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Star complementary characterization of oriented graphs whose skew spectral radius does not exceed 2

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Abstract: We employ the method of star complements to classify all oriented graphs whose skew spectrum lies within the interval $[-2,\ 2]$. At the same time, we provide a structural characterisation of these graphs, showing that, with the sole exception of exactly one graph of order 14, every maximal oriented graph possessing this spectral property is determined by a fixed oriented cycle serving as a star complement for either -2 or 2. The exceptional oriented graph is uniquely determined by a fixed 7-vertex oriented path acting as the star complement. This work may be regarded as a counterpart to [13], where the corresponding oriented graphs were determined via associated signed graphs, without the present characterisation.

Keywords: Star complement, Oriented graph, Skew spectral radius, Prescribed induced subgraph.
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1 Introduction

For a finite simple undirected graph $G = (V, E)$, an *oriented graph* G' is defined as the pair (G, σ'), where σ' is an *edge orientation* function satisfying $\sigma'(ij) \in \{i, j\}$ for every $ij \in E$. If $\sigma'(ij) = i$, we say that the edge ij is oriented from j to i. This orientation is denoted by $i \leftarrow j$ (or equivalently $j \rightarrow i$). In this case, i is called the *head* and j the *tail* of the edge. The number of vertices is denoted by n and called the *order*. The *skew adjacency matrix* $S_{G'} = (s_{ij})$ is the $n \times n$ matrix given by

$$s_{ij} = \begin{cases} 0, & \text{if } ij \notin E, \\ \mathtt{i}, & \text{if } \sigma'(ij) = i, \\ -\mathtt{i}, & \text{if } \sigma'(ij) = j. \end{cases}$$

There is an equivalent approach when the above matrix is multiplied by $-\mathtt{i}$. The eigenvalues of $S_{G'}$ are referred to as the *skew eigenvalues* of G', and, together with their multiplicities, they constitute the *skew spectrum* of G'. Since $S_{G'}$ is Hermitian, all its eigenvalues are real. Moreover, the spectrum is symmetric with respect to the origin, which follows immediately from considering the alternative matrix mentioned above. The *skew spectral radius* of G' is defined as the largest absolute value among the eigenvalues of $S_{G'}$. To simplify the language, the prefix 'skew' will often be omitted from the preceding terminology.

Two oriented graphs are said to be *switching equivalent* if they share the same vertex set and one can be obtained from the other by reversing the orientation of every edge connecting a vertex of a fixed vertex subset to a vertex outside this subset. Switching equivalence, together with graph isomorphism, gives rise to the notion of *switching isomorphism*. Specifically, two oriented graphs are said to be switching isomorphic if one of them is isomorphic to an oriented graph that is switching equivalent to the other. Switching isomorphic signed graphs share the same spectrum and are often treated as identical. This convention will be adopted throughout the paper. In particular, whenever we say that an oriented graph is unique, we mean 'unique up to switching isomorphism'.

An oriented graph is said to be *maximal* relative to a given property if it is not an induced subgraph of any other oriented graph possessing the same property.

If λ is an eigenvalue of an oriented graph G' with multiplicity k, then a *star set* for λ in G' is a set T of k vertices of G' such that λ does not occur in the skew spectrum of the induced subgraph H' obtained by removing the vertices of T. The subgraph H' is then called a *star complement* for λ in G', and G' is referred to as an *extension* (or a *λ-extension*) of H'. In accordance with the previously introduced terminology, an extension is said to be *maximal* if no additional vertex can be added without violating the extension property.

For clarification of the terminology introduced above, we note that an oriented cycle is a maximal connected oriented graph in which every vertex has at most 2. However, as will be shown in the next section, none of the oriented cycles constitute a maximal extension for the eigenvalue 2.

All oriented graphs whose spectral radius does not exceed 2, equivalently, all oriented graphs whose skew spectrum is contained in the interval $[-2,\, 2]$, were determined by the author in [13]. The approach was based on recently established relationships between bipartite oriented graphs and bipartite signed graphs [15], together with the classification of signed graphs possessing the same spectral property, which can be found in any of [5, 9, 14].

In this paper, we present a completely different approach which avoids the use of signed graphs and instead employs the method of star complements to achieve the same goal, namely, the determination of all oriented graphs whose skew spectrum lies within the interval $[-2,\, 2]$.

