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Padovan numbers which are concatenations of three Padovan or Perrin numbers

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Abstract: This paper presents all Padovan numbers that can be written as the concatenation of three Padovan or Perrin numbers under a certain constraint. Namely, we consider the Diophantine equations

$$P_k = 10^{d+l} P_m + 10^l P_n + P_r$$

and

$$P_k = 10^{d+l} R_m + 10^l R_n + R_r,$$

where k, m, n, r, d and l are positive integers satisfying $n \leq m$. The parameters d and l denote the numbers of digits in the integers P_n(or R_n) and P_r (or R_r), respectively. The solutions to these equations can be written in the form $P_{18} = \overline{P_2 P_2 P_6} = 114$ for all $m, n, r \geq 2$ and, similarly, $P_{30} = \overline{R_3 R_3 R_{12}} = 3329$, $P_{33} = \overline{R_7 R_7 R_{13}} = 7739$ for all $m \geq 3$ and $n, r \geq 1$.
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1 Introduction

Let $(P_n)_{n\geq 0}$ and $(R_n)_{n\geq 0}$ be the sequences of Padovan and Perrin numbers given by

$$P_0 = P_1 = P_2 = 1; P_n = P_{n-2} + P_{n-3} \text{ for all } n \geq 3$$

and

$$R_0 = 3, R_1 = 0, R_2 = 2; R_n = R_{n-2} + R_{n-3} \text{ for all } n \geq 3.$$

For further details on the Padovan and Perrin numbers, see references [10, 16]. Using linear forms in logarithms to solve Diophantine equations is widely employed by many researchers as a common tool. For this type of work, references for [5, 14, 17, 19] can be consulted.

Since Bank and Luca's work [4], it has become a popular problem among researchers to investigate whether certain terms of a sequence can be formed as the concatenation of two or three terms either from another sequence or from the sequence itself. For these studies, the reader may refer to [1–3, 6, 11–13]. We then asked the following question: Can a Padovan number be expressed as a concatenation of three Padovan or Perrin numbers? Thus, we have tackled the solutions to the Diophantine equations

$$P_k = \overline{P_m P_n P_r} = 10^{d+l} P_m + 10^l P_n + P_r \tag{1}$$

and

$$P_k = \overline{R_m R_n R_r} = 10^{d+l} R_m + 10^l R_n + R_r, \tag{2}$$

in positive integers (k, m, n, r, d, l) with $n \leq m$, where d and l represent the numbers of digits of (P_n, R_n) and (P_r, R_r), respectively. Since the values of P_0, P_1, and P_2 are the same, we will take $m, n, r \geq 2$ in Equation (1) to avoid trivial solutions. If $m = 1$ in Equation (2), then we have $P_k = \overline{R_n R_r}$. This equation was also solved in [6]. Moreover, since $R_0 = R_3$ and $R_2 = R_4$, we will take $m \geq 3$ and $n, r \geq 1$ in Equation (2).

2 Properties of Padovan and Perrin sequences

In this section, we give some algebraic properties of the Padovan and Perrin sequences. Let $\omega_1 = \sqrt[3]{108 + 12\sqrt{69}}$ and $\omega_2 = \sqrt[3]{108 - 12\sqrt{69}}$. The Binet formulas for the Padovan and Perrin numbers are

$$P_n = t \cdot \alpha^n + s \cdot \beta^n + r \cdot \gamma^n \text{ and } R_n = \alpha^n + \beta^n + \gamma^n.$$

Here,

$$\alpha = \frac{\omega_1 + \omega_2}{6} \text{ and } \overline{\gamma} = \beta = \frac{-\omega_1 - \omega_2 + i\sqrt{3}(\omega_1 - \omega_2)}{12},$$

are the roots of the characteristic equation $x^3 - x - 1 = 0$. Moreover,

$$t = \frac{\alpha + 1}{-\alpha^2 + 3\alpha + 1}, s = \frac{\beta + 1}{-\beta^2 + 3\beta + 1}, r = \frac{\gamma + 1}{-\gamma^2 + 3\gamma + 1}.$$

