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Diophantine equations for additive Pell numbers in Pell, Pell–Lucas, and Modified Pell numbers

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Abstract: This paper investigates the Diophantine equations arising from ternary additive problems of Pell, Pell–Lucas, and Modified Pell numbers. Specifically, we characterize all integer solutions to the equation $P_n + P_m + P_r = X_k$, $X \in \{P, Q, R\}$, where P_i, Q_i, and R_i denote the i-th terms of the Pell, Pell–Lucas, and Modified Pell sequences, respectively. By leveraging recurrence relations, Binet's formulas, and Carmichael's Primitive Divisor Theorem, we provide the first complete classification of solutions to this ternary additive problem. Our results reveal several parametric and singular solutions. Furthermore, we reduce prior results to binary sums of the form $P_n + P_m = X_k$ as special instances of our framework.

Keywords: Pell number, Pell–Lucas number, Primitive divisor, Diophantine equation, Binet's formula.
2020 Mathematics Subject Classification: 11B39, 11D61.

1 Introduction

Pell–like numbers, with their deep roots in number theory and approximation theory, have long captivated mathematicians and scientists due to their recurrence relations and applications in solving the Pell equation. The motivation for such investigations stems from the intriguing research demands of the Pell numbers in areas such as Diophantine equations, matrix-theory applications, and related fields. To align with the existing literature, in what follows, we denote the Pell numbers as $\{P_n\}_{n\geq 0}$, the Pell–Lucas numbers as $\{Q_n\}_{n\geq 0}$, and the Modified Pell numbers as $\{R_n\}_{n\geq 0}$, where n is an integer. It is well-established that these integer sequences are generated from the second-order recurrence relation of the general form $X_{n+2} = 2X_{n+1} + X_n$. The key difference among them lies in their initial conditions: $(P_0, P_1) = (0, 1)$ for the Pell numbers, $(Q_0, Q_1) = (2, 2)$ for the Pell–Lucas numbers, and $(R_0, R_1) = (1, 1)$ for the Modified Pell numbers. Another alternative method for defining these sequences is through what are known as Binet's formulas. Let η and ϕ be the two conjugate roots of the characteristic equation $x^2 - 2x–1 = 0$, where $\eta = 1 + \sqrt{2}$ and $\phi = 1 - \sqrt{2}$. Then one may show, by enforcing the above initial conditions, that

$$P_n = \frac{\eta^n - \phi^n}{\eta - \phi}, \ Q_n = \eta^n + \phi^n, \text{ and } R_n = \frac{\eta^n + \phi^n}{\eta + \phi}, \tag{1}$$

respectively. Each formula has served as an indispensable tool in open literature, just as it does in this work. These integer sequences are strongly interconnected. The following identities serve as compelling evidence of this fact:

$$2P_nR_n = P_{2n}, \ Q_n = 2R_n, \ P_n + P_{n-1} = R_n, \ P_{n+1} - P_n = R_n, \ P_{n+1} + P_{n-1} = Q_n. \tag{2}$$

These identities and many more can be found in references [15] and [6].

