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A note on generalized Zumkeller numbers

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Abstract: A positive integer n is called an e-Zumkeller number if the exponential divisors of n can be partitioned into two disjoint subsets of equal sum. Generalizing the concept of e-Zumkeller numbers, we define multiplicatively e-Zumkeller numbers. In addition, generalizing the concepts of s-Zumkeller numbers and m-Zumkeller numbers, we define two new variants of Zumkeller numbers called $(+s)$-Zumkeller numbers and $(+m)$-Zumkeller numbers, considering even positive divisors. We present some examples in support of these two types of positive integers and study their characteristics.
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1 Introduction

The concept of perfect numbers, one of the best-known concepts of pure mathematics, has been studied since ancient times. A positive integer n is a perfect number if n is equal to the sum of the proper positive divisors of n. The existence of odd perfect numbers has remained yet an open problem. Many researchers have attempted to generalize the concept of perfect numbers over time.

A positive divisor d of a positive integer $n = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ is called an *exponential divisor* (or *e-divisor*) of n if $d = p_1^{b_1} p_2^{b_2} \cdots p_m^{b_m}$ with $b_i | a_i$, for all $i = 1, 2, \ldots, m$. In [14], Straus and Subbarao introduced the concept of exponentially perfect numbers (or e-perfect numbers). In [2], Fabrykowski and Subbarao proved that any e-perfect number not divisible by 3 must be divisible by 2^{117}, greater than 10^{664}, and have at least 118 distinct prime factors. A positive integer n is said to be an *exponentially perfect number* (or *e-perfect number*), if the sum of proper e-divisors of n is equal to n, or equivalently if $\sigma^{(e)}(n) = 2n$, where $\sigma^{(e)}(n)$ denotes the sum of e-divisors of n. Some e-perfect numbers are $2^2 \times 3^2$, $2^2 \times 3^3 \times 5^2$, $2^4 \times 3^2 \times 11^2$, $2^4 \times 3^3 \times 5^2 \times 11^2$.

Sándor and Egri [11] introduced the notion of $+$ perfect numbers considering even divisors. A positive integer n is $+$ *perfect*, if n is equal to the sum of proper even divisors of n, i.e., $\sigma_+(n) = 2n$, where $\sigma_+(n)$ denotes the sum of even positive divisors of n. They used the symbol $+$ to indicate the even divisors of positive integers.

The concept of Zumkeller numbers, one of the generalizations of perfect numbers, was introduced by R. H. Zumkeller in 2003. A positive integer n is said to be a *Zumkeller number* or *integer perfect number*, if the set of positive divisors of n can be partitioned into two disjoint subsets of equal sum, which will be $\frac{\sigma(n)}{2}$, where $\sigma(n)$ denotes the sum of positive divisors of n. One of the most important reasons for investigating the properties of Zumkeller numbers is that all perfect numbers are Zumkeller numbers. Many researchers have investigated various interesting properties of Zumkeller numbers [5, 7, 13].

Recently, Kalita and Saikia [3] generalized the Zumkeller numbers to s-Zumkeller numbers. A positive integer n is said to be *s-Zumkeller number*, if the set of all proper positive divisors of n can be partitioned into two disjoint subsets of equal sum of squares of the proper positive divisors. Some examples of s-Zumkeller numbers are $60, 120, 180, 252, 300, 336, 360, 420$. In 2021, Patodia and Saikia [6] introduced the notion of m-Zumkeller numbers. A positive integer n is called *m-Zumkeller number*, if the positive divisors of n can be partitioned into two disjoint subsets of equal products. The positive integers $6, 8, 10, 14, 15, 16, 21$ are some m-Zumkeller numbers. Another branch of research on unitary Zumkeller numbers was explored in [1]. Generalizing the concept of e-perfect numbers and Zumkeller numbers, Kalita and Saikia [4] have introduced the concept of e-Zumkeller numbers. A positive integer n is said to be an *e-Zumkeller number* if the set of exponential divisors (e-divisors) of n can be partitioned into two subsets of equal sum. Some e-Zumkeller numbers are $36, 180, 252, 396, 468, 612, 684, 828, 900$.

Combining the concept of e-Zumkeller numbers and m-Zumkeller numbers, in this paper we define a new type of sequence of numbers, multiplicatively e-Zumkeller numbers. Additionally, we generalize s-Zumkeller numbers and m-Zumkeller numbers to $(+s)$-Zumkeller numbers and $(+m)$-Zumkeller numbers, respectively, considering their even positive divisors. We also establish related properties of these numbers.

