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On certain arithmetical functions connected with the prime factorization of an integer

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Abstract: If $1 < n = \prod_{i=1}^{r} p_i^{a_i}$ is the prime factorization of integer n, we study the arithmetical functions

$$M(n) = \prod_{i=1}^{r} a_i^{p_i}, \quad F(n) = \prod_{i=1}^{r} p_i^{1/a_i}, \quad G(n) = \prod_{i=1}^{r} a_i^{1/p_i}.$$

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1 Introduction

Let $n = \prod_{i=1}^{r} p_i^{a_i}$ (p_i primes; $a_i \geq 1$) be the prime factorization of $n > 1$. In 1907 O. Meissner [4] (see also [3, p. 448]) introduced the arithmetical function (denoted by him by $Z(n)$, but we will use in his honour the notation $M(n)$ (see [5]).

$$M(n) = \prod_{i=1}^{r} \alpha_i^{p_i}. \tag{1}$$

Meissner studied among others the equation $M(n) = n$, and proved that all solutions are $\prod_{i=1}^{r} p_i^{p_i}$, 16, or $\prod_{i=1}^{r} p_i^{q_i} \cdot q_i^{p_i}$, where p_i, q_i are distinct primes $(i = 1, 2, \ldots, r)$.

The function has been rediscovered by Atanassov (see e.g. [2]), who used the notation $CF(n)$ ("converse factor of n"). In what follows, let $r = \omega(n)$ denote the number of distinct prime factors of n, and $a_1 + \cdots + a_r = \Omega(n)$ the total number of prime factors. Let $\beta(n) = p_1 + \cdots + p_r$ denote the sum of prime divisors of n, and $\gamma(n) = p_1 \cdots p_r$ the product of prime divisors of n ("core function" of n). We will use also the Niven functions $h(n) = \min\{a_1, \ldots, a_r\}$ and $H(n) = \max\{a_1, \ldots, a_r\}$. Other notations, if necessary will be introduced in the text.

Atanassov [1] has introduced also the "irrational factor " of n, $IF(n) = \prod_{i=1}^{r} p_i^{1/a_i}$, which we will denoted simply by

$$F(n) = \prod_{i=1}^{r} p_i^{1/a_i}. \tag{2}$$

We introduce here for the first time the function

$$G(n) = \prod_{i=1}^{r} a_i^{1/p_i}. \tag{3}$$

The aim of this paper is to prove certain new properties of the functions $M(n)$, $F(n)$, $G(n)$ and related functions.

2 Main results

The arithmetical function $M(n)$ is a multiplicative function (as noted also in [2]), we have the following more general result:

Theorem 1. *If a and b are squarefull numbers, then*

$$M(ab) \leq M(a)M(b). \tag{4}$$

If $\gcd(a, b) =$ squarefree, then

$$M(ab) > M(a)M(b). \tag{5}$$

Proof. Let

$$a = \prod_{i=1}^{r} p_i^{a_i} \cdot \prod_{j=1}^{s} q_j^{\beta_j}, \quad b = \prod_{i=1}^{r} p_i^{a'_i} \cdot \prod_{h=1}^{m} r_h^{\gamma_h}$$

be the prime factorizations of a and b ($|$ p_i, q_j, r_h, distinct primes). Then, an easy computation gives

$$\frac{M(ab)}{M(a)M(b)} = \frac{\prod_{i=1}^{r}(a_i + a'_i)^{p_i}}{\prod_{i=1}^{r}(a_i a'_i)^{p_i}.} \tag{6}$$

Now, remark that for $a_i, a'_i \geq 2$ one has $(a_i - 1)(a'_i - 1) \geq 1$, so $a_i + a'_i \leq a_i \cdot a_i$, which implies inequality (4). If $\gcd(a,) = \text{squarefree}$, then $a_i, a'_1 \leq 1$, and in this case, clearly one has $a_i + a'_i > a_i \cdot a'_i$, so relation (5) follows. □

Let us now introduce the arithmetical function $B(n) = \sum_{i=1}^{r} p_i \cdot a_i$. In [2] it is stated that $B(M(n)) = B(n)$, but clearly this holds only when it is supposed that a_i $(i = \overline{1, r})$ are also distinct primes. In the general case one has, however that:

Theorem 2. *One has for all* $n > 1$

$$B(M(n)) \leq B(n). \tag{7}$$

Proof. Let us suppose that $n = p^a \cdot q^b$ (p, q distinct prime; $a, b \geq 1$), as we shall see, the general case follows in the same lines. Then $B(n) = pa + qb$. Let

$$a = \prod_{i=1}^{r} p_i^{a_i} \prod_{j=1}^{s} q_j^{\beta_j}, \quad b = \prod_{i=1}^{r} p_i^{a'_i} \prod_{h=1}^{m} r_h^{\gamma_h}.$$