The minimal polynomial of t over $\mathbb{Z}$ is given by $23x^3 - 23x^2 + 6x - 1$, whose zeros are t, s, r. Furthermore, it can be readily verified that the following estimates are valid:

$$1.32 < \alpha < 1.33, \ 0.86 < |\beta| = |\gamma| < \alpha^{-1/2} < 0.87,$$

and

$$0.72 < |t| < 0.73, 0.24 < |s| = |r| < 0.25.$$

Put

$$e(n) := P_n - t{\cdot}\alpha^n = s \cdot \beta^n + r \cdot \gamma^n$$

and

$$e^{'}(n) := R_n - \alpha^n = \beta^n + \gamma^n.$$

Then $|e(n)| < \frac{0.5}{\alpha^{n/2}}$ and $|e^{'}(n)| < \frac{2}{\alpha^{n/2}}$ for $n \geq 1$. Let $F := \mathbb{Q}(\alpha, \beta)$ be the splitting field of the polynomial ϕ over $\mathbb{Q}$. Then $|\mathrm{Gal}(F/\mathbb{Q})| = [F : \mathbb{Q}] = 6$, $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$ and

$$\mathrm{Gal}(F/\mathbb{Q}) \simeq \{(1), (\alpha\beta), (\alpha\gamma), (\beta\gamma), (\alpha\beta\gamma), (\alpha\gamma\beta)\} \simeq S_3.$$

The relation between P_n, R_n and α is expressed by

$$\alpha^{n-3} \leq P_n \leq \alpha^{n-1} \text{ for all } n \geq 1 \tag{3}$$

and

$$\alpha^{n-2} \leq R_n \leq \alpha^{n+1} \text{ for all } n \geq 2. \tag{4}$$

3 Preliminaries

Before presenting our main results, we first provide a definition and several lemmas in this section.

Definition 3.1. *Let the nonzero algebraic number γ of degree d over $\mathbb{Q}$ with minimal polynomial over $\mathbb{Z}$ be $c_0 \prod_{i=1}^{d} \left(x - \gamma^{(i)}\right)$. Then*

$$h(\gamma) = \frac{1}{d}\left(\log|c_0| + \sum_{i=1}^{d} \log\left(\max\left\{|\gamma^{(i)}|, 1\right\}\right)\right)$$

is called the absolute logarithmic height of γ.

The next lemma states several properties of logarithmic height, as detailed in [18].

Lemma 3.1. *Let $\gamma, \gamma_1, \gamma_2, \ldots, \gamma_n$ be elements of an algebraic closure of $\mathbb{Q}$ and $m \in \mathbb{Z}$. Then*

(1) $h(\gamma_1 \cdots \gamma_n) \leq \sum_{i=1}^{n} h(\gamma_i)$

(2) $h(\gamma_1 + \cdots + \gamma_n) \leq \log n + \sum_{i=1}^{n} h(\gamma_i)$

(3) $h(\gamma^m) = |m| h(\gamma)$.

As stated in [8], the following lemma allows us to find an upper bound for k.

Lemma 3.2. *Let $\gamma_1, \gamma_2, \ldots, \gamma_t$ be positive real algebraic numbers and let $b_1, b_2, \ldots, b_t$ be nonzero integers. Let D be the degree of the number field $\mathbb{Q}(\gamma_1, \gamma_2, \ldots, \gamma_t)$ on $\mathbb{Q}$. Let*

$$B \geq \max\{|b_1|, \ldots, |b_t|\} \text{ and } A_i \geq \max\{Dh(\gamma_i), |\log\gamma_i|, 0.16\}$$

for all $i = 1, \ldots, t$. If $\Gamma := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1$ is not zero, then

$$|\Gamma| > \exp\left(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2 (1 + \log D)(1 + \log B) A_1 A_2 \cdots A_t\right).$$

A version of the Baker–Davenport lemma, stated as the following lemma, appears in [7].

Lemma 3.3. *Let κ be an irrational number and p/q be the convergence of the continued fraction of κ such that $q > 6M$. Let M be a positive integer and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\epsilon := ||\mu q|| - M||\gamma q||$, where $||\cdot||$ denotes the distance from the nearest integer. If $\epsilon > 0$, then there is no positive integer solution (r, s, t) to inequality*

$$0 < |r\kappa - s + \mu| < AB^{-t},$$

subject to the restrictions that $r \leq M$ and $t \geq \dfrac{\log(Aq/\epsilon)}{\log B}$.

The following two lemmas can be found in [9, 15], respectively.