Over the past several centuries, a strikingly rich area has emerged within the broader realm of number theory, generally attributed to the Hellenistic mathematician Diophantus of Alexandria and referred to as Diophantine equations. Building upon this classical heritage, recent work has focused on various extensions and generalizations of these problems. At its core, this field primarily focuses on examining equations where all constants are integers, parameters involve only sums, products, and powers, and whose sought-after solutions are exclusively integers. In recent decades, particular attention has turned to Diophantine equations constructed by selecting parameters from specially defined integer sequences such as Fibonacci, Pell, and Jacobsthal numbers, sparking a wealth of new results and conjectures. This line of inquiry has attracted numerous researchers and holds substantial potential for further breakthroughs. One could cite several exemplary works from the open literature in this regard. Hirata-Kohno and Luca [13] analyzed the equation $F_n^x + F_{n+1}^x = F_m^x$, investigating sums of consecutive Fibonacci powers. Ddamulira *et al.* [9] extended this line of inquiry by characterizing solutions to $F_k = P_mP_n$ and $P_k = F_mF_n$, addressing multiplicative intersections between Fibonacci and Pell sequences. Bravo *et al.* [3] examined ternary Pell sums expressible as powers of 2 via $P_k + P_m + P_n = 2^a$. Luca and Patel [16] classified perfect powers arising from sums of two Fibonacci numbers through $F_n + F_m = y^p$. Regarding Pell numbers, Rihane [18] studied instances where the sum of like powers of two consecutive Pell numbers equals another Pell number, specifically $P_n^x + P_{n+1}^x = P_m$. Tchammou and Togbé [19] derived solutions to the weighted sum $\sum_{j=1}^{k} jP_j{}^p = P_n{}^q$. Patel and Chaves [17] found all solutions to the exponential Diophantine equation $F_{n+1}^x - F_{n-1}^x = F_m$. The problem of finding numbers that are simultaneously generalized k-Fibonacci and generalized k-Pell numbers, $P_n^{(k)} = F_m^{(\ell)}$, was addressed by Bravo *et al.* [4]. Alan and Alan [1] identified Mersenne numbers decomposable into Pell products $M_k = P_n P_m$. Building on the paper of Patel and Chaves [17], Gómez *et al.* [12] introduced an additional exponent into the equation: $F_{n+1}^x - F_{n-1}^x = F_m^y$. Faye *et al.* [11] explored asymmetric power combinations $P_n^x + P_{n+1}^y = P_m^x$ and $P_n^y + P_{n+1}^x = P_m^x$. Daşdemir and Varol [7] investigated Jacobsthal–Modified Pell multiplicative relations $J_k = R_m R_n$ and $R_k = J_m J_n$, while Emin and Ateş [10] solved the Pell-square sum equation $P_m^2 + P_n^2 = 2^a$. Bérczes *et al.* [2] generalized power-sum structures through $F_n^x + F_k^x = F_m^y$, and Daşdemir and Varol [8] established Pell–Lucas multiplicative identities $L_k = P_m P_n$ and $P_k = L_m L_n$. Karaçam *et al.* [14] studied Diophantine equations with Fibonacci and Lucas number coefficients, obtaining several related identities and generalizations.

A comprehensive review of the existing literature reveals a notable gap: while some binary additive equations and multiplicative intersections have been rigorously explored, the ternary additive problem, whether the sum of three distinct Pell numbers can coincide with a term in the Pell, Pell–Lucas, or Modified Pell sequence, remains entirely unresolved. Our investigation fills this void by establishing the first complete characterization of solutions to

$$P_n + P_m + P_r \in \{P_k, Q_k, R_k\}, \tag{3}$$

where P_n, Q_n, and R_n denote n-th term of the Pell, Pell–Lucas and Modified Pell sequences, respectively. Crucially, we uncover parametric families of solutions and demonstrate how these arise from the interplay of recurrence relations, Binet's formulas, and Carmichael's Primitive Divisor Theorem [5]. Furthermore, as special cases of our generalized framework, we unify previously fragmented results on binary sums such that

$$P_n + P_m \in \{P_k, Q_k, R_k\}. \tag{4}$$

By bridging this gap, we advance both the theoretical landscape of Pell numbers and the broader toolkit for Diophantine equations in special sequences.

The additive symmetry complicates the analysis; hence, we assume that $0 \le r \le m \le n$ and $k \ge 0$ without loss of generality.

2 Main results

In this section, we will present our exact results and detailed proofs concerning the Diophantine equations given in Equation (3). Getting started with this section, we list of the following lemmas, which will later be used in the proof process of the target theorems.

Lemma 2.1. *The following inequalities hold for any positive integer n,*

$$\eta^{n-2} \le P_n \le \eta^{n-1},\ \eta^{n-1} \le Q_n \le 2\eta^n,\ \textit{and}\ \eta^{n-1} \le R_n \le \eta^n, \tag{5}$$

which can be proved easily by using the bounds from Binet's formulas.

