

On a family of sums of powers of the floor function and their links with generalized Dedekind sums

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Abstract: In this paper we are concerned with a family of sums involving the floor function. With r a nonnegative integer and n and m positive integers we consider the sums

$$S_r(n, m) := \sum_{k=1}^{n-1} \left[\frac{km}{n} \right]^r.$$

While a formula for S_1 is well known, we provide closed-form formulas for S_2 and S_3 as well as the reciprocity laws they satisfy. Additionally, one can find a closed-form formula for the classical Dedekind sum using the Euclidean algorithm. Finally, we provide a general formula for S_r showing its dependency on generalized Dedekind sums.

Keywords: Sum of powers of the floor function, Dedekind sums, Faulhaber sums, Reciprocity laws, Euclidean algorithm.

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1 Introduction

The analysis of the sums $S_r(n, m)$ defined below (see (1)) was motivated due to their links with Dedekind sums (see Proposition 5.1) which have applications in many areas of mathematics. A broad overview of the Dedekind sums and their applications can be found in the introduction of [4], a reference monograph on the topic by Hans Rademacher and Emil Grosswald.



1.1 Notations

For any real number x we denote by $\lfloor x \rfloor$ the floor function defined as the greatest integer less than or equal to x . For an integer a and a positive integer b , we use $a \bmod b$ to mean the remainder of a when divided by b .

The sums of interest are noted as follows:

$$\mathbf{S}_r(n, m) := \sum_{k=1}^{n-1} \left\lfloor \frac{km}{n} \right\rfloor^r. \quad (1)$$

We also use the following notation for the Faulhaber sums (see [5]):

$$\mathbf{F}_r(n) := \sum_{i=0}^n i^r. \quad (2)$$

For positive coprime integers a and b we use the notation $s(b, a)$ as in Rademacher's book [4] to denote the classical Dedekind sum:

$$s(b, a) := \sum_{k=1}^a \left(\left(\frac{kb}{a} \right) \right) \left(\left(\frac{k}{a} \right) \right) \quad (3)$$

with the symbol $((x))$ defined by

$$((x)) := \begin{cases} \{x\} - \frac{1}{2}, & \text{if } x \notin \mathbb{Z}, \\ 0, & \text{if } x \in \mathbb{Z}. \end{cases} \quad (4)$$

We use as well a definition of generalized Dedekind sums suggested by Don Zagier in the article [6, Eq. 40, p. 157]. In this definition, b and n are positive integers and the coefficients a_i for i from 1 to n are positive integers coprime with b :

$$\delta(b; a_1, \dots, a_n) := 2^n b^{n-1} \sum_{k=1}^{b-1} \prod_{i=1}^n \left(\left(\frac{ka_i}{b} \right) \right). \quad (5)$$

1.2 General considerations

Proposition 1.1. *Let a , b and r be positive coprime integers. If a and b are coprime, then*

$$\mathbf{S}_r(b, a) + \sum_{i=0}^r (-1)^{i+1} \binom{r}{i} (a-1)^{r-i} \mathbf{S}_i(b, a) = 0. \quad (6)$$

Proof. Let k be an integer satisfying $0 < k < b$. Let $u_k := \lfloor \frac{ka}{b} \rfloor$ and $v_k := \lfloor \frac{-ka}{b} \rfloor$. The ratio $\frac{ka}{b}$ is not in \mathbb{Z} and therefore, as a property of the floor function¹, we have $u_k + v_k = -1$:

¹ $\forall x \in \mathbb{R}, \quad \lfloor x \rfloor + \lfloor -x \rfloor = \begin{cases} 0, & \text{if } x \in \mathbb{Z}, \\ -1, & \text{if } x \notin \mathbb{Z}. \end{cases}$

$$\begin{aligned}
\left\lfloor \frac{ka}{b} \right\rfloor^r - \left\lfloor \frac{(b-k)a}{b} \right\rfloor^r &= u_k^r - (a + v_k)^r \\
&= u_k^r - (a - 1 - u_k)^r \\
&= u_k^r - \sum_{i=0}^r \binom{r}{i} (-u_k)^i (a-1)^{r-i} \\
&= u_k^r + \sum_{i=0}^r (-1)^{i+1} \binom{r}{i} (a-1)^{r-i} u_k^i.
\end{aligned}$$

We can take the sum of the last equation for k from 1 to $b-1$. The left-hand side sums to zero as the difference of two equal sums (the two sums index are in reverse order). On the right-hand side we recognize the sums $\mathbf{S}_i(b, a)$ for i from 0 to r . \square