Lemma 3.4. *Let $\rho, \Gamma \in \mathbb{R}$. If $0 < \rho < 1$ and $|\Gamma| < \rho$, then*

$$|\log(1+\Gamma)| < \frac{-\log(1-\rho)}{\rho} \cdot |\Gamma|.$$

Lemma 3.5. *Let η be a real number and $\eta = [a_0; a_1, a_2, a_3, \ldots]$. Put $u, v \in \mathbb{Z}$. If $\left|\eta - \frac{u}{v}\right| < \dfrac{1}{2v^2}$, then $\dfrac{u}{v}$ is the convergent of the continued fraction of η. Furthermore, if W and n are nonnegative integers such that $v_n > W$, then $\left|\eta - \dfrac{u}{v}\right| > \dfrac{1}{(s+2)v^2}$, where $s := \max\{a_i : i = 0, 1, 2, \ldots, n\}$.*

To establish the relationships among the variables in Equations (1) and (2), we present the following two lemmas. Because their proofs are similar, we provide the proof of the first lemma only.

Lemma 3.6. *Assuming the validity of Equation (1), we derive the subsequent inequalities.*

(a) $d < \dfrac{3n+17}{20}$,

(b) $l < \dfrac{3r+17}{20}$,

(c) $P_n < 10^d < 10P_n$,

(d) $P_r < 10^l < 10P_r$,

(e) $m + n + r - 8 < k < m + n + r + 17$,

(f) $k - r \geq 9$.

Proof. (a,c) Since d represents the number of digits of P_n, we have $d = \lfloor \log_{10} P_n \rfloor + 1$. Then, we can write

$$d = \lfloor \log_{10} P_n \rfloor + 1 \leq \log_{10} P_n + 1 \leq \log_{10} \alpha^{n-1} + 1 < \frac{3n+17}{20}$$

and

$$P_n = 10^{\log_{10} P_n} < 10^d \leq 10^{\log_{10} P_n + 1} = 10P_n.$$

Thus, the required proofs for Lemma 3.6(a) and Lemma 3.6(c) have been established.

(b,d) These are demonstrated in a manner similar to the proof of Lemma 3.6(a) and Lemma 3.6(c), respectively.

(e) Considering (3) and using Lemma 3.6(c)(d), we can say

$$\alpha^{k-3} \leq P_k = 100P_nP_rP_m + 10P_rP_n + P_r < 111P_mP_nP_r < \alpha^{m+n+r+14}$$

and

$$\alpha^{k-1} \geq P_k > P_rP_nP_m + P_rP_n + P_r > P_mP_nP_r \geq \alpha^{m+n+r-9}.$$

Thus, we get $m+n+r-8 < k < m+n+r+17$.

(f) First, we consider the inequality

$$P_k > 10P_rP_m + P_rP_n + P_r > 12P_r.$$

If $k - r \leq 8$, then we obtain

$$12P_r < P_k \leq P_{r+8}.$$

This is impossible. So, we can say $k - r \geq 9$. □

Lemma 3.7. *Given that Equation (2) is valid, then the inequalities below hold true.*

(a) $d < \frac{3n+23}{20}$,

(b) $l < \frac{3r+23}{20}$,

(c) $R_n < 10^d < 10R_n$,

(d) $R_r < 10^l < 10R_r$,

(e) $m+n+r-5 < k < m+n+r+23$,

(f) $k - r \geq 11$.

4 Main theorems

Theorem 4.1. *Let d and l be the numbers of digits of P_n and P_r, respectively. The only positive integer solution (k, m, n, r, d, l) of Equation (1) with $m, n, r \geq 2$ and $n \leq m$ is given by*

$$(k, m, n, r, d, l) \in \{(18, 2, 2, 4, 1, 1)\}.$$

Proof. We begin our proof by assuming $k \geq 18$ since $m, n, r \geq 2$. Next, we rewrite Equation (1) as

$$t \cdot \alpha^k - 10^{d+l} \cdot t \cdot \alpha^m = -(s \cdot \beta^k + r \cdot \gamma^k) + 10^{d+l}(s \cdot \beta^m + r \cdot \gamma^m) + 10^l P_n + P_r.$$

Taking into account $P_n < 10^d, P_r < 10^l$ (from Lemma 3.6(c)(d)), together with $k \geq 18$, $m \geq 2$ and $|t| > 0.72$, we get

$$\begin{aligned}\left|\frac{\alpha^{k-m}}{10^{d+l}} - 1\right| &\leq \frac{|s \cdot \beta^k + r \cdot \gamma^k|}{10^{d+l} \cdot |t| \cdot \alpha^m} + \frac{|s \cdot \beta^m + r \cdot \gamma^m|}{|t| \cdot \alpha^m} + \frac{10^l P_n + P_r}{10^{d+l} \cdot |t| \cdot \alpha^m} \\ &\leq \frac{1}{|t| \cdot \alpha^m}\left(\frac{0.5}{10^{d+l} \cdot \alpha^{\frac{k}{2}}} + \frac{0.5}{\alpha^{\frac{m}{2}}} + 1 + \frac{1}{10^d}\right),\end{aligned}$$