Then we can write that

$$\begin{aligned} M(n) &= \prod p_i^{a_i \cdot p} \cdot \prod q_j^{\beta_j \cdot p} \cdot \prod p_i^{a'_i \cdot q} \cdot \prod r_h^{\gamma_h \cdot q} \\ &= \prod p_i^{a_i p + a'_i q} \cdot \prod q_j^{p\beta_j} \cdot \pi r_h^{\gamma_h} \end{aligned}$$

and

$$\begin{aligned} B(M(n)) &= \sum_i (a_i p + a'_i q) p_i + \sum_j p\beta_j q_j + \sum_h q\gamma_h r_h \\ &\leq p \prod p_i^{a_i} \cdot \prod q_j^{\beta_j} + q \prod p_i^{a'_i} \cdot \prod r_h^{\gamma_h} \end{aligned}$$

by the following remarks. One has

$$\prod_i p_i^{a_i} \cdot \prod_j q_j^{\beta_j} \geq \sum_i a_i p_i + \sum_j \beta_j \cdot q_j$$

since $p_i^{a_i} \geq p_i a_i$ (this is true, as $p_i^{a_i - 1} \geq 2^{a_i - 1} \geq a_i$ for $a_i \geq 1$).

Now, it is sufficient to show that $\prod p_i \cdot a_i \cdot \prod q_j \beta_j \geq \sum_i p_i a_i + \sum_j \beta_j a_j$, or more generally $(x_1 \cdots x_r) \cdot (y_1 \cdots y_m) \geq (x_1 + \cdots + x_r)(y_1 + \cdots | y_m)$, where $x_i = p_i a_i$, $y_j = q_j \beta_j$. It will be sufficient to show that

$$x_1 \cdots x_r \geq x_1 + \cdots + x_r \tag{8}$$

where $x_i \geq 2$ $(i = 1, 2, \ldots, r)$. This can be proved easily by mathematical induction. □

Theorem 3. *For* $n > 1$, *one has*

$$h(n) \leq \frac{\beta(n)}{K(n)} \leq M(n)^{1/\beta(n)} \leq \frac{B(n)}{\beta(n)} \leq H(n), \tag{9}$$

where

$$B(n) = \sum_{i=1}^{r} p_i a_i, \quad K(n) = \sum_{i=1}^{r} \frac{p_i}{a_i}.$$

Proof. We will apply the weighted arithmetic mean-geometric mean inequality:

$$\prod_{i=1}^{r} x_i^{\lambda_i} \le \prod_{i=1}^{r} \lambda_i x_i, \tag{10}$$

where $x_i > 0$, $\lambda_i > 0$, $\sum_{i=1}^{r} \lambda_i = 1$. Let $x_i = a_i$ and $\lambda_i = p_i/\beta(n)$ $(i = \overline{1,r})$. Then, by (10) we get the third inequality in (9). As $B(n) = \sum_{i=1}^{r} a_i p_i \le \max\{a_1, \dots, a_i\} \cdot \sum_{i=1}^{r} p_i = H(n) \cdot \beta(n)$, the last inequality of (9) follows, too.

Now, applying (10) to $\frac{1}{x_i} > 0$ in place of x_i, we get the weight geometric-harmonic mean inequality:

$$\prod_{i=1}^{r} x_i^{\lambda_i} \ge \frac{1}{\sum_{i=1}^{r} \lambda_i / x_i}. \tag{11}$$

Letting again $x_i = a_i$, $\lambda_i = p_i/\beta(n)$ in (11), we get the second inequality of (9). As $\frac{1}{a_i} \le \frac{1}{h(n)}$, we get that $\sum_{i=1}^{r} \frac{p_i}{a_i} \le \frac{\beta(n)}{h(n)}$, so the first inequality of (9) follows, too. □

Corollary 1. *The sequence of general terms*

$$x_n = \frac{(M(n))^{1/\beta(n)}}{\log n} (n > 1)$$

is dense in the interval $(0, 1/\log 2)$.

Proof. From inequality (9) particularly we get

$$h(n) \le x_n \le H(n). \tag{12}$$

As $n \ge 2^{a_1 + \cdots + a_r} \ge 2^{a_r}$, with $a_r = H(n)$, we get $H(n) \le \frac{\log n}{\log 2}$, so $\frac{H(n)}{\log n} \le \frac{1}{\log 2}$.

Now, Schinzel and Šalát [7] have proved that the sequences defined by $u_n = \frac{h(n)}{\log n}$ and $v_n = \frac{H(n)}{\log n}$ are dense in the interval $(0, 1/\log 2)$.