More precisely, we show that, apart from a single exceptional case, every maximal oriented graph whose skew spectral radius does not exceed 2 can be obtained as a maximal extension of a fixed t-vertex oriented cycle, which serves as a star complement for either -2 or 2. The sole exception is a 14-vertex oriented graph containing any but fixed 7-vertex oriented path, which likewise acts as a star complement for the same eigenvalue. This yields an elegant descriptive characterization of oriented graphs under consideration stating that each is an induced subgraph of a maximal extension of a cycle in the role of a star complement for -2, unless a star complement has 7 vertices when it is a path.

The results presented in this paper are part of a long-term investigation into the spectra of matrices associated with oriented graphs. In this context, our work is closely related to several lines of research: the study of skew spectra of oriented graphs in [2, 11]; skew spectra of random oriented graphs in [6, 7, 11]; skew spectra of products of oriented graphs in [8]; and connections with the spectra of signed graphs, as explored in [15].

In Section 2 we give a very brief introduction to star complements in oriented graphs. Our contribution is reported in Section 3. A comparison with the results of [15] is given in Section 4.

2 Star complements in oriented graphs

In what follows, we recall all the material required for the subsequent section. For further background and a more detailed account of the theory of star complements in oriented graphs, the reader is referred to the recent paper [16].

We begin with the following theorem, carried over from the theory of unoriented graphs.

Theorem 2.1 ([16], cf. [3, Theorem 5.1.7]). *Let*

$$\begin{pmatrix} A_T & B^* \\ B & A_{H'} \end{pmatrix}$$

be the adjacency matrix of an oriented graph G', where A_T is the $k \times k$ adjacency matrix of the subgraph induced by a star set T for an eigenvalue λ and, H' is the corresponding star complement. Then

$$\lambda I - A_T = B^*(\lambda I - A_{H'})^{-1}B.$$

This result is referred to as the *Reconstruction Theorem*. Let H' be a graph of order t that acts as a star complement for the eigenvalue λ. For any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{C}^t$, we define the quadratic form as follows:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^*(\lambda I - A_{H'})^{-1}\mathbf{y}.$$

As a direct consequence of Theorem 2.1, we have that if λ is not an eigenvalue of an oriented graph H', then an oriented graph G' contains H' as a star complement for λ if and only if

$$\langle \mathbf{b}_i, \mathbf{b}_i \rangle = \lambda \quad \text{and} \quad \langle \mathbf{b}_i, \mathbf{b}_j \rangle \in \{\mathtt{i}, 0, -\mathtt{i}\}$$

for all distinct $i, j \in T = V(G') \setminus V(H')$, where $\mathbf{b}_i$ and $\mathbf{b}_j$ define neighbourhoods of i and j in H', respectively. Evidently, the vectors $\mathbf{b}_i$ form the submatrix B of Theorem 2.1. Also,

$$\text{if } \langle \mathbf{b}_i, \mathbf{b}_j \rangle = \begin{cases} 0, & \text{then } i \nsim j; \\ \mathtt{i}, & \text{then } i \to j; \\ -\mathtt{i}, & \text{then } i \leftarrow j. \end{cases} \tag{1}$$

We also need the following result, adapted from the theory of unoriented graphs to be found in [4, Section 5.1] and explicitly proved in [16].

Theorem 2.2. *The following statements hold:*

(i) *A connected oriented graph has a connected star complement for any fixed skew eigenvalue.*

(ii) *If $\lambda \neq 0$, then the matrix B of Theorem 2.1 does not contain a zero-column.*

(iii) *The matrix B of Theorem 2.1 does not contain columns that are equal up to negation.*

3 Results

Since the spectrum of an oriented graph is symmetric, every star complement for -2 also serves as a star complement for 2. Consequently, the spectral radius of any corresponding extension does not exceed 2. We first consider oriented cycles as star complements for -2.

Lemma 3.1. *Up to switching isomorphism, there exists exactly one oriented cycle of each odd order, and exactly two oriented cycles of each even order.*

Proof. Let C'_n be an oriented cycle with vertices labelled $0, 1, \dots, n-1$ in natural order. We now apply a sequence of switching operations generated by the following algorithm:

```
FOR j = 0 to n-2
IF j is a head of (j,j+1)
make a switch at j+1
ENDIF
ENDFOR
```

In this way, we obtain the oriented cycle in which the first $n-1$ edges are oriented as

$$0 \to 1 \to 2 \to \cdots \to n-3 \to n-2.$$

The remaining edge between $n-1$ and 0 can be oriented in either direction, giving exactly two oriented cycles: C'_n, in which $n-1 \to 0$, and C''_n, in which $n-1 \leftarrow 0$.