Proposition 1.2. *Let m, n and r be positive integers. Let $d = \gcd(n, m)$ such that there exist two positive coprime integers b and a such that $n = db$ and $m = da$. We have*

$$\mathbf{S}_r(n, m) = a^r \mathbf{F}_r(d-1) + \sum_{k=0}^r \binom{r}{k} a^k \mathbf{F}_k(d-1) \mathbf{S}_{r-k}(b, a). \quad (7)$$

Proof.

$$\begin{aligned}
\mathbf{S}_r(n, m) &= \sum_{k=0}^{n-1} \left\lfloor \frac{ka}{b} \right\rfloor^r \\
&= \sum_{i=0}^{d-1} \sum_{j=0}^{b-1} \left\lfloor \frac{(ib+j)a}{b} \right\rfloor^r \\
&= \sum_{i=0}^{d-1} \sum_{j=0}^{b-1} \sum_{k=0}^r \binom{r}{k} i^k a^k \left\lfloor \frac{ja}{b} \right\rfloor^{r-k}.
\end{aligned}$$

The case $k = r$ needs attention since $\sum_{j=0}^{b-1} \left\lfloor \frac{ja}{b} \right\rfloor^{r-r} = 1 + \mathbf{S}_0(b, a)$:

$$\begin{aligned}
\mathbf{S}_r(n, m) &= a^r \mathbf{F}_r(d-1) (1 + \mathbf{S}_0(b, a)) + \sum_{k=0}^{r-1} \binom{r}{k} a^k \mathbf{F}_k(d-1) \mathbf{S}_{r-k}(b, a) \\
&= a^r \mathbf{F}_r(d-1) + \sum_{k=0}^r \binom{r}{k} a^k \mathbf{F}_k(d-1) \mathbf{S}_{r-k}(b, a). \quad \square
\end{aligned}$$

Equation (7) shows that in general (whether m and n are coprime or not) the formula of $\mathbf{S}_r(n, m)$ only depends on $\mathbf{S}_i(b, a)$ for $0 \leq i \leq r$ and some known Faulhaber sums. Therefore it is enough to focus on studying $\mathbf{S}_r(b, a)$ with b and a positive coprime integers.

2 Formulas for \mathbf{S}_i for $0 \leq i \leq 3$

2.1 A formula for $\mathbf{S}_1(n, m)$

A formula for $\mathbf{S}_1(n, m)$ is provided and proved in [3] and also in [2, p. 94]. If m and n are positive integers and if $d = \gcd(m, n)$, then

$$\mathbf{S}_1(n, m) = \frac{(m-1)(n-1)}{2} + \frac{d-1}{2}. \quad (8)$$

We give here a first² alternative proof of Equation (8). Let a and b be the positive coprime integers defined by $m = da$ and $n = db$. From Proposition 1.1 with $r = 1$ we have

$$2\mathbf{S}_1(b, a) - (a - 1)\mathbf{S}_0(b, a) = 0.$$

It is clear that

$$\mathbf{S}_0(b, a) = b - 1 \tag{9}$$

and therefore

$$\mathbf{S}_1(b, a) = \frac{(a - 1)(b - 1)}{2}. \tag{10}$$

We write now Equation (7) from Proposition 1.1 with $r = 1$:

$$\mathbf{S}_1(n, m) = a\mathbf{F}_1(d - 1) + \mathbf{F}_0(d - 1)\mathbf{S}_1(b, a) + a\mathbf{F}_1(d - 1)\mathbf{S}_0(b, a).$$

This equation gives Equation (8) knowing that $\mathbf{F}_0(d - 1) = d$ and $\mathbf{F}_1(d - 1) = \frac{(d - 1)d}{2}$.