2 Multiplicatively e-Zumkeller numbers

We first present some preliminary arithmetic functions and preliminary results that are necessary for our work.

The arithmetic functions $\sigma^{(e)}(n)$ and $\tau^{(e)}(n)$ denote the sum of all positive e-divisors of n and the number of positive e-divisors of n, respectively. The function $T_e(n)$ denotes the product of the e-divisors of n; it was first published in [8] and later applied also in [9]. These results and many related ones can be found also in the monograph [10].

Lemma 2.1. [3] *Let* $n = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ *be a positive integer. Then* $\tau^{(e)}(n) = \tau(a_1)\tau(a_2)\cdots\tau(a_m)$. *Also,* $\sigma^{(e)}(n) = \prod_{i=1}^{m} \sigma^{(e)}(p_i^{a_i}) = \prod_{j=1}^{m}(\sum_{b_j|a_j} p_i^{b_j})$.

Lemma 2.2. [8, 9] *If* $p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ *is the prime factorization of* n, *then* $T_e(n) = (t(n))^{\frac{\tau^e(n)}{2}}$ *while the arithmetical function* $t(n)$ *is given by* $t(1) = 1$ *and* $t(n) = p_1^{2^{\frac{\sigma(a_1)}{\tau(a_1)}}} p_2^{2^{\frac{\sigma(a_2)}{\tau(a_2)}}} \cdots p_m^{2^{\frac{\sigma(a_m)}{\tau(a_m)}}}$.

Now we define multiplicatively e-Zumkeller numbers.

Definition 2.1. *A positive integer* n *is called multiplicatively* e*-Zumkeller number if the set of all* e*-divisors of* n *can be partitioned into two subsets of equal product. Equivalently, a positive integer* n *is called a multiplicatively* e*-Zumkeller number if the set* D *of all positive* e*-divisors of* n *can be partitioned as* $\{A, B\}$ *such that*

$$\prod_{d \in A} d = \prod_{d \in B} d = \sqrt{T_e(n)}.$$

Example 2.1.1. *The positive integers* $36, 64, 100, 180, 729$ *are some examples of multiplicatively* e*-Zumkeller numbers.*

Now we present some properties of multiplicatively e-Zumkeller numbers.

Proposition 2.2. *If* n *is a multiplicatively* e*-Zumkeller number, then*

(i) $\sqrt{T_e(n)} \geq n$, *and*

(ii) $\tau^e(n) \geq 4$.

Proof. (i) By definition, it is obvious.

(ii) Let n be a multiplicatively e-Zumkeller number. By definition, it is clear that $\tau^e(n) \geq 3$. If $\tau^e(n) = 3$, then n is of the form p^{r^2} (where p is a prime and r is a positive integer greater than 1). Then the e-divisors of n are p, p^r and p^{r^2}. It is impossible to partition these e-divisors of n into two subsets of equal product. Therefore, $\tau^e(n) \geq 4$. □

Proposition 2.3. *A positive integer* n*, with* $T_e(n) \geq 4$*, is multiplicatively* e*-Zumkeller number if and only if* $\frac{\sqrt{T_e(n)}}{n}$ *is equal to* 1 *or equal to the product of some* e*-divisors of* n *excluding* n.

Proof. Let n be a multiplicatively e-Zumkeller number with partition $\{A, B\}$. Without loss of generality, we may assume that $n \in A$.

- Case 1. If A contains only n, then by the definition of multiplicatively e-Zumkeller numbers, $\sqrt{T_e(n)} = n$. This implies $\frac{\sqrt{T_e(n)}}{n} = 1$.
- Case 2. If A contains elements other than n, then the product of the remaining elements is $\frac{\sqrt{T_e(n)}}{n}$. Hence $\frac{\sqrt{T_e(n)}}{n}$ is the product of some e-divisors of n excluding n.

Conversely, suppose that $\frac{\sqrt{T_e(n)}}{n}$ is equal to 1 or equal to the product of some e-divisors of n excluding n. If we augment this set with n, then we have a set of some positive e-divisors of n including n, whose product will be $\sqrt{T_e(n)}$. Therefore, n is a multiplicatively e-Zumkeller number. □

Proposition 2.4. *If α is a Zumkeller number, then $n = (p_1p_2)^\alpha$ is a multiplicatively e-Zumkeller number.*