We show that for odd n, these cycles are switching isomorphic. By taking C''_n and performing a switch at every even vertex, we obtain the orientation $0 \leftarrow 1 \leftarrow 2 \leftarrow \cdots \leftarrow n-2 \leftarrow n-1 \leftarrow 0$. It is clear that this cycle is isomorphic to C'_n.

For even n, the oriented cycles C'_n and C''_n are not switching isomorphic, as can be verified through a straightforward combinatorial argument. This also follows from the fact that their spectra are different (see below discussion). □

According to the previous proof, every odd oriented cycle is switching isomorphic to

$$0 \to 1 \to 2 \to \cdots \to n-2 \to n-1 \to 0, \tag{2}$$

whereas every even oriented cycle is switching isomorphic to either

$$0 \to 1 \to 2 \to \cdots \to n-2 \to n-1 \to 0 \quad \text{or} \quad 0 \to 1 \to 2 \to \cdots \to n-2 \to n-1 \leftarrow 0 \tag{3}$$

For n odd, the spectral radius of C_n' is less than 2. Similarly, for n even, the spectral radius of the first (respectively second) cycle of (3) is less than 2 if and only if $n \equiv 0 \pmod 2$ ($n \equiv 0 \pmod 4$). These conclusions are results of a short algebraic computation; alternatively, the entire skew spectrum of an oriented cycle is computed in [1].

In what follows, we compute $(-2I - S)^{-1}$, where S is the skew adjacency matrix of an oriented cycle whose spectral radius does not attain 2. It is always assumed that their vertices are labelled $0, 1, \ldots, n-1$ and edges are oriented as in (2) or (3). We also write $(r_0, r_1, \ldots, r_{n-1})$ to denote the first row of $(-2I - S)^{-1}$.

Let $P = (p_{ij})$ denote the cyclic permutation matrix, $p_{ij} = \delta_{j,i+1 \pmod n}$. For the consistent edge orientation, the skew adjacency matrix is

$$S_c = -\mathtt{i}(P - P^{\mathsf{T}}).$$

For the one-edge reversed orientation, it reads

$$S_r = S_c - 2\mathtt{i}\big(E_{0,n-1} - E_{n-1,0}\big),$$

where $E_{0,n-1}$ is the matrix with a single 1 in position $(0, n-1)$ and zeros elsewhere.

The matrix P is diagonalized by the unitary discrete Fourier transformation matrix F with entries

$$f_{ij} = \frac{1}{\sqrt{n}}\omega^{ij}, \quad \omega = e^{2\pi\mathtt{i}/n}.$$

If $\mathbf{v}_k^* = \frac{1}{\sqrt{n}}(1, \omega^{-k}, \omega^{-2k}, \ldots, \omega^{-(n-1)k}), 0 \le k \le n-1$, is the k-th eigenvector of P, for the consistent orientation we have

$$S_c\mathbf{v}_k = -\mathtt{i}(\omega^k - \omega^{-k})\mathbf{v}_k = 2\sin\left(\frac{2\pi k}{n}\right)\mathbf{v}_k,$$

hence the eigenvalues of $M_c = -2I - S_c$ are

$$\alpha_k = -2 - 2\sin\left(\frac{2\pi k}{n}\right), \quad 0 \le k \le n-1.$$

The matrix M_c is circulant and invertible, so

$$M_c^{-1} = F\,\mathrm{diag}(\alpha_k^{-1})\,F^*, \;\text{ giving }\; r_d = \frac{1}{n}\sum_{k=0}^{n-1}\frac{\omega^{kd}}{\alpha_k}, \quad 0 \le d \le n-1.$$

