψ-quadratic triples of the form $(2^k, 2^k, 2^{k-1})$.

Proof. We consider three cases.

- **Case 1.** Let (a, b, c) be a triple such that

$$a = b = 2^k \,, \quad c = 2^{k-1} \,, \quad k \geq 1 \,. \tag{24}$$

From (1) and (24), we get

$$\psi^2(a) = \psi^2(b) = 9 \cdot 2^{2(k-1)} = 2^{2k} + 2^{2k} + 2^{2(k-1)} \,. \tag{25}$$

Consequently $(2^k, 2^k, 2^{k-1})$ is a ψ-quadratic triple.

- **Case 2.** Let (a, b, c) be a ψ-quadratic triple such that

$$a = b = p_1^{a_1} \cdots p_r^{a_r} \,, \quad r \geq 1 \,, \tag{26}$$

where $p_1, \ldots, p_r$ are odd prime numbers. Using (1) and (26), we write

$$\psi^2(a) - 2a^2 = p_1^{2(a_1-1)} \cdots p_r^{2(a_r-1)} (p_1+1)^2 \cdots (p_r+1)^2 - 2p_1^{2a_1} \cdots p_r^{2a_r} \,. \tag{27}$$

Now (23) and (27) yield

$$p_1^{2(a_1-1)} \cdots p_r^{2(a_r-1)} \big[(p_1+1)^2 \cdots (p_r+1)^2 - 2p_1^2 \cdots p_r^2 \big] = c^2 \,.$$

Therefore

$$F = (p_1+1)^2 \cdots (p_r+1)^2 - 2p_1^2 \cdots p_r^2 \tag{28}$$

must be a perfect square. Put

$$A = (p_1+1) \cdots (p_r+1) \,, \qquad B = p_1 \cdots p_r \,. \tag{29}$$

By (28) and (29), we have

$$F = A^2 - 2B^2 \,.$$

Since A is even and B is odd, we deduce

$$F \equiv 0 - 2 \equiv 2 \pmod 4 \,,$$

which means that F is not a perfect square, because the squares are congruent only to 0 or $1 \pmod 4$.

- **<u>Case 3.</u>** Let (a,b,c) be a ψ-quadratic triple such that

$$a=b=2^k p_1^{a_1}\cdots p_r^{a_r}\,,\quad k\geq 1\,,\quad r\geq 1\,. \tag{30}$$

From (1) and (30), we derive

$$\psi^2(a)-2a^2=9\cdot 2^{2(k-1)}p_1^{2(a_1-1)}\cdots p_1^{2(a_r-1)}(p_1+1)^2\cdots(p_r+1)^2-2^{2k+1}p_1^{2a_1}\cdots p_r^{2a_r}\,. \tag{31}$$

Now (23) and (31) imply

$$2^{2(k-1)}p_1^{2(a_1-1)}\cdots p_1^{2(a_r-1)}\big[9(p_1+1)^2\cdots(p_r+1)^2-8p_1^2\cdots p_r^2\big]=c^2\,.$$

Hence

$$H=9(p_1+1)^2\cdots(p_r+1)^2-8p_1^2\cdots p_r^2 \tag{32}$$

must be a perfect square. Set

$$P=(p_1+1)\cdots(p_r+1)\,,\qquad Q=p_1\cdots p_r\,. \tag{33}$$

By (32) and (33), we write

$$H=9P^2-8Q^2\,.$$

  - **Case 3.1.**

$$P\equiv 0\pmod 4\,.$$

Since Q is odd, we obtain

$$H\equiv 0-8\equiv 8\pmod{16}\,,$$

which means that H is not a perfect square, because the squares are congruent only to $0,1,4$ or $9\pmod{16}$.

  - **Case 3.2.**

$$P\equiv 2\pmod 4\,.$$

We have

$$P^2\equiv 4\pmod{16}\,.$$

Since Q is odd, we get

$$H\equiv 9\cdot 4-8\equiv 28\equiv 12\pmod{16}\,,$$

which means that H is not a perfect square, because the squares are congruent only to $0,1,4$ or $9\pmod{16}$.

This completes the proof of Theorem 4.1. □

Several ψ-quadratic triples are listed below.