By (12), Corollary 2 follows. □

Corollary 2. *One has the following asymptotic result:*

$$\sum_{n \le x} M(n)^{1/\beta(n)} \sim x \quad as \quad x \to \infty. \tag{13}$$

Proof. By (9), particularly, one has

$$h(n) \le M(n)^{1/\beta(n)} \le \frac{B(n)}{\beta(n)}. \tag{14}$$

Now, the following asymptotic results are known:

$$\sum_{n\geq x} \frac{B(n)}{\beta(n)} \sim x \tag{15}$$

(see [8, p. 144]); and

$$\sum_{n\leq x} h(n) \sim x \tag{16}$$

(see [7, p. 330]). By (14), (15) and (16), relation (13) follows. □

Corollary 3. *Let* $\Omega_1(n) = \sum_{i=1}^{r} 1/a_i$ *(introduced in [6]).*

Then

$$\limsup_{x\to\infty} \frac{1}{x} \cdot \sum_{n\leq x} \Omega_1(n) \cdot M(n)^{1/\beta(n)} \geq 1. \tag{17}$$

Proof. Let $P(n)$ denote the greatest prime factor of n. Then $K(n) = \sum_{i=1}^{r} p_i/a_i \leq P(n) \cdot \Omega_1(n)$, so by (9) we get $\Omega_1(n) \cdot M(n)^{1/\beta(n)} \geq \frac{\beta(n)}{P(n)}$. Now, it is known that $\sum_{n\geq x} \frac{\beta(n)}{P(n)} \sim x$ $(x \to \infty)$ (see [8, p. 145]). Then relation (17) follows.

The definition of "irrational factor" for $F(n)$ follows from the fact that $\prod_{i=n}^{r} p_i^{1/a_i}$ is an irrational number, if p_i are distinct primes, and $a_i \geq 1$ are integers.

We prove for simplicity the case $r = 2$: Let us suppose that

$$p^{1/a} \cdot q^{1/b} = \frac{A}{B} = \text{ rational number.} \tag{18}$$

Then $p^b \cdot q^a = \frac{A^{ab}}{B^{ab}}$. Now, it follows by the unique factorization theorem that if $u^n \mid v^n$ (for $u, v, n \geq 1$ integers), then $u \mid v$. Thus, as $B^{ab} \mid A^{ab}$, we get that $B \mid A$, so $A/B = t$, and we get from (18) that $p^b \cdot q^a = t^{ab}$ $(t \geq 1$ integer). As $p^b \mid (t)^{ab}$ we get that $p \mid t^a$, and as p is prime, we get $p \mid t$. Similarly we get $q \mid t$, so $t = mpq$ and $t^{ab} = (mpq)^{ab} > p^b q^a$, so we get a contradiction. □

Theorem 4. *One has*

$$F(n) \geq 2^{\Omega_1(n)}; \tag{19}$$

If n *is squarefull, then*

$$F(n) < M(n). \tag{20}$$

If n *is squarefree, then*

$$F(n) = \gamma(n) > M(n) = 1. \tag{21}$$

Proof. Relation (19) follows by $p_i \geq 2$ and the definition of $\Omega_1(n)$. If all $a_i \geq 2$, then $p_i^{1/a_i} < a_i^{p_i}$ can be rewritten as $a_i^{a_i} \geq p^{1/p}$. Now $a_i^{a_i} \geq 2^2 = 4$ and $p_i^{1/p_i} < 4$ as $4^{p_i} > p_i$ is clearly true. If n is squarefreee, i.e. all $a_i = 1$, then clearly $F(n) = p_i \cdots p_r = \gamma(n)$ and $M(n) = 1$, so (21) trivially follows. □

Theorem 5. *Let us denote by* $\beta^*(n) = \prod_{i=1}^{r} a_i$ $(n > 1)$. *Then*

$$G(n) \cdot M(n) \geq (\beta^*(n))^2; \tag{22}$$

$$G(n) \leq \sqrt{\beta^*(n)}. \tag{23}$$

If $p_i \geq a_i$ $(i = 1, \dots, r)$, *then*

$$F(n) \geq G(n). \tag{24}$$

Proof. Remarking that $G(n)M(n) = \prod_{i=1}^{r} a_i^{(p_i+1/p_i)}$, and by the inequality $p_i + 1/p_i \geq 2$, relation (22) follows.