Pairing k with $n-k$ and simplifying yields the following closed forms. If $n \equiv 1 \pmod 4$:

$$r_d = \begin{cases} -\dfrac{n-d}{2} - \dfrac{\mathtt{i}d}{2}, & d \equiv 0 \pmod 4, \\ -\dfrac{d}{2} + \dfrac{\mathtt{i}(n-d)}{2}, & d \equiv 1 \pmod 4, \\ \dfrac{n-d}{2} + \dfrac{\mathtt{i}d}{2}, & d \equiv 2 \pmod 4, \\ \dfrac{d}{2} - \dfrac{\mathtt{i}(n-d)}{2}, & d \equiv 3 \pmod 4. \end{cases}$$

If $n \equiv 3 \pmod 4$:

$$r_d = \begin{cases} -\dfrac{n-d}{2} + \dfrac{\mathtt{i}d}{2}, & d \equiv 0 \pmod 4, \\ \dfrac{d}{2} + \dfrac{\mathtt{i}(n-d)}{2}, & d \equiv 1 \pmod 4, \\ \dfrac{n-d}{2} - \dfrac{\mathtt{i}d}{2}, & d \equiv 2 \pmod 4, \\ -\dfrac{d}{2} - \dfrac{\mathtt{i}(n-d)}{2}, & d \equiv 3 \pmod 4. \end{cases}$$

Similarly, for $n \equiv 2 \pmod 4$, by setting $n = 4m+2$ and $d_0 = \min\{d, n-d\}$, we find

$$r_d = \begin{cases} (-1)^{t+1} \, \frac{n-2d_0}{4}, & d_0 \text{ even}, \\ (-1)^t \, (s\,\mathtt{i}) \, \frac{n-2d_0}{4}, & d_0 \text{ odd}, \end{cases}$$

for $0 \le d \le n-1$, where $t = \lfloor d_0/2 \rfloor$, along with $s = 1$ for $d \le n/2$ and $s = -1$, otherwise.

In each case, n odd and $n \equiv 2 \pmod 4$, the inverse is a circulant matrix and every row is a cyclic shift of the first row.

For $n = 4m$, since one edge is flipped, the inverse S_r is no longer circulant but the considered inverse carries on this property. To see this, one may apply a similar diagonalization, or simply solve the recurrence

$$\mathtt{i}x_{j+1} - 2x_j - \mathtt{i}x_{j-1} = 0, \quad 1 \le j \le n-2,$$

with the boundary conditions

$$\mathtt{i}x_1 + \mathtt{i}x_{n-1} - 2x_0 = 1, \quad -\mathtt{i}x_{n-2} - 2x_{n-1} - \mathtt{i}x_0 = 0, \tag{4}$$

to show that the first row of $(-2I - S)^{-1}$ is given by

$$r_{2d} = -(m-d)(-1)^d, \; r_{2d+1} = (-1)^{d+1}\mathtt{i}\left(m - d - \tfrac{1}{2}\right), \quad 0 \le d \le 2m-1. \tag{5}$$

Indeed, by substituting a solution of the form $x_j = \xi^j$ into the homogeneous recurrence, we obtain the characteristic equation $(\xi - i)^2 = 0$, which indicates that the general solution has the form $x_j = (\alpha + \beta j)i^j$. Applying the boundary conditions then determines the coefficients α and β, yielding the desired solution. The remaining rows are handled by a slight modification of the preceding argument.

We are ready to determine all maximal extensions for -2.

Theorem 3.1. *Let C_t' be a t-vertex oriented cycle with no -2 in the skew spectrum. If G' is a maximal extension of C_t' for -2, then G' switches to one of oriented graphs S_8', Q_4', $T_{2t}', t \geq 3$, illustrated in Figure 1.*