ψ-Quadratic triples
$(2,2,1)$, $(4,4,2)$, $(8,8,4)$, $(16,16,8)$, $(32,32,16)$, $(64,64,32)$, $(128,128,64)$, $(256,256,128)$, $(512,512,256)$, $(1024,1024,512)$, $(2048,2048,1024)$, $(4096,4096,2048)$, $(8192,8192,4096)$, $(16384,16384,8192)$, $(32768,32768,16384)$, $(65536,65536,32768)$, $(131072,131072,65536)$, $(262144,262144,131072)$

List 1

Question 1. *Are there any ψ-quadratic triples other than those of the form*

$$(2^k, 2^k, 2^{k-1}), \quad k \geq 1\,?$$

5 ψ-Quadratic quadruples

The positive integers a, b, c and d with $a \leq b \leq c$ form a ψ-quadratic quadruple if

$$\psi^2(a) = \psi^2(b) = \psi^2(c) = a^2 + b^2 + c^2 + d^2\,.$$

Several ψ-quadratic quadruples are listed below.

ψ-Quadratic quadruples
$(6,6,6,6)$, $(12,12,12,12)$, $(18,18,18,18)$, $(18,22,22,2)$, $(24,24,24,24)$, $(36,36,36,36)$, $(36,44,44,4)$, $(48,48,48,48)$, $(54,54,54,54)$, $(72,72,72,72)$, $(72,88,88,8)$, $(96,96,96,96)$, $(108,108,108,108)$, $(144,144,144,144)$, $(144,176,176,16)$, $(162,162,162,162)$, $(192,192,192,192)$, $(216,216,216,216)$, $(288,288,288,288)$, $(288,352,352,32)$, $(324,324,324,324)$, $(384,384,384,384)$, $(432,432,432,432)$, $(486,486,486,486)$, $(576,576,576,576)$, $(576,704,704,64)$, $(648,648,648,648)$, $(768,768,768,768)$, $(864,864,864,864)$, $(972,972,972,972)$ $(1152,1152,1152,1152)$, $(1152,1408,1408,128)$, $(1296,1296,1296,1296)$

List 2

The OEIS [20] sequence A391289 lists the first elements of ψ-quadratic quadruples.

6 ψ-Cubic triples

The positive integers a, b and c with $b \leq c$ form a ψ-cubic triple if

$$\psi^3(a) = a^3 + b^3 + c^3\,.$$

Several ψ-cubic triples are listed below.

ψ-Cubic triples
$(4,3,5)$, $(5,3,4)$, $(6,8,10)$, $(8,6,10)$, $(12,16,20)$, $(16,12,20)$, $(18,24,30)$, $(24,32,40)$, $(25,15,20)$, $(32,24,40)$, $(36,48,60)$, $(48,64,80)$, $(53,12,19)$, $(54,72,90)$, $(58,59,69)$, $(64,48,80)$, $(72,96,120)$, $(96,128,160)$, $(102,26,208)$, $(102,117,195)$, $(108,144,180)$, $(116,118,138)$, $(118,116,138)$, $(125,75,100)$, $(128,96,160)$, $(144,192,240)$, $(162,216,270)$, $(192,256,320)$, $(204,52,416)$, $(204,234,390)$, $(216,288,360)$, $(232,236,276)$, $(236,232,276)$, $(256,192,320)$, $(258,126,504)$, $(288,384,480)$, $(306,78,624)$, $(306,351,585)$, $(324,432,540)$, $(384,512,640)$, $(408,104,832)$, $(408,468,780)$, $(426,6,828)$, $(426,646,668)$, $(432,576,720)$, $(1615,1065,1670)$

List 3

The sequence A391290 in the OEIS [20] consists of the first members of ψ-cubic triples.

7 ψ-Cubic quadruples

The positive integers a, b, c, d with $a \leq b$ and $c \leq d$ form a ψ-cubic quadruple if

$$\psi^3(a) = \psi^3(b) = a^3 + b^3 + c^3 + d^3 \,.$$

Several ψ-cubic quadruples are listed below.