2.2 A formula for $\mathbf{S}_2(n, m)$

In this section we carry out a direct calculation of $\mathbf{S}_2(n, m)$ and establish an equation involving another sum of interest that will be studied separately. We have the positive integers m, n, a, b, d and k such that $m = da, n = db$ and a and b are coprime. We write now Equation (7) from Proposition 1.2 with $r = 2$:

$$\begin{aligned} \mathbf{S}_2(n, m) &= a^2\mathbf{F}_2(d - 1) + \\ &\quad \mathbf{F}_0(d - 1)\mathbf{S}_2(b, a) + 2a\mathbf{F}_1(d - 1)\mathbf{S}_1(b, a) + a^2\mathbf{F}_2(d - 1)\mathbf{S}_0(b, a). \end{aligned}$$

Since all is known apart from $\mathbf{S}_2(b, a)$, we have

$$\mathbf{S}_2(n, m) = \frac{(d - 1)m}{6} ((2d - 1)ab + 3(a - 1)(b - 1)) + d\mathbf{S}_2(b, a).$$

Since we have

$$\left[\frac{ka}{b} \right] = \frac{ka}{b} - \left\{ \frac{ka}{b} \right\}, \tag{11}$$

summing the square of Equation (11) for k from 1 to $b - 1$ leads to

$$\begin{aligned} \mathbf{S}_2(b, a) &= \sum_{k=1}^{b-1} \left(\frac{ka}{b} \right)^2 + \sum_{k=1}^{b-1} \left\{ \frac{ka}{b} \right\}^2 - 2 \sum_{k=1}^{b-1} \frac{ka}{b} \left\{ \frac{ka}{b} \right\} \\ &= \frac{1 + a^2}{b^2} \mathbf{F}_2(b - 1) - 2 \frac{a}{b^2} \sum_{k=1}^{b-1} k(ka \bmod b). \end{aligned}$$

Let us define the function $\mathbf{W}_n(a, b)$ by the following sum:

$$\mathbf{W}_n(a, b) := \sum_{k=1}^{n-1} (ak \bmod n)(bk \bmod n). \tag{12}$$

² A second proof is given in Section 5.

We also give a sense to this function when at least one of its arguments is equal to one through this definition

$$\mathbf{W}_n(a) := \mathbf{W}_n(a, 1). \quad (13)$$

With this definition we get to

$$\mathbf{S}_2(b, a) = \frac{(1 + a^2)(b - 1)(2b - 1)}{6b} - 2\frac{a}{b^2}\mathbf{W}_b(a), \quad (14)$$

and therefore

$$\mathbf{S}_2(n, m) = \frac{d}{6b} \left((b - 1)(2b - 1) + a^2(n - 1)(2n - 1) \right) - \frac{m}{2}(d - 1)(b - 1) - 2\frac{m}{b^2}\mathbf{W}_b(a). \quad (15)$$

This formula together with Equation (22) from Section 3.3 provides a closed-form formula for $\mathbf{S}_2(n, m)$.

2.3 A formula for $\mathbf{S}_3(n, m)$

Let us write Equation (6) from Proposition 1.1 with $r = 3$:

$$2\mathbf{S}_3(b, a) - (a - 1)^3\mathbf{S}_0(b, a) + 3(a - 1)^2\mathbf{S}_1(b, a) - 3(a - 1)\mathbf{S}_2(b, a) = 0.$$

By means of Equations (9), (10) and (14), the above equation gives:

$$\mathbf{S}_3(b, a) = \frac{1}{4b}(b - 1)(a - 1) \left((b - 1)(1 + a^2) + 2ab \right) - \frac{3}{b^2}a(a - 1)\mathbf{W}_b(a). \quad (16)$$

We now write Equation (7) from Proposition 1.2 with $r = 3$. There is

$$\begin{aligned} \mathbf{S}_3(n, m) &= a^3\mathbf{F}_3(d - 1) + \mathbf{F}_0(d - 1)\mathbf{S}_3(b, a) + \\ &\quad 3a\mathbf{F}_1(d - 1)\mathbf{S}_2(b, a) + 3a^2\mathbf{F}_2(d - 1)\mathbf{S}_1(b, a) + a^3\mathbf{F}_3(d - 1)\mathbf{S}_0(b, a). \end{aligned}$$

Using known formulas for Faulhaber sums, as well as Equations (9), (10), (14) and (16), we get to

$$\begin{aligned} \mathbf{S}_3(n, m) &= \frac{1}{4}(d - 1)am \left((d - 1)bm + (2d - 1)(a - 1)(b - 1) \right) \\ &\quad + \frac{1}{4b}d(1 + a^2)(b - 1) \left((d - 1)a(2b - 1) + (b - 1)(a - 1) \right) \\ &\quad + \frac{1}{2}m(b - 1)(a - 1) - \frac{3}{b^2}m(m - 1)\mathbf{W}_b(a). \quad (17) \end{aligned}$$

This formula together with Equation (22) from Section 3.3 provides a closed-form formula for $\mathbf{S}_3(n, m)$.