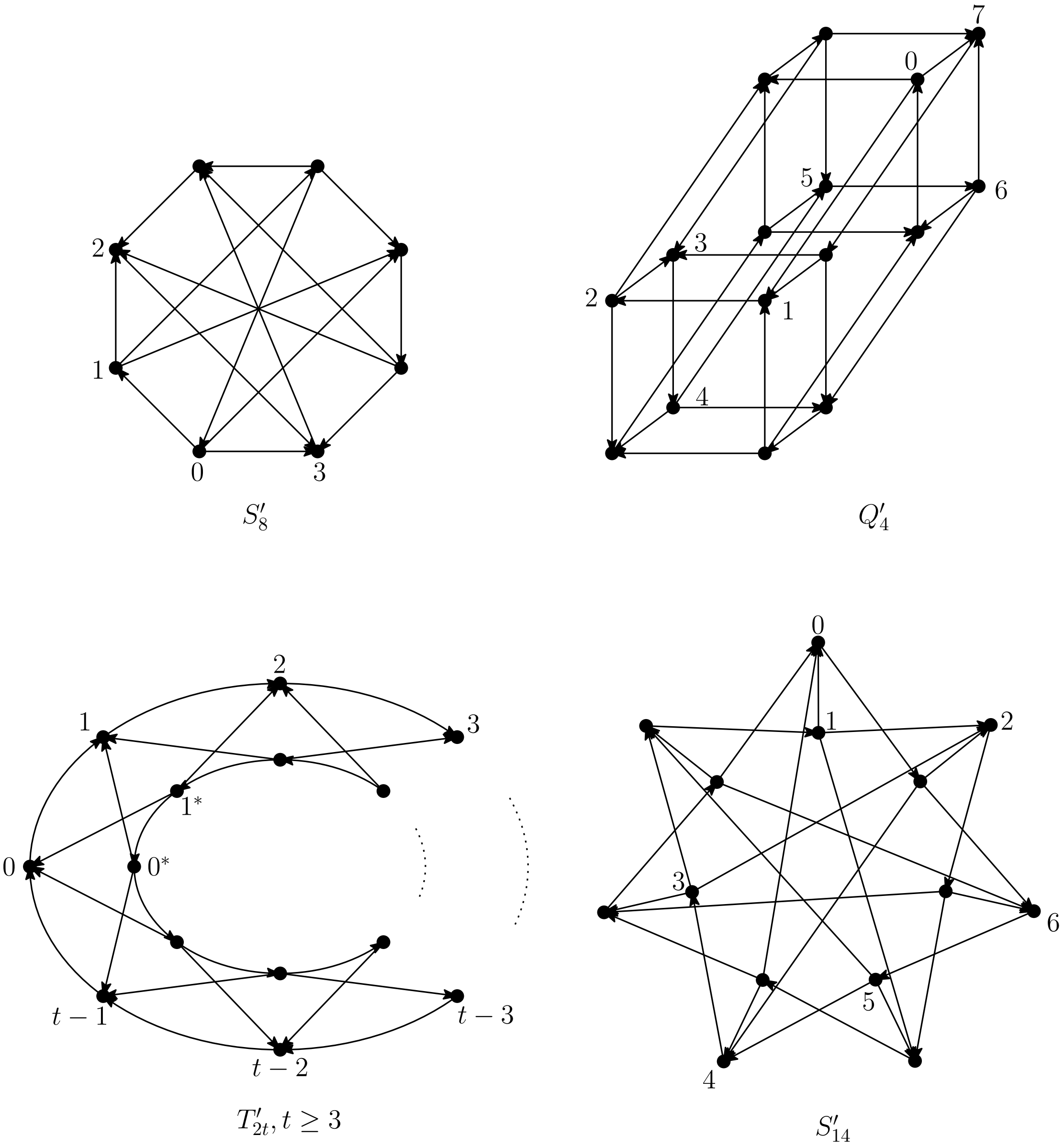

Figure 1. Maximal oriented graphs whose spectral radius does not exceed 2.

Proof. It is sufficient to consider the following star complements in the role of C_t': the cycle of (2) for odd t; the first cycle of (3) for $t \equiv 2 \pmod 4$; and the second cycle of (3) for $t \equiv 0 \pmod 4$. Let C_t' denote any of these, and let u be a star set vertex of a (-2)-extension G'.

Since the proof is rather long, we first provide a brief overview to guide the reader. We begin by determining the C_t'-neighbourhoods of the vertex u, which lays the foundation for understanding the local structure of the graph. Next, we examine the compatibility of the star set vertices by checking the equalities (1) to identify which configurations are possible. Finally, using the information gathered in the previous steps, we construct the maximal extensions, thereby completing the characterization.

We claim that u is adjacent to at most two consecutive vertices of C'_t. Indeed, if u is adjacent to at least four consecutive vertices, then either $t = 4$, or G' contains an induced subgraph consisting of a vertex joined to every vertex of a 4-vertex oriented path. In either case, regardless of the particular edge orientation, the spectral radius exceeds 2. If u is adjacent to three consecutive vertices, then either $t = 4$, or G' contains an induced subgraph consisting of a vertex joined to the first three vertices of a 4-vertex oriented path. These cases are handled in the same way. Hence, the claim follows.