The following result follows directly from Carmichael's primitive prime divisor theorem [5].

Theorem 2.1 (Carmichael's Primitive Divisor Theorem [5])**.** *For every integer $a > 1$, the a-th Pell number P_a has at least one primitive prime divisor, i.e., a prime number p that divides P_a but does not divide any Pell number P_b for $b < a$.*

The following is our first result.

Theorem 2.2. *All the solutions to the equation $P_n + P_m + P_r = P_k$ are*

$$(n, m, r, k) \in \{(t, 0, 0, t), (t + 1, t + 1, t, t + 2)\},$$

where t is a natural number.

Proof. Consider the case where $r = 0$. If $m = 0$, our problem is reduced to $P_n = P_k$. By Carmichael's Primitive Divisor Theorem (PDT), there is no solution except for $(n, m, r, k) = (n, 0, 0, n)$. Now, let us focus on the case where $1 \leq m \leq n$, which implies that $k \geq 2$. If $n = 1$, then $P_1 + P_1 = 1 + 1 = 2 = P_2$, which gives the solution $(n, m, r, k) = (1, 1, 0, 2)$. For now, suppose that $n \geq 2$. By Equation (5), we can write

$$\eta^{k-2} \leq P_k = P_n + P_m \leq \eta^{n-1} + \eta^{m-1} \leq 2\eta^{n-1} < \eta^n,$$

and

$$\eta^{n-2} \leq P_n < P_n + P_m = P_k \leq \eta^{k-1},$$

which implies that $n - 1 < k < n + 2$. If $k = n$, then, $P_m = 0$, which is absurd due to fact that $m \geq 1$. Substituting $k = n + 1$ into the problem, we get $P_m = P_{n+1} - P_n$. By Equation (2), we can write $P_m = R_n = P_{2n}/(2P_n)$, which means that $2P_nP_m = P_{2n}$. As a result, we conclude that there is no solution due to Carmichael's PDT.

Now, consider the case where $1 \leq r \leq m \leq n$. It is easy to see no solution for $n = 1$. Now suppose that $n \geq 2$. Using Equation (5), we can write

$$\eta^{k-2} \leq P_k = P_n + P_m + P_r \leq 3P_n \leq 3\eta^{n-1} < \eta^2\eta^{n-1} = \eta^{n+1}$$

and

$$\eta^{n-2} \leq P_n < P_k \leq \eta^{k-1},$$

which implies that $k \in \{n, n + 1, n + 2\}$. If $k = n$, our problem is reduces to $P_m + P_r = 0$ which implies $m = r = 0$, contradicting $1 \leq r \leq m \leq n$. Taking $k = n+1$ yields $P_n + P_m + P_r = P_{n+1}$, leading to $P_m + P_r = P_n + P_{n-1}$. Since $r \leq m \leq n$, the parameter m cannot take a value smaller than n due to the parity condition. This implies $r = n - 1$ and $m = n$, which gives the parametric solution $(n, m, r, k) = (n, n, n - 1, n + 1)$. This includes the particular solution above. Following a similar idea to the above, if $k = n + 2$, we get $P_n + P_m + P_r = P_{n+2}$. So, $P_m + P_r = P_{n+1} + P_{n+1}$, that is $m = r = n + 1$, which contradicts $1 \leq r \leq m \leq n$. This completes the proof. □

The next theorem establishes results related to the Pell–Lucas numbers.

Theorem 2.3. *All the solutions to the equation $P_n + P_m + P_r = Q_k$ are*

$$(n, m, r, k) \in \{(1, 1, 0, 0), (1, 1, 0, 1), (2, 0, 0, 0), (2, 2, 2, 2), (4, 1, 1, 3), (t + 2, t, 0, t + 1)\},$$

where t is a non-negative integer.