Proof. Let α be a Zumkeller number. Suppose that the divisors of α are $d_1, d_2, \dots, d_m, d'_1, d'_2, \dots, d'_n$ such that

$$\sum_{i=1}^{m} d_i = \sum_{j=1}^{n} d'_j = \frac{\sigma(\alpha)}{2}.$$

Then the e-divisors of n are $p_1^{d_i}p_2^{d_i}$, $p_1^{d_i}p_2^{d'_j}$, $p_1^{d'_j}p_2^{d_i}$ and $p_1^{d'_j}p_2^{d'_j}$ ($i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$). We can partition this set of e-divisors as $\{A, B\}$, where $A = \{p_1^{d_i}p_2^{d_i}, p_1^{d'_j}p_2^{d'_j}\}$ and $B = \{p_1^{d_i}p_2^{d'_j}, p_1^{d'_j}p_2^{d_i}\}$ such that

$$\begin{aligned}\prod_{d\in A} d = p_1^{m\sum_{i=1}^m d_i} p_2^{m\sum_{i=1}^m d_i} \times p_1^{n\sum_{j=1}^n d'_j} p_2^{n\sum_{j=1}^n d'_j} &= p_1^{m\frac{\sigma(\alpha)}{2}} p_2^{m\frac{\sigma(\alpha)}{2}} \times p_1^{n\frac{\sigma(\alpha)}{2}} p_2^{n\frac{\sigma(\alpha)}{2}} \\ &= (p_1p_2)^{(m+n)\frac{\sigma(\alpha)}{2}}.\end{aligned}$$

Also,

$$\begin{aligned}\prod_{d\in B} d = p_1^{n\sum_{i=1}^m d_i} p_2^{m\sum_{j=1}^n d'_j} \times p_1^{m\sum_{j=1}^n d'_j} p_2^{n\sum_{i=1}^m d_i} &= p_1^{n\frac{\sigma(\alpha)}{2}} p_2^{m\frac{\sigma(\alpha)}{2}} \times p_1^{m\frac{\sigma(\alpha)}{2}} p_2^{n\frac{\sigma(\alpha)}{2}} \\ &= (p_1p_2)^{(m+n)\frac{\sigma(\alpha)}{2}}.\end{aligned}$$

Thus, $\prod_{d\in B} d = \prod_{d\in B} d$. Therefore, n is a multiplicatively e-Zumkeller number. □

Proposition 2.5. *If n is a multiplicatively e-perfect number, then n is a multiplicatively e-Zumkeller number.*

Proof. Let n be a multiplicatively e-perfect number. Therefore, $T_e(n) = n^2$. This implies that $\sqrt{T_e(n)} = n$ and so $\frac{\sqrt{T_e(n)}}{n} = 1$. Therefore, by Proposition 2.3, n is a multiplicatively e-Zumkeller number. □

However, the converse of the above proposition does not hold. For example, 2^{12} is a multiplicatively e-Zumkeller number (since 12 is a Zumkeller number), but not a multiplicatively e-perfect number.

Proposition 2.6. *If $n = \prod_{i=1}^m p_i^{\alpha_i}$ is a multiplicatively e-Zumkeller number, then $\sigma(\alpha_i)\tau^{(e)}(n) \equiv 0 \pmod{2\tau(a_i)}$.*

Proof. By Lemma 2.2, $T_e(n) = p_1^{\frac{\tau^e(n)\sigma(a_1)}{\tau(a_1)}} p_2^{\frac{\tau^e(n)\sigma(a_2)}{\tau(a_2)}} \cdots p_m^{\frac{\tau^e(n)\sigma(a_m)}{\tau(a_m)}}$. Since n is a multiplicatively e-Zumkeller number, $T_e(n)$ must be a perfect square number. So $2 \mid \frac{\tau^e(n)\sigma(a_i)}{\tau(a_i)}$, for all $i = 1, 2, \dots, m$. Therefore, $2\tau(a_i) \mid \tau^e(n)\sigma(a_i)$. Hence, $\sigma(\alpha_i)\tau^{(e)}(n) \equiv 0 \pmod{2\tau(a_i)}$, for all $i = 1, 2, \dots, m$. □

Proposition 2.7. *Any positive integer n of the form $p_1^{\alpha_1}p_2^{\alpha_2}$, where α_1 and α_2 are prime numbers, is a multiplicatively e-Zumkeller number.*

Proof. Let $n = p_1^{\alpha_1}p_2^{\alpha_2}$, where α_1 and α_2 be prime numbers. Therefore, the divisors of α_1 are 1 and α_1. Similarly, the divisors of α_2 are 1 and α_2. This implies that the e divisors of $n = p_1^{\alpha_1}p_2^{\alpha_2}$ are p_1p_2, $p_1p_2^{\alpha_2}$, $p_1^{\alpha_1}p_2$, and $p_1^{\alpha_1}p_2^{\alpha_2}$. Therefore, the set of e divisors of n, can be partitioned as $\{p_1p_2, p_1^{\alpha_1}p_2^{\alpha_2}\}$ and $\{p_1p_2^{\alpha_2}, p_1^{\alpha_1}p_2\}$, where the product of each subset is the same. Hence n is a multiplicatively e-Zumkeller number. □