We now claim that if u is adjacent to two consecutive vertices, then $t \in \{3, 4\}$. Indeed, for $t \geq 5$, the oriented graph G' would contain an induced subgraph consisting of a vertex joined to the two middle vertices of a 4-vertex path, which is resolved as before.

The next claim is that u is adjacent to at most three vertices of C'_t. Indeed, u cannot be adjacent to five or more vertices, since this would produce an induced 6-vertex star, whose spectral radius exceeds 2. The case where u has four neighbours implies $t \leq 9$; otherwise, G' would contain a 5-vertex star with an additional vertex attached to a pendant vertex, which is resolved as before. The remaining case, $t \leq 9$, is handled directly.

If u is adjacent to three vertices, then $t \leq 10$; otherwise, G' would contain an oriented tree violating the spectral radius condition, as follows from the examination of Smith trees [12]. By inspecting the star complements with at most 10 vertices, we find that three neighbours are possible if and only if $t = 8$.

Assume that u is adjacent to exactly two vertices of C'_t, and let $\mathbf{b} = -(b_0, b_1, \dots, b_{t-1})^*$ be the corresponding column of the matrix B, which contains exactly two non-zero entries. Without loss of generality, we may assume that the non-zero entries of $\mathbf{b}$ are at positions 0 and a, with $b_0 = s_1$, $b_a = s_2$, $s_j \in \{\pm\mathtt{i}\}$. Write $s_j = \varepsilon_j \mathtt{i}$ with $\varepsilon_j \in \{\pm 1\}$. Since $(-2I - S)^{-1}$ is Hermitian with a constant main diagonal (as follows from the formulas derived prior this statement), with first row $(r_0, r_1, \dots, r_{t-1})$, we have

$$\begin{aligned}\langle \mathbf{b}, \mathbf{b} \rangle &= \overline{s}_1 s_1 r_0 + \overline{s}_1 s_2 r_a + \overline{s}_2 s_1 r_{-a} + \overline{s}_2 s_2 r_0 \\ &= 2r_0 + (\overline{s}_1 s_2) r_a + (\overline{s}_2 s_1) \overline{r}_a = 2r_0 + 2\sigma\, \Re(r_a),\end{aligned}$$

where $\sigma = \overline{s}_1 s_2 = \varepsilon_1 \varepsilon_2 \in \{\pm 1\}$. (For clarification, in the computation of the preceding quadratic form, only four entries of the inverse matrix are relevant: $(0, 0)$, $(0, a)$, $(a, 0)$, and (a, a), which directly yields the result stated above.) Hence,

$$\langle \mathbf{b}, \mathbf{b} \rangle = -2 \iff 2r_0 + 2\sigma\, \Re(r_a) = -2 \iff \sigma\, \Re(r_a) = -1 - r_0. \tag{6}$$

From the explicit formulas for the first row given prior to the formulation of this theorem, one has:

$$r_0 = \begin{cases} -\frac{t}{2}, & t \text{ odd}, \\ -\frac{t}{4}, & t \text{ even}. \end{cases}$$

Substituting into (6) gives

$$\sigma\, \Re(r_a) = \begin{cases} \frac{t-2}{2}, & t \text{ odd}, \\ \frac{t-4}{4}, & t \text{ even}, \end{cases}$$

in circulant cases. Similarly, $\sigma\, \Re(r_a) = -\frac{t+1}{4}$ holds in the remaining case.

By direct inspection of the piecewise expressions for r_d, we deduce the following:

- If t is odd or $t \equiv 2 \pmod 4$, $\Re(r_a)$ attains the required value only for $a \in \{2, t-2\}$, so $\langle \mathbf{b}, \mathbf{b} \rangle = -2$ if and only if either $t = 3$ or the entries are at cyclic distance 2. In each case, the entries and equal ($\sigma = +1$).
- If $t \equiv 0 \pmod 4$, $\Re(r_a)$ attains the required value if and only if the entries are at cyclic distance 2. The entries are equal except for the pairs $(0, t-2)$ and $(1, t-1)$ due to the last edge reversed in the star complement.

Finally, u is adjacent to exactly one vertex of C'_t if and only if $r_0 = -2$, which implies $t = 8$ (see (5)). (Naturally, u has at least one neighbour in C'_t; this follows directly and also from Theorem 2.2(ii).)