i.e.,

$$\left|\frac{\alpha^{k-m}}{10^{d+l}}-1\right|<\frac{2.06}{\alpha^m}. \tag{5}$$

We apply Lemma 3.2 with the pairs $(\gamma_1,b_1):=(\alpha,k-m)$ and $(\gamma_2,b_2):=(10,-d)$. We can take $\mathbb{K}=\mathbb{Q}(\alpha)$, for which $D=3$. We now demonstrate that $\Gamma_1:=\frac{\alpha^{k-m}}{10^{d+l}}-1\neq 0$. If Γ_1 were equal to zero, it would follow that $\alpha^{k-m}=10^{d+l}$. After applying an automorphism σ to both sides and taking the absolute values, we arrive at

$$\left|10^{d+l}\right|=\left|\sigma(\alpha^{k-m})\right|=|\beta|^{k-m}<1,$$

which is impossible. Given that $h(\gamma_1)=h(\alpha)=\frac{\log\alpha}{3}$ and $h(\gamma_2)=h(10)=\log 10$, we can choose $A_1:=\log\alpha$ and $A_2:=3\log 10$. Additionally, we can take $B:=k+1$. By Lemma 3.6(a)(b)(e), we have the inequality

$$d+l<\frac{3(n+r)+34}{20}<\frac{3(k-m+8)+34}{20}<k-m+3\leq k+1, \tag{6}$$

for $m\geq 2$. Let $K=-1.4\cdot 30^5\cdot 2^{4.5}\cdot 3^2\cdot(1+\log 3)$. Thus, combining inequality (5) and Lemma 3.2, we conclude that

$$2.06\cdot\alpha^{-m}>|\Gamma_1|>\exp\left(K\cdot(1+\log(k+1))\cdot\log\alpha\cdot 3\log 10\right)$$

or

$$m\log\alpha-\log 2.06<2.83\cdot 10^{10}\cdot(1+\log(k+1)). \tag{7}$$

We now modify Equation (1) as

$$t\cdot\alpha^k\cdot(1-\alpha^{r-k})-10^l(10^dP_m+P_n)=-(s\cdot\beta^k+r\cdot\gamma^k)+(s\cdot\beta^r+r\cdot\gamma^r).$$

By making the required adjustments, the above equation transforms into

$$\left|1-\frac{10^l(10^dP_m+P_n)}{t\cdot\alpha^k\cdot(1-\alpha^{r-k})}\right|\leq\frac{1}{\alpha^k}\left|\frac{1}{t\cdot(1-\alpha^{-(k-r)})}\right|\left(\frac{0.5}{\alpha^{\frac{k}{2}}}+\frac{0.5}{\alpha^{\frac{r}{2}}}\right),$$

i.e.,

$$\left|\frac{10^l(10^dP_m+P_n)}{t\cdot\alpha^k\cdot(1-\alpha^{r-k})}-1\right|<\frac{0.63}{\alpha^k}, \tag{8}$$

where we used the fact that $k-r\geq 9$ (from Lemma 3.6(f)), $k\geq 18$, $r\geq 2$ and $|t|>0.72$. Put $(\gamma_1,b_1):=(\alpha,-k)$, $(\gamma_2,b_2):=(10,l)$ and

$$(\gamma_3,b_3):=\left((10^dP_m+P_n)\cdot(t\cdot(1-\alpha^{(r-k)}))^{-1},1\right).$$

Furthermore, $D=3$. Our goal is to show that

$$\Gamma_2:=\frac{10^l(10^dP_m+P_n)}{t\cdot\alpha^k\cdot(1-\alpha^{r-k})}-1$$

is nonzero. If $\Gamma_2=0$, then we have $10^l(10^dP_m+P_n)=t\cdot(\alpha^k-\alpha^r)$. We apply an automorphism σ to both sides of this equation and take the absolute values. Then, we get

$$\left|10^l(10^dP_m+P_n)\right|=\left|\sigma(t\left(\alpha^k-\alpha^r\right))\right|\leq|s|\left(|\beta|^k+|\beta|^r\right)<1,$$

which is impossible. Thus, $\Gamma_2 \neq 0$. We can choose $A_1 := \log\alpha$, $A_2 := 3\log 10$ and $A_3 := 17.4 + 8m\log\alpha$, since

$$\begin{aligned} h(\gamma_3) &\leq d \cdot h(10) + h(P_m) + h(P_n) + h(t) + (k-r)h(\alpha) + 2\log 2 \\ &< \frac{3n+17}{20}\log 10 + 2(m-1)\frac{\log\alpha}{3} + \frac{1}{3}\log 23 + (2m+17)\frac{\log\alpha}{3} + 2\log 2 \\ &< \frac{3m}{20}\log 10 + \frac{4m}{3}\log\alpha + 5.8 \\ &< 5.8 + \frac{8m}{3}\log\alpha. \end{aligned}$$