Proposition 2.8. *If n is a multiplicatively e-Zumkeller number such that the numbers of e-divisors of each subset of Zumkeller partitions are equal, then for any positive integer l and any prime p with $(n,p) = 1$, np^l is a multiplicatively e-Zumkeller number.*

Proof. Let n be a multiplicatively e-Zumkeller number such that the numbers of e-divisors of each subset of partitions are equal. Let the multiplicatively e-Zumkeller partition be $\{A, B\}$. Therefore, $\prod_{d\in A} d = \prod_{d\in B} d = \sqrt{T_e(n)}$. Let the divisors of l be $d_1, d_2, \ldots, d_k$. Then the set of e-divisors of np^l is $p^{d_1}A \cup p^{d_2}A \cup \cdots \cup p^{d_k}A \cup p^{d_1}B \cup p^{d_2}B \cup \cdots \cup p^{d_k}B$, which can be partitioned as $\{A', B'\}$, where $A' = p^{d_1}A \cup p^{d_2}A \cup \cdots \cup p^{d_k}A$ and $B' = p^{d_1}B \cup p^{d_2}B \cup \cdots \cup p^{d_k}B$. Clearly, $\prod_{d\in A'} d = \prod_{d\in B'} d$. Hence np^l is a multiplicatively e-Zumkeller number. □

3 $(+s)$-Zumkeller numbers

The arithmetic functions $\sigma_+(n)$ and $\sigma_{2+}(n)$ denote the sum of even positive divisors of n and the sum of squares of all even positive divisors of n, respectively. At first, we present some preliminary facts about the functions σ_+ and σ_{2+}.

Lemma 3.1. [12] *If n is odd, then $\sigma_+(n) = 0$. If n is even of the form $n = 2^kN$, $k \geq 1$ with N odd, then $\sigma_+(n) = 2(2^k - 1)\sigma(N)$.*

Lemma 3.2. *If n is odd, then $\sigma_{2+}(n) = 0$. If n is even, $n = 2^kN = 2^kp_1^{k_1}p_2^{k_2}\cdots p_m^{k_m}$, $k \geq 1$ with $p_1, p_2, \ldots, p_m$ are odd primes, then*

$$\sigma_{2+}(n) = \frac{4}{3}(2^{2k} - 1)\sigma_2(N) \quad \textit{and} \quad \frac{\sigma_{2+}(n)}{n^2} < \frac{4}{3}(1 - \frac{1}{2^{2k}})\prod_{i=1}^{m}\frac{p_i^2}{p_i^2 - 1}.$$

Proof. If n is odd, then obviously $\sigma_{2+}(n) = 0$.
If n is even, $n = 2^kN$, $k \geq 1$ with N odd, then $\sigma_{2+}(n) = \{2^2 + (2^2)^2 + (2^3)^2 + \cdots + (2^k)^2\}\sigma_2(N)$. This implies that $\sigma_{2+}(n) = \frac{4}{3}(2^{2k} - 1)\sigma_2(N)$.

Since $\sigma_2(N) = \sigma_2(p_1^{k_1}p_2^{k_2}\cdots p_m^{k_m}) = \prod_{i=1}^{m}\frac{p_i^{2(k_i+1)}-1}{p_i^2-1}$, therefore,

$$\sigma_{2+}(n) = \frac{4}{3}(2^{2k} - 1)\prod_{i=1}^{m}\frac{p_i^{2(k_i+1)} - 1}{p_i^2 - 1}.$$

and

$$\frac{\sigma_{2+}(n)}{n^2} = \frac{4}{3}\frac{2^{2k} - 1}{2^{2k}}\prod_{i=1}^{m}\frac{p_i^{2(k_i+1)} - 1}{p_i^{2k_i}(p_i^2 - 1)} < \frac{4}{3}(1 - \frac{1}{2^{2k}})\prod_{i=1}^{m}\frac{p_i^2}{p_i^2 - 1}.$$ □

Now we define $(+s)$-Zumkeller numbers and present some properties of these numbers.