We have now identified all possible star set vertices and proceed to examine their compatibility in order to construct the maximal extensions. Let $\mathbf{b}$ and $\mathbf{c}$ be distinct columns of B.

For $t = 4$, based on the previous part of the proof and Theorem 2.2(iii), we deduce the existence of 8 possible star set vertices. By computing $\langle \mathbf{b}, \mathbf{c} \rangle$ and equating it with 0, $\mathtt{i}$, or $-\mathtt{i}$, we find that all 4 vertices adjacent to consecutive vertices of C'_4 are mutually compatible, while none of them is compatible with any of the remaining vertices. The first conclusion yields the graph S'_8 shown in Figure 1, where the star complement vertices are labelled $0, 1, 2, 3$. The star set vertices that are not adjacent to consecutive vertices are examined immediately thereafter, following the analysis of the remaining configurations.

For $t = 8$, there are 24 possible star set vertices. Considering extensions that include at least one vertex adjacent to either 1 or 3 vertices of the star complement, we obtain a unique maximal extension. This oriented graph is Q'_4 in the figure, with the star complement labelled as before.

It remains to consider extensions that include star set vertices with exactly 2 neighbours in $C'_t,\ t \geq 3$. Let the j-th entry of $\mathbf{b}$ and $\mathbf{c}$ be

$$b_j = \mathtt{i}(\delta_{j,q} + \delta_{j,q+2}) \ \text{ and } \ c_j = \mathtt{i}(\delta_{j,p} + \delta_{j,p+2}),$$

and put $a \equiv q - p \pmod t$. A direct computation gives

$$\langle \mathbf{b}, \mathbf{c} \rangle = -\big(2r_a + r_{a-2} + r_{a+2}\big).$$

Thus, it suffices to analyse the function $S(a) = 2r_a + r_{a-2} + r_{a+2}$.

For the circulant part we have

$$\mathtt{i}r_{j+1} - 2r_j - \mathtt{i}r_{j-1} = 0 \quad (\text{for } j \notin X),$$

where $X = \varnothing$ for $t \not\equiv 0 \pmod 4$ and $X = \{0, n-1\}$, otherwise. From the previous recurrence, we obtain

$$r_{j+2} = -3r_j - 2\mathtt{i}r_{j-1}, \quad r_{j-2} = r_j + 2\mathtt{i}r_{j-1},$$

valid whenever the required indices avoid X. Substituting these two identities into $S(a)$ gives the exact cancellation:

$$S(a) = 2r_a + (r_a + 2\mathtt{i}r_{a-1}) + (-3r_a - 2\mathtt{i}r_{a-1}) = 0.$$

Hence, $S(a) = 0$ for every displacement a such that the indices $a-2, a-1, a, a+1, a+2$ do not hit the exceptional set X. In particular, for the circulant cases $n \not\equiv 0 \pmod 4$ this proves $S(a) = 0$ whenever $a \notin \{0, \pm 1\}$. In addition, for $a = \pm 1$, using the same recurrence, we compute $S(\pm 1) = \pm \mathtt{i}$, whereas $a = 0$ does not occur since $\mathbf{b} \neq \mathbf{c}$ by Theorem 2.2(iii).

The case in which some of the aforementioned indices belongs to X is treated by employing the boundary conditions (4). Carrying out these short substitutions reproduces the same values as in the circulant case except when the interlacing arc between the supports straddles the reversed edge: in that configuration the local algebra flips the sign of $S(1)$ (equivalently, $S(-1)$). Thus, one finds

$$S(1) = \begin{cases} \mathtt{i}, & \text{if the interlacing does not cross the reversed edge,} \\ -\mathtt{i}, & \text{if the interlacing crosses the reversed edge,} \end{cases}$$

and similarly for $S(-1) = -S(1)$.