In view of inequality (6) and since $d \geq 1$, we are allowed to take $B := k$. Then, from Lemma 3.2 and inequality (8), it follows that

$$0.63 \cdot \alpha^{-k} > |\Lambda_2| > \exp\left(L \cdot (\log\alpha)\,(\log 1000)\,(17.4 + 8m\log\alpha)\,(1 + \log k)\right),$$

i.e.,

$$k\log\alpha - \log 0.63 < 5.26 \cdot 10^{12} \cdot (1 + \log k)\,(17.4 + 8m\log\alpha)\,, \tag{9}$$

where $L = -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 3^2 \cdot (1 + \log 3)$. Combining inequalities (7) and (9), we obtain $k < 1.85 \cdot 10^{28}$. Let

$$\varsigma_1 := (k-m)\log\alpha - d\log 10$$

and $\Gamma_1 := e^{\varsigma_1} - 1$. From (5), we get

$$|\Gamma_1| = |e^{\varsigma_1} - 1| < 2.06 \cdot \alpha^{-m} < 0.9$$

for $m \geq 3$. According to Lemma 3.4, we have

$$|\varsigma_1| < -\frac{\log 0.1}{0.9} \cdot \frac{2.06}{\alpha^m} < (5.28) \cdot \alpha^{-m}$$

and so

$$0 < \left|\frac{\log\alpha}{\log 10} - \frac{d}{k-m}\right| < \frac{2.3}{(k-m)\cdot\alpha^m}. \tag{10}$$

Now, suppose $m \geq 249$. In this case, we can write

$$\frac{\alpha^m}{4.6} > 5.57 \cdot 10^{29} > k > k-m,$$

i.e.,

$$\left|\frac{\log\alpha}{\log 10} - \frac{d}{k-m}\right| < \frac{2.3}{(k-m)\cdot\alpha^m} < \frac{1}{2(k-m)^2}.$$

It follows from Lemma 3.5 that the rational number $\frac{d}{k-m}$ is a convergent to $\eta = \frac{\log\alpha}{\log 10}$. Let us denote the continued fraction expansion of η by $[a_0; a_1, a_2, \ldots]$, and let $\frac{u_r}{v_r}$ be its r-th convergent. Assume that $\frac{d}{k-m} = \frac{u_t}{v_t}$ for some integer t. In that case, since $v_{58} > 1.9 \cdot 10^{28} > k > k-m$, it follows that $t \in \{0, 1, 2, \ldots, 57\}$. Furthermore, $s = \max\{a_i | i = 0, 1, 2, \ldots, 58\} = 49$. Hence, by Lemma 3.5, we obtain

$$\left|\eta - \frac{u_t}{v_t}\right| > \frac{1}{51 \cdot (k-m)^2}.$$

This leads to

$$\frac{8.98}{10^{31}} > \frac{2.3}{\alpha^m} > \frac{1}{51\cdot(k-m)} > \frac{1}{51\cdot 1.9\cdot 10^{28}} = \frac{1}{9.69\cdot 10^{29}},$$

a contradiction. So, $m \le 248$. Consequently, choosing $m \le 248$ and substituting this upper bound into inequality (9), we derive $k < 4.49\cdot 10^{17}$. Repeating the same steps using inequality (10) yields $m \le 159$ and $k < 2.9\cdot 10^{17}$. Now, put

$$\varsigma_2 := l\log 10 - k\log\alpha + \log\left(\frac{10^d P_m + P_n}{t\cdot(1-\alpha^{r-k})}\right)$$

and $\Gamma_2 := e^{\varsigma_2} - 1$. For $k \ge 18$, it follows directly from inequality (8) that

$$|\Gamma_2| = |e^{\varsigma_2} - 1| < (0.63)\cdot\alpha^{-k} < 0.004.$$

Applying Lemma 3.4, we get

$$|\varsigma_2| < -\frac{\log(0.996)}{0.004}\cdot\frac{0.63}{\alpha^k} < (0.64)\cdot\alpha^{-k}.$$