3 Analysis of \mathbf{W}

The objective of this section is to give a closed-form formula for $\mathbf{W}_n(m)$ in order to finalize the calculation of $\mathbf{S}_2(n, m)$ in Equation (15) and of $\mathbf{S}_3(n, m)$ in Equation (17). The analysis of \mathbf{W} provides an elementary proof³ of the simplest form of Dedekind's reciprocity law.

³ Although it is not fundamentally a new proof, one can see it in Section 4.1.

3.1 Basic properties of \mathbf{W}

In the previous section we introduced the function \mathbf{W} in Equations (12) and (13). The objective of this section is to provide some of its properties.

Proposition 3.1. *Let n, a, b and c be any positive integers, then we have the following:*

- (i) $\mathbf{W}_n(a, b) = \mathbf{W}_n(a \bmod n, b \bmod n)$,
- (ii) If $\gcd(c, n) = 1$, we have $\mathbf{W}_n(ac, bc) = \mathbf{W}_n(a, b)$,
- (iii) If $ab \bmod n = 1$, we have $\mathbf{W}_n(a) = \mathbf{W}_n(b)$,
- (iv) $\mathbf{W}_n(a) + \mathbf{W}_n(n - a) = \frac{1}{2}n^2(n - 1)$.

Proof. (i) comes from $ak \bmod n = ((a \bmod n)k) \bmod n$.

(ii) Whenever the positive integer c is coprime with n , the application $x \mapsto cx \bmod n$ is a bijection of $\{1, \dots, n - 1\}$. In that case, we have $\mathbf{W}_n(ac, bc) = \mathbf{W}_n(a, b)$.

(iii)

$$\begin{aligned} \mathbf{W}_n(a) &= \sum_{k=1}^{n-1} (bk \bmod n)(a(bk \bmod n) \bmod n) \\ &= \sum_{k=1}^{n-1} (bk \bmod n)((ab \bmod n)k \bmod n) \\ &= \mathbf{W}_n(b). \end{aligned}$$

For the second equality we use the fact that $ab \bmod n = 1$ implies that $\gcd(b, n) = 1$ and therefore $k \mapsto bk \bmod n$ is a bijection of A_n .

(iv)

$$\mathbf{W}_n(a) + \mathbf{W}_n(n - a) = \sum_{k=1}^{n-1} k \{(ak \bmod n) + (-ak \bmod n)\}. \quad \square$$

Proposition 3.2. *Let $d = \gcd(m, n)$, where $m = da$ and $n = db$ with a and b coprime. We have the following equation*

$$\mathbf{W}_n(m) = d^2 \mathbf{W}_b(a) + \frac{1}{4}n^2(d - 1)(b - 1). \quad (18)$$

Proof.

$$\begin{aligned} \mathbf{W}_n(m) &= \sum_{k=1}^{db-1} k(dak \bmod db) \\ &= d \sum_{j=0}^{d-1} \sum_{k=0}^{b-1} (k + jb)(a(k + jb) \bmod b) \\ &= d^2 \mathbf{W}_b(a) + db \sum_{j=0}^{d-1} j \sum_{k=0}^{b-1} (ak \bmod b) \\ &= d^2 \mathbf{W}_b(a) + \frac{1}{4}n^2(d - 1)(b - 1). \quad \square \end{aligned}$$

Note that when m and n are coprime, then $d = 1$ and the equation is obviously satisfied. We can now focus on calculating $\mathbf{W}_b(a)$ when b and a are coprime.

3.2 Calculation of $W_b(a)$ when a and b are coprime

Given that from property (ii) $W_b(a) = W_b(a \bmod b)$ and that $\gcd(a, b) = 1$ implies $\gcd(a \bmod b, b) = 1$, we can work under the assumption that $0 < a < b$ even if it means considering $a \bmod b$ instead of a . According to Definition (12),

$$W_b(a) = \sum_{k=1}^{b-1} k(ak \bmod b).$$

Note that the term $k(ak \bmod b)$ inside the sum is zero for $k = b$. In particular, we can write $W_b(a)$ in a slightly different way:

$$W_b(a) = \sum_{j=0}^{a-1} \sum_{k=\lfloor \frac{jb}{a} \rfloor + 1}^{\lfloor \frac{(j+1)b}{a} \rfloor} k(ak \bmod b).$$