Definition 3.1. *A positive integer n is said to be a $(+s)$-Zumkeller number, if the set of all even proper positive divisors of n can be partitioned into two disjoint subsets of equal sum of the squares of the proper even positive divisors.*

In other words, a positive integer n is called a $(+s)$-Zumkeller number if the set D of even proper positive divisors of n can be partitioned as $\{A, B\}$, such that

$$\sum_{d\in A} d^2 = \sum_{d\in B} d^2 = \frac{\sigma_{2^+}(n) - n^2}{2}.$$

These numbers are defined only for even numbers, not for odd numbers.

Example 3.1.1. $120, 240, 360, 480, 504, 600, 672, 720, 840, 960,$ *etc., are some $(+s)$-Zumkeller numbers.*

Theorem 3.1. *If n is a positive integer divisible by 4, then n is a $(+s)$-Zumkeller number if and only if $\frac{2\sigma_{2^+}(n)-3n^2}{4}$ is a sum (possibly an empty sum) of square of distinct proper even positive divisors of n excluding $\frac{n}{2}$.*

Proof. Since n is divisible by 4, $\frac{n}{2}$ is an even divisor of n. Assume that n is a $(+s)$-Zumkeller number with Zumkeller partition {A,B}. Without loss of generality, we may assume that $\frac{n}{2} \in A$. Then the sum of squares of the remaining elements of A will be

$$\frac{\sigma_{2^+}(n) - n^2}{2} - \left(\frac{n}{2}\right)^2 = \frac{2\sigma_{2^+}(n) - 3n^2}{4}.$$

Thus $\frac{2\sigma_{2^+}(n)-3n^2}{4}$ is a sum of squares of distinct even proper positive divisors of n excluding $\frac{n}{2}$.

Conversely, suppose that $\frac{2\sigma_{2^+}(n)-3n^2}{4}$ is a sum of squares of distinct even proper positive divisors of n excluding $\frac{n}{2}$. If we augment this set with $\frac{n}{2}$, we have a set of even positive divisors of n, whose sum of squares is equal to

$$\frac{2\sigma_{2^+}(n) - 3n^2}{4} + \left(\frac{n}{2}\right)^2 = \frac{\sigma_{2^+}(n) - n^2}{2}.$$

Therefore, the sum of the squares of the even proper positive divisors of n of the complementary set of above mentioned augmented set will be equal. Hence n is a $(+s)$-Zumkeller number. □

Theorem 3.2. (i) *If $n = 2^k p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$ is a $(+s)$-Zumkeller number, where $k > 1$ and $p_1 < p_2 < \cdots < p_m$, then $(1 - \frac{1}{2^{2k}}) \prod_{i=1}^{m} \frac{p_i^2}{p_i^2-1} > \frac{9}{8}$.*

(ii) *If $n = 2p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$ is a $(+s)$-Zumkeller number, then $\prod_{i=1}^{m} \frac{p_i^2}{p_i^2-1} > \frac{2}{p_1^2} + 1$.*

Proof. (i) Let $n = 2^k N = 2^k p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$ be a $(+s)$-Zumkeller number, where $k > 1$ and $p_1 < p_2 < \cdots < p_m$. So, $\frac{n}{2}$ is the greatest even proper positive divisor of n. Then $\frac{\sigma_{2^+}(n)-n^2}{2} \geq (\frac{n}{2})^2$. This implies $\sigma_{2^+}(n) \geq \frac{2n^2}{4} + n^2$. Then

$$\frac{3}{2} \leq \frac{\sigma_{2^+}(n)}{n^2}. \tag{1}$$

Again, by Lemma 3.2

$$\frac{\sigma_{2+}(n)}{n^2} < \frac{4}{3}\left(1-\frac{1}{2^{2k}}\right)\prod_{i=1}^{m}\frac{p_i^2}{p_i^2-1}. \tag{2}$$

Hence, by (1) and (2)

$$\frac{3}{2} < \frac{4}{3}\left(1-\frac{1}{2^{2k}}\right)\prod_{i=1}^{m}\frac{p_i^2}{p_i^2-1}, \text{ which implies that } \left(1-\frac{1}{2^{2k}}\right)\prod_{i=1}^{m}\frac{p_i^2}{p_i^2-1} > \frac{9}{8}.$$

(ii) Using a similar method, for $n = 2N = 2p_1^{k_1}p_2^{k_2}\cdots p_m^{k_m}$, we can easily prove that

$$\prod_{i=1}^{m}\frac{p_i^2}{p_i^2-1} > \frac{2}{p_1^2}+1.$$

□

Theorem 3.3. *If $n = 2^k p_1^{k_1}p_2^{k_2}\cdots p_m^{k_m}$ (for $k > 1$) is a $(+s)$-Zumkeller number, then n contains at least*