So, we have

$$0 < \left| d\frac{\log 10}{\log\alpha} - k + \frac{\log\left(\frac{10^d P_m + P_n}{t\cdot(1-\alpha^{r-k})}\right)}{\log\alpha}\right| < (2.28)\cdot\alpha^{-k}. \tag{11}$$

Based on inequality (11), we may apply the following in order to apply Lemma 3.3:

$$\kappa := \frac{\log 10}{\log\alpha}, \mu := \frac{\log\left(\frac{10^d P_m + P_n}{t\cdot(1-\alpha^{r-k})}\right)}{\log\alpha}, A := 2.28, B := \alpha, t := k$$

and $d < k < M := 2.89\cdot 10^{17}$. We observed that q_{49}, the denominator of the 49-th convergent of κ is greater than $6M$. Moreover, for $2 \le n \le m \le 159, 1 \le d < \frac{3n+17}{20}$ and $9 \le k-r < m+n+17$ we calculate $\epsilon := ||\mu q_{49}|| - M||\gamma q_{49}|| > 1.3\cdot 10^{-7}$. This leads to the inequality

$$k < \frac{\log(Aq_{49}/\epsilon)}{\log B} < 246.7,$$

which yields $k \le 246$. Finally, refining our bounds once more using inequality (10), we get $m \le 32$. Substituting this into inequality (11), we further conclude $k \le 100$. For $k \le 100$, it also follows that $r \le 91$. Therefore, we find the only solution to Equation (1) as $P_{18} = \overline{P_2P_2P_6} = 114$ for $2 \le n \le m \le 159, 1 \le r \le 91, 1 \le d < \frac{3n+17}{20}, 1 \le l < \frac{3r+17}{20}$ and $18 \le k \le 100$. □

Now, we present the other theorem.

Theorem 4.2. *Let* d *and* l *be the numbers of digits of* R_n *and* R_r*, respectively. Then all positive integer solutions* (k, m, n, r, d, l) *of Equation (2) with* $m \ge 2, n, r \ge 1$ *and* $n \le m$ *are given by*

$$(k, m, n, r, d, l) \in \{(30, 3, 3, 12, 1, 2), (33, 7, 7, 13, 1, 2)\}.$$

Proof. As $m \geq 3$ and $n, r \geq 1$, we can take $k \geq 17$. We regulate Wquation (2) as

$$t \cdot \alpha^k - 10^{d+l} \cdot \alpha^m = -(s \cdot \beta^k + r \cdot \gamma^k) + 10^{d+l}(\beta^m + \gamma^m) + 10^l R_n + R_r.$$

The above equality leads to

$$\left|\frac{t \cdot \alpha^{k-m}}{10^{d+l}} - 1\right| \leq \frac{\left|s \cdot \beta^k + r \cdot \gamma^k\right|}{10^{d+l} \cdot \alpha^m} + \frac{|\beta^m + \gamma^m|}{\alpha^m} + \frac{10^l R_n + R_r}{10^{d+l} \cdot \alpha^m}$$
$$\leq \frac{1}{\alpha^m}\left(\frac{0.5}{10^{d+l} \cdot \alpha^{\frac{k}{2}}} + \frac{2}{\alpha^{\frac{m}{2}}} + 1 + \frac{1}{10^d}\right),$$

or

$$\left|t \cdot \alpha^{k-m} \cdot 10^{-(d+l)} - 1\right| < \frac{2.42}{\alpha^m}. \tag{12}$$

Here, we have used the fact that $R_r < 10^l$, $R_n < 10^d$ (from Lemma 3.7(c)(d)), $k \geq 17$, $d, l \geq 1$ and $m \geq 3$. To apply Lemma 3.2, we take

$$(\gamma_1, b_1) := (\alpha, k - m), (\gamma_2, b_2) := (10, -(d + l)), (\gamma_3, b_3) := (t, 1).$$