The integer variable k of the inner sum satisfies

$$\frac{jb}{a} < \left\lfloor \frac{jb}{a} \right\rfloor + 1 \leq k \leq \left\lfloor \frac{(j+1)b}{a} \right\rfloor \leq \frac{(j+1)b}{a},$$

which implies

$$0 < ka - jb \leq b.$$

It should be noted that the right-hand side inequality is always a strict inequality apart from the case when $j = a - 1$ and $k = b$. Indeed, when $0 \leq j < a - 1$, the ratio $\frac{(j+1)b}{a}$ is never an integer. If that was the case, knowing that a and b are coprime, the Gauss lemma would imply that a divides $j + 1$ which is not possible since $0 < j + 1 < a$. That means that apart from the case $j = a - 1$ and $k = b$ we have

$$ka \bmod b = ka - jb.$$

Now we can write

$$W_b(a) = \left(\sum_{j=0}^{a-1} \sum_{k=\lfloor \frac{jb}{a} \rfloor + 1}^{\lfloor \frac{(j+1)b}{a} \rfloor} k(ka - jb) \right) - b^2.$$

Note that when $j = a - 1$ and $k = b$, the expressions $k(ka - jb) = b^2$ and $k(ka \bmod b) = 0$ are not equal, the reason why we need to subtract b^2 .

The first part of the sum is easily simplified

$$\begin{aligned} \sum_{j=0}^{a-1} \sum_{k=\lfloor \frac{jb}{a} \rfloor + 1}^{\lfloor \frac{(j+1)b}{a} \rfloor} ak^2 &= a \sum_{k=1}^b k^2 \\ &= a \frac{b(b+1)(2b+1)}{6}. \end{aligned}$$

We are now left with the calculation of the second term of the sum:

$$A := -b \sum_{j=0}^{a-1} j \sum_{k=\lfloor \frac{jb}{a} \rfloor + 1}^{\lfloor \frac{(j+1)b}{a} \rfloor} k.$$

We have

$$\begin{aligned} A &= -\frac{b}{2} \sum_{j=0}^{a-1} j \left(\left\lfloor \frac{(j+1)b}{a} \right\rfloor \left(\left\lfloor \frac{(j+1)b}{a} \right\rfloor + 1 \right) - \left\lfloor \frac{jb}{a} \right\rfloor \left(\left\lfloor \frac{jb}{a} \right\rfloor + 1 \right) \right) \\ &= -\frac{b}{2} \sum_{j=1}^a (j-1) \left\lfloor \frac{jb}{a} \right\rfloor \left(\left\lfloor \frac{jb}{a} \right\rfloor + 1 \right) + \frac{b}{2} \sum_{j=0}^{a-1} j \left\lfloor \frac{jb}{a} \right\rfloor \left(\left\lfloor \frac{jb}{a} \right\rfloor + 1 \right) \\ &= \frac{b}{2} \mathbf{S}_2(a, b) + \frac{b}{2} \mathbf{S}_1(a, b) - \frac{b}{2} (a-1)b(b+1). \end{aligned}$$

Using Equation (10), after some simplifications we get to

$$\mathbf{W}_b(a) = \frac{b}{2} \mathbf{S}_2(a, b) + \frac{b}{12} (b-1)(2b-1)(3-a). \quad (19)$$

3.3 Formula for $\mathbf{W}_a(b)$ using the Euclidean algorithm

The purpose of this section is to provide a closed-form formula for $\mathbf{W}_a(b)$ for two positive coprime integers a and b with $a < b$. From Equation (24) and using property (ii) we can write

$$\mathbf{W}_a(b) = f(a, b) - \left(\frac{a}{b}\right)^2 \mathbf{W}_b(a \bmod b) \quad (20)$$

with f being the following function

$$f(x, y) := \frac{x}{12y} \left((1+x^2)(1+y^2) - xy(x-3)(y-3) \right). \quad (21)$$

Let $(u_n)_{n \in \mathbb{N}}$ be the sequence defined by the first two terms, $u_0 = a$, $u_1 = b$, and the following induction equation $u_{i+2} = u_i \bmod u_{i+1}$ for $i \geq 0$. This sequence is the sequence of remainders of Euclid's algorithm (see [1]). We know that $(u_n)_{n \in \mathbb{N}}$ is strictly decreasing until it reaches $u_N = 1 = \gcd(a, b)$ for a specific index $N \geq 1$. Then for any $i > N$ we have $u_i = 0$. For i from 0 to $N-1$ we have $\gcd(u_i, u_{i+1}) = 1$ and we can write N times Equation (20) for $\mathbf{W}_{u_i}(u_{i+1})$. Compounding these N equations leads to

$$\mathbf{W}_{u_0}(u_1) = \left(\sum_{k=0}^{N-1} (-1)^k \left(\frac{u_0}{u_k} \right)^2 f(u_k, u_{k+1}) \right) + (-1)^N \left(\frac{u_0}{u_N} \right)^2 \mathbf{W}_{u_N}(u_{N+1}).$$