Proof. Take $r = 0$. So, our problem is reduced to $P_n + P_m = Q_k$. If $m = 0$, we get $P_n = Q_k$, which can be written in the form $P_n P_k = P_{2k}$ by Equation (2). From Carmichael's PDT, there is no solution except for $k > 1$, which yields the trivial solutions $(n, m, r, k) \in \{(2, 0, 0, 0), (2, 0, 0, 1)\}$. If $n = 1$, then $Q_k = 2$, which gives the trivial solutions $(n, m, r, k) \in \{(1, 1, 0, 0), (1, 1, 0, 1)\}$. If $n \geq 2$, by Equation (5), we can write

$$\eta^{k-1} \leq Q_k = P_n + P_m \leq \eta^{n-1} + \eta^{m-1} \leq 2\eta^{n-1} < \eta^n$$

and

$$\eta^{n-2} \leq P_n < P_n + P_m = Q_k \leq 2\eta^k < \eta^{k+1},$$

which means that $k \in \{n-2, n-1, n\}$. Taking $k = n$ reduces our problem to $P_m = P_n + 2P_{n-1}$ by Equation (2), which is a contradiction with $P_m \leq P_n$. Similarly, if $k = n - 1$, we have $P_m = P_{n-2}$, which gives the parametric solution $(n, m, r, k) = (n, n-2, 0, n-1)$. If $k = n-2$, we have $P_n + P_m = Q_{n-2}$, which is a absurd with the fact that $P_n \geq Q_{n-2}$ for $n \geq 2$.

Now, let us focus on the case where $1 \leq r \leq m \leq n$. When $n \leq 4$, it is easy to observe the solutions $(n, m, r, k) \in \{(4, 1, 1, 3), (2, 2, 2, 2)\}$. Using Equation (5), we can write

$$\eta^{k-1} \leq Q_k = P_n + P_m + P_r \leq 3P_n \leq 3\eta^{n-1} < \eta^2\eta^{n-1} = \eta^{n+1}$$

and

$$\eta^{n-2} \leq P_n < Q_k \leq 2\eta^k < \eta^{k+1},$$

which give that $k \in \{n-2, n-1, n, n+1\}$.

If $k = n - 2$, our problem is reduced to $P_r + P_m + 3P_{n-2} = 0$, which is a contradiction with $1 \leq r \leq m \leq n$. Taking $k = n - 1$ yields $P_r + P_m = P_{n-2}$, which concludes that there is no solution by Theorem 2.2. If $k = n$, we get $P_r + P_m = P_n + 2P_{n-1}$. Here, it is clear that $m \neq n$. Indeed, if the contrary were true, the last equation would become $P_r = 2P_{n-1}$, for which there is no solution according to Carmichael's PDT. From this, we have $2P_m \geq P_r + P_m = P_n + 2P_{n-1} = 2P_{n-1} + P_{n-2} + 2P_{n-1} > 4P_{n-1}$, which gives that $m \geq n + 1$, contradicting with $m \leq n$. If $k = n + 1$, our problem is reduced to $P_r + P_m = P_{n+2}$. By Equation (5), we can write

$$\eta^n < P_{n+2} = P_m + P_r \leq 2P_m \leq 2\eta^{m-1} < \eta^m.$$

From this we have that $n < m$, which is a contradiction with $m \leq n$. This completes the proof. □

The following theorem presents our final results concerning the Modified Pell numbers.

Theorem 2.4. *All the solutions to the equation $P_n + P_m + P_r = R_k$ are*

$$(n, m, r, k) \in \{(1, 0, 0, 0), (1, 1, 1, 2), (3, 1, 1, 3), (t + 1, t, 0, t + 1)\},$$

where t is a natural number.