Relation (23) follows by $\frac{1}{p_i} \leq \frac{1}{2}$, while (24) is a consequence of the definitions of $F(n)$ and $G(n)$. □

Theorem 6. *For* $n > 1$ *one has*

$$h(n) \leq \frac{\log \gamma(n)}{\log F(n)} \leq H(n) \tag{25}$$

and

$$h(n) \leq (G(n))^{1/\beta_{-1}(n)} \leq H(n), \tag{26}$$

where $\beta_{-1}(n) = \sum_{i=1}^{r} 1/p_i$ *(for this notation, see [6]).*

Proof. Let $a_1 \leq \cdots \leq a_r$. Then $1/a_1 \geq \cdots \geq 1/a_r$, so $F(n) = p_1^{1/a_1} \cdots p_r^{1/a_r} \geq (p_1 \cdots p_r)^{\frac{1}{H(n)}}$ and $F(n) \leq (\gamma(n))^{1/h(n)}$. Thus, (25) follows. In the same manner, $G(n) = a_1^{1/p_1} \cdots a_r^{1/p_r}$ $\leq a_r^{1/p_1+\cdots+1/p_r} = H(n)^{\beta_{-1}(n)}$ and $G(n) \geq a_1^{\beta_{-1}(n)=h(n)^{\beta_{-1}(n)}}$. Therefore, relation (26) follows, too. □

Corollary 4. *The sequences of general terms* $(\log \gamma(n))/(\log n \cdot \log F(n))$ *and* $(((G(n))^{1/\beta_{-1}(n)})/\log n)$ *are dense in the interval* $(0, 1/\log 2)$.

Proof. This is similar to the proof of Corollary 1. □

Theorem 7. *If* $n > 1$ *is squarefull, then*

$$(F(n))^3 \cdot G(n) \leq M(n), \tag{27}$$

with equality only for $n = 4$.

Proof. We will prove the inequality

$$(p^{1/a})^3 \cdot a^{1/p} \leq a^p \tag{28}$$

for $p \geq 2$ and $a \geq 2$.

By taking logarithms, (28) can be rewritten as

$$\frac{p^2-1}{p\log p} \geq \frac{3}{a\log a}. \tag{29}$$

Now, the function $f(p) = \dfrac{p^2-1}{p\log p}$ has a derivative $f'(p) = (p^2\log p - p^2 + \log p + 1)/p^2\log^2 p > 0$ for $p \geq 2$ as $p^2\log p - p^2 + \log p + 1 > 0$ for $p \geq 2$. Thus, the function $f(p)$ is strictly increasing, implying $f(p) \geq f(2) = \dfrac{3}{2\log 2}$. For $a \geq 2$ one has $\dfrac{3}{a\log a} \leq \dfrac{3}{2\log 2}$, so relation (29) is proved. Now, the inequality (27) follows by letting $p = p_i$, $a = a_i$; and by taking the products of these inequalities for $i = 1, 2, \ldots, r$. □

Finally, we introduce here two more arithmetic function, namely: For $n > 1$, let

$$Q(n) = \prod_{i=1}^{r} a_i^{a_i/p_i}, \tag{30}$$

$$N(n) = \prod_{i=1}^{r} p_i^{a_i/p_i}. \tag{31}$$

Theorem 8. *One has*

$$(G(n))^{h(n)} \leq Q(n) \leq (G(n))^{H(n)} \tag{32}$$

and

$$(A(n))^{h(n)} \leq N(n) \leq (A(n))^{H(n)}, \tag{33}$$

where

$$A(n) = \prod_{i=1}^{r} p_i^{1/p_i}. \tag{34}$$

Proof. The proof is similar to the proof of Theorem 6 and we omit the details. □

Theorem 9. *One has*

$$\frac{N(n)}{Q(n)} < e^{\omega(n)/e}. \tag{35}$$

Proof. We will prove first the inequality

$$x^{1/x} \leq e^{1/e},\ x > 0. \tag{36}$$

By letting the function $g(x) = \dfrac{\log x}{x}$, we get $g'(x) = \dfrac{1-\log x}{x^2}$, so $g(x)$ has a maximum point at $x = e$, and thus $g(x) \leq g(e) = \dfrac{1}{e}$, so (36) follows. Now, letting $x = p_i/a_i$ in (36), and by using the fact that e is an irrational number (so $p_i/a_i \neq e$), we get the strict inequality (35). □

Theorem 10. *For the function $A(n)$ defined by* (34) *one has*

$$\left(\frac{\beta_{-2}(n)}{\beta_{-1}(n)}\right)^{\beta_{-1}(n)} \leq A(n) \leq \left(\frac{\omega(n)}{\beta_{-1}(n)}\right)^{\beta_{-1}(n)}, \tag{37}$$

where $\beta_{-2}(n) = \displaystyle\sum_{i=1}^{r} \frac{1}{p_i^2}$.

Proof. Apply inequalities (10) and (11) for $x_i = p_i$ and $\lambda_i = \frac{1}{p_i \beta_{-1}(n)}$. As $\beta_{-1}(n) = \sum_{i=1}^{r} 1/p_i$, we get $\sum_{i=1}^{r} \lambda_i = 1$. The right side of (37) follows by (10), while the left side, by inequality (11). □

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