In summary, the vertices with two neighbours in C'_t are pairwise compatible, and therefore yield a unique maximal extension for each $t \geq 3$. Combining the previous conclusions, we deduce that this extension is T'_{2t}. For clarity, the vertices of the star complement are labelled as before, with the exception that for $t \equiv 0 \pmod 4$, vertex 0 is replaced by 0^*. This completes the proof. □

The second oriented graph in Figure 1 is denoted by Q'_4 because its underlying graph is the 4-dimensional cube, commonly denoted Q_4; here, the number 4 refers to the dimension of the cube rather than the number of vertices. The third graph in the same figure actually represents an infinite family of 4-regular oriented graphs, also known as toral tessellations, which are obtained by repeating a fundamental pattern consisting of the four initial vertices labelled 0, 0^*, 1, and 1^*. The remaining graphs depicted in the figure are sporadic solutions arising from the computations.

By Theorem 2.2(i), every connected oriented graph with eigenvalue -2 contains a connected star complement corresponding to this eigenvalue. The preceding theorem lists all such graphs in which the star complement is an oriented cycle. Let H' be an oriented graph that is not a cycle. The least eigenvalue of H' is greater than -2 if and only if either (a) H' is an induced subgraph of an oriented graph obtained by attaching an arbitrary oriented path to each of two non-adjacent vertices of the 4-vertex cycle whose edges are oriented as in the second cycle of (3), or (b) H' has at most 8 vertices. This conclusion follows from the examination of induced Smith trees with least eigenvalue -2.

The star complements defined in (a) are treated as in Theorem 3.1, and they yield no new extensions except when their order is equal to 7. The details of the computation are omitted from the presentation, as they closely follow the approach of Theorem 3.1, together with a similar algebraic analysis. We emphasize, however, that for $t \geq 9$, the identity $\langle \mathbf{b}, \mathbf{b} \rangle = -2$ holds if and only if the corresponding star set vertex satisfies one of the following conditions: it is either adjacent to exactly two vertices at distance 2, or adjacent to exactly two vertices of degree 1 in the star complement. Applying these conditions, the remaining computations lead directly to the oriented graph T'_{2t} illustrated in Figure 1. For the cases $t \leq 8$, the analysis must be performed individually. This process yields another maximal extension, namely the oriented graph S'_{14}, which is also depicted in Figure 1. A closer inspection of S'_{14} reveals the existence of several star complements. Notably, one of these complements is the 7-vertex oriented path explicitly indicated in the figure, highlighting the underlying structure that emerges from the star complement construction. The star complements defined in (b) are examined individually and yield no additional extensions. We summarise these observations in the following theorem.

Theorem 3.2. *Every connected oriented graph whose skew spectral radius does not exceed* 2 *is switching equivalent to an (not necessarily proper) induced subgraph of at least one of the oriented graphs shown in Figure 1.*

Proof. If the spectral radius of a connected oriented graph G' is less than 2, then G' is an oriented graph described in (2) or (3), or corresponds to one of the cases (a) or (b) stated prior the formulation of this theorem. In the preceding discussion, all such graphs were identified as subgraphs of the oriented graphs shown in Figure 1. If the spectral radius of G' is equal to 2, then G' possesses a connected star complement for the eigenvalue -2. The desired assertion follows since the maximal extensions of all connected star complements appear in Figure 1. □

The paper is concluded with the following characterization.

Theorem 3.3. *Every maximal oriented graph whose skew spectral radius is equal to* 2 *contains a star complement for the skew eigenvalue* -2 *which, up to switching, is either*

(i) the unique oriented cycle whose skew spectral radius is distinct from 2 *and whose order is distinct from* 7*, or*

(ii) the oriented path of order 7*.*

Proof. This result is a consequence of Theorem 3.1 and the subsequent discussion. □

4 Concluding remarks

This paper presents an alternative approach to proving the results of [13], in which the existing characterization of signed graphs with spectrum contained in $[-2, 2]$ is combined with recent connections between the spectra of signed graphs and the skew spectra of associated oriented graphs to obtain a complete characterization of all oriented graphs whose skew spectrum lies within the same interval.

Without commenting on the quality of the approach or the elegance of the proofs, we note that the method presented here offers three key benefits. First, it completely avoids the use of signed graphs, providing a conceptually simpler and more direct framework entirely within the theory of oriented graphs. Second, it yields an elegant and structurally transparent characterization of all oriented graphs whose spectral radius does not exceed 2: with a single exception, these graphs are precisely those whose star complements are isomorphic to fixed oriented cycles. Finally, it demonstrates the broad applicability and effectiveness of the star complement technique in the setting of oriented graphs, extending its utility beyond the well-established contexts of ordinary and signed graphs.

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