

A note on the self-convolution of the Tribonacci sequence

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Abstract: We present a simple formula for the self-convolution of the Tribonacci numbers. The resulting identity is considerably simpler than that obtained in a recent publication.

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1 Motivation and Preliminaries

Convolutions and self-convolutions of sequences are of the current interest in the research topic among various authors. Dresden and Wang [1] obtained some properties of iterated convolution of the Fibonacci numbers while Frontczak [3] considered convolution of the Fibonacci and the Lucas sequence with the Tribonacci and the Tribonacci-Lucas sequence.

Komatsu [4, 5] obtained the following identity for the self-convolution of the Tribonacci sequence:

$$\sum_{k=0}^n T_k T_{n-k} = \sum_{\ell=1}^{n-1} \left(\sum_{i=0}^{\lfloor \frac{n-\ell-1}{3} \rfloor} 2^{i-1} \left((-1)^{\frac{n-\ell-i-1}{2}} + (-1)^{\frac{3(n-\ell-i-1)}{2}} \right) \binom{\frac{n-\ell-i-1}{2}}{i} \right) \ell T_{\ell}.$$



Although this formula provides an expansion in terms of $n - 1$ consecutive Tribonacci numbers, the coefficients in this expansion are quite involved, and the representation itself is not very efficient. Indeed, a sum of an arbitrary number of consecutive Tribonacci numbers can always be reduced to a sum of at most three consecutive terms.

In this article, we prove the following identity, which provides a clear and short formula for the convolution considered by Komatsu.

Theorem 1.1. *For any $n \geq 0$ we have*

$$\sum_{k=0}^n T_k T_{n-k} = \left(\frac{5}{22}n + \frac{5}{22}\right) T_{n+2} - \left(\frac{3}{22}n + \frac{5}{22}\right) T_{n+1} - \left(\frac{2}{11}n + \frac{4}{11}\right) T_n. \quad (1)$$

Recall that the Tribonacci numbers T_n are defined via the recurrence relation

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n$$

with initial conditions $T_{-1} = T_0 = 0$ and $T_1 = 1$. The ordinary generating function (throughout abbreviated o.g.f.) of this sequence is

$$T(x) = \frac{x}{1 - x - x^2 - x^3} = \sum_{k=0}^n T_k x^k.$$

We also have the following set of generating functions:

$$\sum_{n=0}^{+\infty} T_{n+1} x^n = \frac{1}{1 - x - x^2 - x^3}, \quad (2)$$

$$\sum_{n=0}^{+\infty} T_{n+2} x^n = \frac{1 + x + x^2}{1 - x - x^2 - x^3}, \quad (3)$$

$$\sum_{n=0}^{+\infty} n T_n x^n = \frac{x + x^3 + 2x^4}{(1 - x - x^2 - x^3)^2}, \quad (4)$$

$$\sum_{n=0}^{+\infty} n T_{n+1} x^n = \frac{x + 2x^2 + 3x^3}{(1 - x - x^2 - x^3)^2}, \quad (5)$$

$$\sum_{n=0}^{+\infty} n T_{n+2} x^n = \frac{x^5 + 2x^4 + 3x^3 + 4x^2 + 2x}{(1 - x - x^2 - x^3)^2}. \quad (6)$$

For more information on how to transform $T(x)$ into the above set we refer to for example the excellent book [8].

We note that in the classic paper by Rabinowitz [7] the following formula for the convolution of the Tribonacci sequence is given:

$$22 \sum_{k=0}^n T_k T_{n-k} = 5(n+1)T_{n+2} - 2(n+1)T_{n+1} - 4(n+2)T_n.$$

Rabinowitz credited this formula to David Zeitlin via personal correspondence. However, the formula contains an incorrect middle term. This appeared again in Frontczak's paper [2] with the comment that "The formula, however, apparently contains a typo as becomes obvious by numerical inspection", but the author does not provide the correct statement.

2 Proof of Theorem 1.1

In order to proof our result we employ the generating function approach. Since $T(x)$ is the o.g.f. of the Tribonacci sequence, we have, using the Cauchy product,

$$(T(x))^2 = \sum_{n=0}^{+\infty} \left(\sum_{k=0}^n T_k T_{n-k} \right) x^n,$$

$$(T(x))^2 = \frac{x^2}{(1-x-x^2-x^3)^2}.$$

Thus it is sufficient to show that the o.g.f. of the sequence $\sum_{k=0}^n T_k T_{n-k}$ and the sequence

$$\left(\frac{5}{22}n + \frac{5}{22} \right) T_{n+2} - \left(\frac{3}{22}n + \frac{5}{22} \right) T_{n+1} - \left(\frac{2}{11}n + \frac{4}{11} \right) T_n.$$

coincide. Let $S(x)$ denote the o.g.f. the latter. We have by (2)–(3) that

$$\begin{aligned} \sum_{k=0}^n \left(\frac{5}{22}T_{n+2} - \frac{5}{22}T_{n+1} - \frac{4}{11}T_n \right) x^n &= \frac{5}{22} \cdot \frac{1+x+x^2}{1-x-x^2-x^3} - \frac{5}{22} \cdot \frac{1}{1-x-x^2-x^3} \\ &\quad - \frac{4}{11} \cdot \frac{x}{1-x-x^2-x^3} \\ &= \frac{5x^2 - 3x}{22(1-x-x^2-x^3)}. \end{aligned}$$

Similarly, we have by (4)–(6) that

$$\begin{aligned} \sum_{k=0}^n \left(\frac{5}{22}nT_{n+2} - \frac{3}{22}nT_{n+1} - \frac{2}{11}nT_n \right) x^n &= \frac{5}{22} \cdot \frac{x^5 + 2x^4 + 3x^3 + 4x^2 + 2x}{(1-x-x^2-x^3)^2} \\ &\quad - \frac{3}{22} \cdot \frac{x + 2x^2 + 3x^3}{(1-x-x^2-x^3)^2} - \frac{2}{11} \cdot \frac{x + x^3 + 2x^4}{(1-x-x^2-x^3)^2} \\ &= \frac{5x^5 + 2x^4 + 2x^3 + 14x^2 + 3x}{22(1-x-x^2-x^3)^2}. \end{aligned}$$

Therefore

$$S(x) = \frac{5x^5 + 2x^4 + 2x^3 + 14x^2 + 3x}{22(1-x-x^2-x^3)^2} + \frac{5x^2 - 3x}{22(1-x-x^2-x^3)},$$

which, after minor computation, simplifies to

$$S(x) = \frac{22x^2}{22(1-x-x^2-x^3)^2} = (T(x))^2.$$

This concludes the theorem and proves (1).

3 Conclusion

Komatsu [4], and later Komatsu and Li [6], obtained the following identities, respectively:

$$\sum_{k=0}^{n-3} T_k(T_{n-k} + T_{n-2-k} + 2T_{n-k-3}) = (n-2)T_{n-1} - T_{n-2},$$
$$\sum_{k=0}^{n-4} Q_k(Q_{n-k} + Q_{n-2-k} + 2Q_{n-3-k} + 3Q_{n-4-k}) = (n-2)Q_n - Q_{n-2} - 3Q_{n-3}.$$

Here Q_n denotes the Tetranacci numbers. They also provided a similarly structured but more complicated formula for the self-convolution of the Tetranacci numbers. It would be of interest to find an analogue of (1), that is, a compact identity or simple closed formula for $\sum_{k=0}^n Q_k Q_{n-k}$ and higher order Fibonacci-like sequences.

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