ψ-Cubic quadruples
$(14,16,5,19)$, $(28,32,10,38)$, $(30,45,43,56)$, $(42,48,40,86)$, $(54,68,58,84)$, $(56,64,20,76)$, $(60,72,63,129)$, $(84,96,80,172)$, $(90,135,129,168)$, $(108,136,116,168)$, $(112,128,40,152)$, $(120,126,144,258)$, $(124,161,52,95)$, $(126,144,120,258)$, $(150,225,215,280)$, $(168,192,160,344)$, $(174,200,12,322)$, $(180,216,189,387)$, $(216,272,232,336)$, $(224,256,80,304)$, $(240,252,288,516)$, $(252,288,240,516)$, $(270,405,387,504)$, $(308,322,78,504)$, $(336,384,320,688)$, $(348,400,24,644)$, $(360,378,432,774)$, $(378,432,360,774)$, $(432,544,464,672)$, $(448,512,160,608)$, $(450,675,645,840)$, $(480,504,576,1032)$, $(504,576,480,1032)$

List 4

The OEIS [20] sequence A391534 lists the first components of ψ-cubic quadruples.

8 ψ-Cubic quintuples

The positive integers a, b, c, d, e with $a \leq b \leq c$ and $d \leq e$ form a ψ-cubic quintuple if

$$\psi^3(a) = \psi^3(b) = \psi^3(c) = a^3 + b^3 + c^3 + d^3 + e^3 \,.$$

Several ψ-cubic quintuples are listed below.

ψ-Cubic quintuples
$(6,9,9,3,3)$, $(12,14,16,7,17)$, $(18,27,27,9,9)$, $(24,28,32,14,34)$, $(30,36,40,48,50)$, $(30,55,55,11,23)$, $(40,44,46,12,50)$, $(40,46,51,29,38)$, $(45,46,51,21,35)$, $(48,56,64,28,68)$, $(56,63,77,7,13)$, $(62,62,69,4,43)$, $(54,81,81,27,27)$, $(60,72,72,75,117)$, $(60,72,80,96,100)$, $(66,72,72,45,123)$, $(66,72,115,2,93)$, $(66,88,92,62,100)$, $(70,88,119,12,65)$, $(70,92,99,21,96)$, $(80,88,92,24,100)$, $(92,92,94,36,82)$, $(78,78,98,82,132)$, $(96,112,128,56,136)$, $(96,124,128,69,123)$, $(930,1280,2101,74,379)$, $(960,1152,1152,1200,1872)$, $(960,1152,1280,1536,1600)$, $(960,1528,1532,117,1611)$, $(1056,1152,1152,720,1968)$, $(1056,1408,1472,992,1600)$

List 5

The OEIS [20] sequence A391535 consists of the first elements of ψ-cubic quintuples.

9 ψ-Quartic quintuples

The positive integers a, b, c, d, e with $b \le c \le d \le e$ form a ψ-quartic quintuple if

$$\psi^4(a) = a^4 + b^4 + c^4 + d^4 + e^4 .$$

Several ψ-quartic quintuples are listed below.

ψ-Quartic quintuples
$(538,96,532,548,648)$, $(34432,6144,34048,35072,41472)$, $(68864,12288,68096,70144,82944)$, $(137728,24576,136192,140288,165888)$, $(275456,49152,272384,280576,331776)$, $(550912,98304,544768,561152,663552)$

List 6

The OEIS [20] sequence A391536 lists the first components of ψ-quartic quintuples.

10 ψ-Quintic quintuples

The positive integers a, b, c, d, e with $b \le c \le d \le e$ form a ψ-quintic quintuple if

$$\psi^5(a) = a^5 + b^5 + c^5 + d^5 + e^5 .$$

Several ψ-quintic quintuples are listed below.

ψ-Quintic quintuples
$(46,19,43,47,67)$, $(92,38,86,94,134)$, $(94,38,86,92,134)$, $(946,418,1012,1034,1474)$, $(1139,323,731,782,799)$

List 7

The OEIS [20] sequence A390006 consists of the first elements of ψ-quintic quintuples.

11 Conclusion

For each of the k-tuples from Section 5 to Section 10, the question was raised whether there exist infinitely many k-tuples of this kind.

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