Proof. Set first $r = 0$, concluding $P_n + P_m = R_k$. If $m = 0$, we obtain $P_n = R_k$, which by Equation (2) takes the form $2P_nP_k = P_{2k}$. Carmichael's PDT proves that there is no solution for $k > 1$, giving the trivial solutions $(n, m, r, k) \in \{(1, 0, 0, 0), (1, 0, 0, 1)\}$. When $n = 1$, we have $R_k = 2$, which is impossible. Now assume $n \geq 2$. By Equation (5), we can write

$$\eta^{k-1} \leq R_k = P_n + P_m < 2P_n \leq 2\eta^{n-1} < \eta^n$$

and likewise,

$$\eta^{n-2} \leq P_n < P_n + P_m = R_k \leq \eta^k.$$

Hence, we conclude that $k \in \{n-1, n\}$. If $k = n-1$, using the relation $R_{n-1} = P_n - P_{n-1}$ given by Equation (2), the problem reduces to $P_m = -P_{n-1}$, which is a contradiction. Choosing $k = n$ reduces the equality to $P_m = P_{n-1}$ by Equation (2), giving the parametric form $(n, m, r, k) = (n, n-1, 0, n)$. This also includes the second of the above solutions.

Now consider the range $1 \leq r \leq m \leq n$. When $n \leq 3$, a direct inspection gives the solutions $(n, m, r, k) \in \{(1, 1, 1, 2), (3, 1, 1, 3)\}$. Focus on the case $n \geq 4$. By Equation (5),

$$\eta^{k-1} \leq R_k = P_n + P_m + P_r < 3P_n \leq 3\eta^{n-1} < \eta^{n+1}$$

and also

$$\eta^{n-2} \leq P_n < P_n + P_m + P_r = R_k \leq \eta^k.$$

Thus, $k \in \{n-1, n, n+1\}$.

Consider first take $k = n-1$. By Equation (5), we have $P_n + P_m + P_r = R_{n-1} = P_n - P_{n-1}$, which means that $P_m + P_r + P_{n-1} = 0$, contradicting the assumption $1 \leq r \leq m \leq n$. So, there is no solution. Taking $k = n$ or $k = n+1$ leads to $P_m + P_r \in \{P_{n-1}, P_{n+1}\}$ by Equation (5), which has no solution for $n \geq 4$ by Theorem 2.2. All possible cases have thus been exhausted, completing the proof. □

3 Conclusion

In this paper, we have systematically investigated the Diophantine equations involving additive combinations of Pell, Pell–Lucas, and Modified Pell numbers. By employing a combination of recurrence relations, Binet's formulas, and Carmichael's Primitive Divisor Theorem, we have provided a complete characterization of solutions to the ternary additive problem $P_n + P_m + P_r = X_k$, $X \in \{P, Q, R\}$. Our work not only fills a critical gap in the literature but also implies some results regarding binary additive equations. Below, we summarize our key findings on the binary Diophantine equations $P_n + P_m = X_k$, highlighting explicit solutions:

- For the equation $P_n + P_m = P_k$, we identified parametric and explicit solutions:

$$(n, m, k) \in \{(0, 0, 0), (1, 1, 2), (t, 0, t)\}$$

- For the equation $P_n + P_m = Q_k$, we uncovered sporadic and parametric solutions:

$$(n, m, k) \in \{(1, 1, 0), (1, 1, 1), (2, 0, 0), (t+2, t, t+1)\}$$

- For the equation $P_n + P_m = R_k$, we determined sporadic and parametric solutions:

$$(n, m, k) \in \{(1, 0, 0), (t + 1, t, t + 1)\}$$

Our findings show that combining recurrence relations with Binet's formulas and primitivity arguments provides an effective framework for solving additive Diophantine equations in Pell-type sequences. Unlike earlier work, we tackle the three-term additive case for the first time. This not only deepens our understanding of Pell numbers but also offers a clear methodology for similar investigations in other special sequences.

Looking ahead, one might consider extensions to sums involving more than three terms, to exponential variants, or to generalized Pell sequences. Exploring hybrid equations, i.e., those mixing Pell and Fibonacci numbers, could also prove fruitful. In each case, the methods presented here highlight the continued vitality of Diophantine problems in number theory and their role in uniting classical and contemporary approaches.

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