(i) 3 *distinct odd primes, for $k = 2$.*

(ii) 2 *distinct odd primes, for $k = 3$.*

Proof. Let $n = 2^k p_1^{k_1}p_2^{k_2}\cdots p_m^{k_m}$ $(k > 1)$ be a $(+s)$-Zumkeller number. Then, by Theorem 3.2(i),

$$\left(1-\frac{1}{2^{2k}}\right)\prod_{i=1}^{m}\frac{p_i^2}{p_i^2-1} > \frac{9}{8}. \tag{3}$$

(i) If $k = 2$, then

$$\prod_{i=1}^{m}\frac{p_i^2}{p_i^2-1} > \frac{6}{5}. \tag{4}$$

For $m < 4$

$$\prod_{i=1}^{m}\frac{p_i^2}{p_i^2-1} \le \frac{3^2}{3^2-1}\times\frac{5^2}{5^2-1}\times\frac{7^2}{7^2-1} = \frac{9}{8}\times\frac{25}{24}\times\frac{49}{48} < \frac{6}{5},$$

which contradicts (4).

But for $m = 4$, inequality (4) holds, as

$$\frac{3^2}{3^2-1}\times\frac{5^2}{5^2-1}\times\frac{7^2}{7^2-1}\times\frac{11^2}{11^2-1} = \frac{9}{8}\times\frac{25}{24}\times\frac{49}{48}\times\frac{121}{120} > \frac{6}{5}.$$

Hence for $k = 2$, there must exist at least 4 distinct odd primes.

(ii) If $k = 3$, then

$$\prod_{i=1}^{m}\frac{p_i^2}{p_i^2-1} > \frac{8}{7}. \tag{5}$$

For $m < 2$,

$$\prod_{i=1}^{m}\frac{p_i^2}{p_i^2-1} \le \frac{3^2}{3^2-1} = \frac{9}{8} < \frac{8}{7},$$

which contradicts (5).

But for $m = 2$, inequality (4) holds, as

$$\frac{3^2}{3^2-1} \times \frac{5^2}{5^2-1} = \frac{9}{8} \times \frac{25}{24} > \frac{8}{7}.$$

Hence for $k = 3$, there must exist at least 2 distinct odd primes. $\square$

From the above theorem, we get the following Table 1 (for $1 < k \leq 5$) of the least number of distinct odd primes, for $(+s)$-Zumkeller numbers of the form $2^k p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$.

Table 1. Least number of distinct odd primes for a $(+s)$-Zumkeller number of the form $2^k p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$.

k	**Least number of odd primes (m)**
2	4
3	2
4	2
5	2

Theorem 3.4. *If $n = 2p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$ is a $(+s)$-Zumkeller number and 3 divides n, then n contains at least 7 distinct odd primes.*

Proof. By Theorem 3.2(ii),

$$\prod_{i=1}^{m} \frac{p_i^2}{p_i^2-1} > \frac{2}{p_1^2} + 1.$$

If $p_1 = 3$, then

$$\prod_{i=1}^{m} \frac{p_i^2}{p_i^2-1} > \frac{11}{9}. \tag{6}$$

For $m < 7$,

$$\prod_{i=1}^{m} \frac{p_i^2}{p_i^2-1} \leq \frac{3^2}{3^2-1} \times \frac{5^2}{5^2-1} \times \frac{7^2}{7^2-1} \times \frac{11^2}{11^2-1} \times \frac{13^2}{13^2-1} \times \frac{17^2}{17^2-1} < \frac{11}{9},$$

which contradicts the inequality (6).

But for $m = 7$, the inequality holds as:

$$\frac{3^2}{3^2-1} \times \frac{5^2}{5^2-1} \times \frac{7^2}{7^2-1} \times \frac{11^2}{11^2-1} \times \frac{13^2}{13^2-1} \times \frac{17^2}{17^2-1} \times \frac{19^2}{19^2-1} > \frac{11}{9}.$$

Hence n must contain at least 7 distinct odd primes. $\square$

From the above theorem and following the proof of result (ii), we get the following Table 2 (for $p_1 \leq 13$) for the number $n = 2p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$.

Table 2. Least number of distinct odd primes for a $(+s)$-Zumkeller number of the form $2p_1^{k_1}p_2^{k_2}\cdots p_m^{k_m}$.

Smallest odd prime divisor (p_1)	**Least number of odd primes (m)**
3	7
5	5
7	6
11	3
13	3

4 $(+m)$-Zumkeller numbers

In this section, we define some preliminary arithmetic functions and results that are needed to study the characteristics of the $(+m)$-Zumkeller numbers. Then we define $(+m)$-Zumkeller numbers and investigate their characteristics.