For this choice, we have $A_1 := \log \alpha, A_2 := 3 \log 10$ and $A_3 := \log 23$, because we know that $h(\gamma_1) = h(\alpha) = \frac{\log \alpha}{3}, h(\gamma_2) = h(10) = \log 10$ and $h(\gamma_3) = h(t) = \frac{\log 23}{3}$. Considering Lemma 3.7(a)(b)(e), we write

$$d + l < \frac{3(n + r) + 46}{20} < \frac{3(k - m + 5) + 46}{20} < k - m + 3 \leq k, \tag{13}$$

and so $B := k$. Moreover, $D = 3$. On the other hand, $\Gamma_1 := t \cdot \alpha^{k-m} \cdot 10^{-(d+l)} - 1$ is nonzero. In contrast to this, we assume that $\Gamma_1 = 0$. Then we get $t \cdot \alpha^{k-m} = 10^{d+l}$. We apply an automorphism σ to both sides of this equation and take the absolute values. Then, we obtain

$$\left|10^{d+l}\right| = \left|\sigma(t \cdot \alpha^{k-m})\right| = |s| \cdot |\beta|^{k-m} < 1,$$

which is not possible. Thus, we make use of Lemma 3.2 together with inequality (12), to get

$$(2.42) \cdot \alpha^{-m} > |\Gamma_1| > \exp\left(U \cdot (1 + \log k) \cdot \log \alpha \cdot 3 \log 10 \cdot \log 23\right),$$

where $U = -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 3^2 \cdot (1 + \log 3)$. This yields to

$$m \log \alpha - \log 2.42 < 1.65 \cdot 10^{13} \cdot (1 + \log k). \tag{14}$$

We rearrange Equation (2) as

$$\alpha^k \cdot (t - \alpha^{r-k}) - 10^l(10^d R_m + R_n) = -(s \cdot \beta^k + r \cdot \gamma^k) + (\beta^r + \gamma^r).$$

We obtain

$$\left|1 - \frac{10^l(10^d R_m + R_n)}{\alpha^k \cdot (t - \alpha^{r-k})}\right| < \frac{1}{\alpha^k}\left|\frac{1}{t - \alpha^{r-k}}\right|\left(\frac{0.5}{\alpha^{\frac{k}{2}}} + \frac{2}{\alpha^{\frac{r}{2}}}\right)$$

by first dividing both sides of the above equality by $\alpha^k \cdot (t - \alpha^{r-k})$, and then taking the absolute value of both sides. The above inequality leads to

$$\left|\frac{10^l(10^d R_m + R_n)}{\alpha^k \cdot (t - \alpha^{r-k})} - 1\right| < \frac{2.59}{\alpha^k}, \tag{15}$$

where we used the fact that $k - r \geq 11$ (from Lemma 3.7(f)), $k \geq 17$, and $r \geq 1$. To apply Lemma 3.2, we choose

$$(\gamma_1, \gamma_2, \gamma_3) := \left(\alpha, 10, (10^d R_m + R_n) \cdot (t - \alpha^{r-k})^{-1}\right) \text{ and } (b_1, b_2, b_3) := (-k, l, 1)\,.$$

We can take $\mathbb{K} = \mathbb{Q}(\alpha)$, which has degree $D = 3$. Let

$$\Gamma_2 := \frac{10^l(10^d R_m + R_n)}{\alpha^k \cdot (t - \alpha^{r-k})} - 1.$$

It can be easily shown that $\Gamma_2 \neq 0$, in the same way as we have proved that $\Gamma_1 \neq 0$. Considering inequality (13) for $d \geq 1$, we conclude that

$$\begin{aligned} h(\gamma_3) &\leq d \cdot h(10) + h(R_m) + h(R_n) + h(t) + (k-r)h(\alpha) + 2\log 2 \\ &< \frac{3n+23}{20}\log 10 + 2(m+1)\frac{\log\alpha}{3} + \frac{1}{3}\log 23 + (2m+22)\frac{\log\alpha}{3} + 2\log 2 \\ &< \frac{3m}{20}\log 10 + \frac{4m}{3}\log\alpha + 7.33 \\ &< 7.33 + \frac{8m}{3}\log\alpha, \end{aligned}$$

which leads us to the choice $(A_1, A_2, A_3, B) := (\log\alpha, 3\log 10, 22 + 8m\log\alpha, k)$. Thus, we deduce via inequality (15) and Lemma 3.2 that

$$(2.59) \cdot \alpha^{-k} > |\Gamma_2| > \exp\left(U \cdot (1 + \log k) \cdot \log\alpha \cdot 3\log 10 \cdot (22 + 8m\log\alpha)\right).$$

This shows that

$$k\log\alpha - \log(2.59) < 5.26 \cdot 10^{12} \cdot (1 + \log k) \cdot (22 + 8m\log\alpha)\,. \tag{16}$$

Inequalities (14) and (16) tell us that $k < 1.31 \cdot 10^{31}$. Put

$$\varsigma_1 := (k-m)\log\alpha - (d+l)\log 10 + \log t$$

and $\Gamma_1 := e^{\varsigma_1} - 1$. For $m \geq 4$, it follows by (12) that

$$|\Gamma_1| = |e^{\varsigma_1} - 1| < 2.42 \cdot \alpha^{-m} < 0.8.$$

Thanks to Lemma 3.4, we have that

$$|\varsigma_1| < -\frac{\log 0.2}{0.8} \cdot \frac{2.42}{\alpha^m} < 4.87 \cdot \alpha^{-m}.$$

From this, we obtain

$$0 < \left|(k-m)\frac{\log\alpha}{\log 10} - (d+l) + \frac{\log t}{\log 10}\right| < \frac{2.12}{\alpha^m}. \tag{17}$$

By applying Lemma 3.3, we can choose

$$\kappa := \frac{\log\alpha}{\log 10}, \mu := \frac{\log t}{\log 10}, A := 2.12, B := \alpha, t := m.$$

In addition, using the bound $k-m<k<M:=1.31\cdot 10^{31}$, we find that $q_{68}>6M$ for κ and compute $\epsilon:=||\mu q_{68}||-M||\gamma q_{68}||>0.2$. Thus, we conclude that

$$m<\frac{\log(Aq_{68}/\epsilon)}{\log B}<271.22$$

from Lemma 3.3. This leads to $m\leq 271$. Inserting the upper bound for m into (16) yields $k<4.94\cdot 10^{17}$. Taking $M=4.94\cdot 10^{17}$ and considering (17), a further reduction yields $m\leq 169$ and $k<3.11\cdot 10^{17}$. Let

$$\varsigma_2:=l\log 10-k\log\alpha+\log\left(\frac{10^d R_m+R_n}{t-\alpha^{r-k}}\right) \tag{18}$$

and $\Gamma_2:=e^{\varsigma_2}-1$. From (15), it is evident that

$$|\Gamma_2|=|e^{\varsigma_2}-1|<(2.59)\cdot\alpha^{-k}<0.03$$

for $k\geq 17$. Therefore, by applying Lemma 3.4, we get

$$|\varsigma_2|<-\frac{\log(0.97)}{0.03}\cdot\frac{2.59}{\alpha^k}<(2.63)\cdot\alpha^{-k}.$$

Thus, we have

$$0<\left|l\frac{\log 10}{\log\alpha}-k+\frac{\log\left(\frac{10^d R_m+R_n}{t-\alpha^{r-k}}\right)}{\log\alpha}\right|<(2.63)\cdot\alpha^{-k}. \tag{19}$$

By applying Lemma 3.3, we can choose

$$\kappa:=\frac{\log 10}{\log\alpha},\mu:=\frac{\log\left(\frac{10^d R_m+R_n}{t-\alpha^{r-k}}\right)}{\log\alpha},A:=2.63,B:=\alpha,t:=k$$

and $l<k<M:=3.11\cdot 10^{17}$. We find that $q_{47}>6M$ for κ and $\epsilon:=||\mu q_{47}||-M||\gamma q_{47}||>4.2\cdot 10^{-7}$ for $3\leq m\leq 169, 1\leq d<\frac{3n+23}{20}$ and $11\leq k-r<m+n+22$. Hence, Lemma 3.3 tells us that

$$k<\frac{\log(Aq_{47}/\epsilon)}{\log B}<248.83,$$

and thus $k\leq 248$. Finally, if we reduce once more inequality (17), we get $m\leq 38$, and from inequality (19) we have $k\leq 103$. Since $k\leq 103$, we can conclude that $r\leq 90$. Therefore, we find the only solution of Equation (2) as $P_{30}=\overline{R_3R_3R_{12}}=3329$ and $P_{33}=\overline{R_7R_7R_{13}}=7739$ for $3\leq n\leq m\leq 38, 1\leq r\leq 90, 1\leq d<\frac{3n+23}{20}, 1\leq l<\frac{3r+23}{20}$ and $17\leq k\leq 103$. □

5 Conclusion

This paper has explored the representation of Padovan numbers as the concatenation of three Padovan or Perrin numbers under specific digit-based constraints. By examining all admissible combinations where the middle number does not exceed the first and accounting for the digit lengths of the components, we identified all such Padovan numbers that satisfy the given conditions. The findings highlight interesting structural relationships within the Padovan and Perrin sequences and contribute to the broader study of number concatenation in linear recurrence sequences.

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