The function $\tau_+(n)$ denotes the number of even positive divisors of n.

Lemma 4.1. *If $n = 2^\alpha p_1^{k_1}p_2^{k_2}\cdots p_m^{k_m}$, then $\tau_+(n) = \alpha\tau(p_1^{k_1}p_2^{k_2}\cdots p_m^{k_m})$.*

The function $T(n)$ denotes the product of all positive divisors of n.

Lemma 4.2. [6] *If $n = \Pi_{i=1}^m p_i^{k_i}$ is a positive integer, then $T(n) = n^{\frac{\tau(n)}{2}} = (\Pi_{i=1}^m p_i^{k_i})^{\frac{\tau(n)}{2}}$.*

The function $T_+(n)$ denotes the product of all even positive divisors of n.

Lemma 4.3. *If n is a positive integer, then $T_+(n) = (2n)^{\frac{\tau_+(n)}{2}}$.*

Proof. Let $n = 2^\alpha N = 2^\alpha p_1^{k_1}p_2^{k_2}\cdots p_m^{k_m}$, where $N = p_1^{k_1}p_2^{k_2}\cdots p_m^{k_m}$. Then we have

$$T_+(n) = 2^{(1+2+\cdots+\alpha)(1+k_1)(1+k_2)\cdots(1+k_m)}(T(\Pi_{i=1}^m p_i^{k_i}))^\alpha.$$

This implies $T_+(n) = 2^{\frac{\alpha}{2}(\alpha+1)\tau(N)}(\Pi_{i=1}^m p_i^{k_i})^{\frac{\alpha\tau(N)}{2}}$. Then $T_+(n) = (2\times 2^\alpha\Pi_{i=1}^m p_i^{k_i})^{\frac{\alpha\tau(N)}{2}}$. Since $\alpha\tau(N) = \tau_+(2^\alpha N) = \tau_+(n)$, $T_+(n) = (2n)^{\frac{\tau_+(n)}{2}}$. □

Now we define $(+m)$-Zumkeller numbers with suitable examples.

Definition 4.1. *A positive integer n is called a (+m)-Zumkeller number if the even positive divisors of n can be partitioned into two disjoint subsets of equal product. $(+m)$-Zumkeller numbers are defined only for even positive integers.*

Example 4.1.1. *The integers $8, 12, 16, 20$, etc., are some $(+m)$-Zumkeller numbers.*

It is obvious that a positive integer n is a $(+m)$-Zumkeller if and only if $T_+(n)$ is a perfect square.

Theorem 4.1. *A positive integer $n = 2^\alpha\Pi_{i=1}^m p_i^{\alpha_i}$ is a $(+m)$-Zumkeller number if and only if $4 \mid (\alpha+1)\tau_+(n)$ and $4 \mid \alpha_i\tau_+(n)$, for any $i = 1, 2, 3, \ldots, m$.*

Proof. Let, $n = 2^\alpha \Pi_{i=1}^m p_i^{\alpha_i}$. Then by Lemma 4.3, $T_+(n) = (2n)^{\frac{\tau_+(n)}{2}}$. This implies

$$T_+(n) = (2 \times 2^\alpha \Pi_{i=1}^m p_i^{\alpha_i})^{\frac{\tau_+(n)}{2}} = (2^{(\alpha+1)} \Pi_{i=1}^m p_i^{\alpha_i})^{\frac{\tau_+(n)}{2}} = 2^{\frac{(\alpha+1)\tau_+(n)}{2}} \Pi_{i=1}^m p_i^{\frac{\alpha_i \tau_+(n)}{2}}.$$

Therefore, n is a $(+m)$-Zumkeller number if and only if n is perfect square. Hence n is a $(+m)$-Zumkeller number if and only if $4|(\alpha+1)\tau_+(n)$ and $4|\alpha_i \tau_+(n)$, for all $i = 1, 2, \dots, m$. □

Corollary 4.1.1. *If p, q are odd primes and k is any positive integer, then 2^{4k}, $2^{(4k-1)}$, $2^{(4k-2)}p$, $2p^{(4k-1)}$, $2^2p^{(4k-3)}$, $2^3p^{(4k-1)}$, etc., are $(+m)$-Zumkeller numbers.*

Example 4.1.2. (i) $16 = 2^4$ *is a $(+m)$-Zumkeller number, where the partitions are* $\{2, 16\}$ *and* $\{4, 8\}$.

(ii) $8 = 2^{(4-1)}$ *is a $(+m)$-Zumkeller number, where the partitions are* $\{2, 4\}$ *and* $\{8\}$.

(iii) $12 = 2^{(4-2)} \times 3$ *is a $(+m)$-Zumkeller number, where the partitions are* $\{2, 12\}$ *and* $\{4, 6\}$.

(iv) $54 = 2 \times 3^{(4-1)}$ *is a $(+m)$-Zumkeller number, where the partitions are* $\{2, 54\}$ *and* $\{6, 18\}$.

(v) $216 = 2^3 \times 3^{(4-1)}$ *is a $(+m)$-Zumkeller number, where the partitions are* $\{2, 4, 6, 72, 108, 216\}$ *and* $\{8, 12, 18, 24, 36, 54\}$.

Corollary 4.1.2. *If $n = 2^\alpha \Pi_{i=1}^m p_i, (m > 1)$, then n is a $(+m)$-Zumkeller number.*

Proof. Here $\tau_+(n) = \alpha \times \underbrace{2 \times 2 \times \cdots \times 2}_{m \text{ times}}$. Since $m > 1$, so clearly $4|\tau_+(n)$. Also $4|(\alpha+1)\tau_+(n)$. Therefore, by Theorem 4.1, n is a $(+m)$-Zumkeller number. □

Theorem 4.2. *If n is a $(+m)$-Zumkeller number and p is an odd prime with $(n, p) = 1$, then for any positive integer l, np^l is a $(+m)$-Zumkeller number if and only if $\tau_+(n)$ is even or $l \equiv 0 \pmod 4$ or $l \equiv 3 \pmod 4$.*

Proof. Let $n = 2^\alpha \Pi_{i=1}^m p_i^{\alpha_i}$, where p_i's are odd primes. Since $(n, p) = 1$, so $\tau_+(np^l) = \tau_+(n)(l+1)$. So np^l is a $(+m)$-Zumkeller number if and only if $4|(\alpha+1)\tau_+(np^l)$, $4|\alpha_i\tau_+(np^l)$ and $4|l(l+1)\tau_+(n)$. Since n is a $(+m)$-Zumkeller number, $4|(\alpha+1)\tau_+(np^l)$ and $4|\alpha_i\tau_+(np^l)$ are obvious. Therefore, np^l is a $(+m)$-Zumkeller number if and only if $4|l(l+1)\tau_+(n)$, that is, $2|\tau_+(n)$ or $l \equiv 0 \pmod 4$ or $l \equiv 3 \pmod 4$. Hence, if n is a $(+m)$-Zumkeller number, then np^l is a $(+m)$-Zumkeller number if and only if $\tau_+(n)$ is even or $l \equiv 0 \pmod 4$ or $l \equiv 3 \pmod 4$. □

From the above theorem we get the following corollary.

Corollary 4.1.3. *If n is a $(+m)$-Zumkeller number such that $\tau_+(n)$ is even and $\gcd(n, m) = 1$, then nm is a $(+m)$-Zumkeller number.*

Theorem 4.3. *Let $c_0, c_1 \dots, c_k$ be non-negative integers. If a is a $(+m)$-Zumkeller number such that $a|c_i$, (for all $1 \le i \le k$), $\gcd(a, c_0) = 1$ and $\tau_+(n)$ is even, then the polynomial*

$$p(n) = ac_0 + ac_1n + ac_2n^2 + \cdots + ac_kn^k$$

is a $(+m)$-Zumkeller number for all $n \in \mathbb{N}$.

Proof. Let a be a $(+m)$-Zumkeller number such that $a \mid c_i$, for all $1 \leq i \leq k$ and $\gcd(a, c_0) = 1$. So, for all positive integers n,

$$\gcd(a, c_0 + c_1 n + c_2 n^2 + \cdots + c_k n^k) = 1.$$

Therefore, by Corollary 4.1.3,

$$a(c_0 + c_1 n + c_2 n^2 + \cdots + c_k n^k) = ac_0 + ac_1 n + ac_2 n^2 + \cdots + ac_k n^k$$

is a $(+m)$-Zumkeller number. □

5 Conclusion and future scope

In this paper, we have defined and studied the properties of multiplicatively e-Zumkeller numbers, $(+s)$-Zumkeller numbers and $(+m)$-Zumkeller numbers. The concept of graph labeling using Zumkeller numbers is one of the most prominent research ideas for mathematicians. In future, we can investigate the graph labeling for different graphs using the notation of these newly defined variants of Zumkeller numbers.

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