Given that $u_{N+1} = 0$, we have $\mathbf{W}_{u_N}(u_{N+1}) = 0$ and we are left with

$$\mathbf{W}_a(b) = \frac{a^2}{12} \sum_{k=0}^{N-1} (-1)^k \left(\frac{(1+u_k^2)(1+u_{k+1}^2)}{u_k u_{k+1}} - (u_k-3)(u_{k+1}-3) \right). \quad (22)$$

As a consequence, we have closed-form formulas⁴ for $\mathbf{S}_2(b, a)$, $\mathbf{S}_2(n, m)$, $\mathbf{S}_3(b, a)$, $\mathbf{S}_3(n, m)$, $\mathbf{W}_n(m)$ and the classical Dedekind sum $s(b, a)$ respectively from Equations (14), (15), (16), (17), (18) and (26).

⁴ Don Zagier in [6, p. 166] had already noticed that the classical Dedekind sum was fully determined from their properties and the use of the Euclidean algorithm.

4 Reciprocity laws

With positive and coprime integers a and b , the consideration of Equations (19) and (14) yields easily to the following symmetrical equations that could be considered as reciprocity laws:

Theorem 4.1 (Reciprocity law for \mathbf{S}_2). *If a and b are positive coprime integers, then*

$$a\mathbf{S}_2(a, b) + b\mathbf{S}_2(b, a) = \frac{1}{6}(a-1)(2a-1)(b-1)(2b-1). \quad (23)$$

Proof. In Equation (14), we replace $\mathbf{W}_b(a)$ by its expression from Equation (19). \square

Theorem 4.2 (Reciprocity law for \mathbf{W}). *If a and b are positive coprime integers, then*

$$a^2\mathbf{W}_b(a) + b^2\mathbf{W}_a(b) = \frac{ab}{12}((1+a^2)(1+b^2) - ab(a-3)(b-3)). \quad (24)$$

Proof. In Equation (14), we swap a and b and inject the expression of $\mathbf{S}_2(a, b)$ in Equation (19). \square

Theorem 4.3 (Reciprocity law for \mathbf{S}_3). *If a and b are positive coprime integers, then*

$$a(a-1)\mathbf{S}_3(a, b) + b(b-1)\mathbf{S}_3(b, a) = \frac{1}{4}(a-1)^2(b-1)^2((a-1)(b-1) + ab). \quad (25)$$

Proof. In Equation (6) for $r = 3$, we replace \mathbf{S}_0 and \mathbf{S}_1 according to their formulas in Equations (9) and (10) and get an Equation between \mathbf{S}_2 and \mathbf{S}_3 . With this equation and the reciprocity law for \mathbf{S}_2 in (23) we easily get Equation (25). \square

4.1 A proof of Dedekind's reciprocity law

The proof of Dedekind's reciprocity law⁵ that we give here is in essence the same as the one given in [6, p. 153] although it is presented differently.

In the definition Equation (3) of the classical Dedekind sum, the summand for $k = a$ is equal to 0. For $0 < k < a$ both $\frac{kb}{a}$ and $\frac{k}{a}$ are not in \mathbb{Z} :

$$\begin{aligned} s(b, a) &= \sum_{k=1}^{a-1} \left(\frac{kb \bmod a}{a} - \frac{1}{2} \right) \left(\frac{k}{a} - \frac{1}{2} \right) \\ &= \frac{1}{a^2} \mathbf{W}_a(b) - \frac{1}{4}(a-1), \end{aligned}$$

that is

$$\mathbf{W}_a(b) = a^2 \left(s(b, a) + \frac{a-1}{4} \right). \quad (26)$$

This reciprocity law satisfied by the classical Dedekind sum results from the reciprocity law satisfied by \mathbf{W} (Equation (24)) and the relation between \mathbf{W} and the classical Dedekind sum (Equation (26)). Combining these two equations yields:

$$\begin{aligned} a^2b^2 \left(s(b, a) + \frac{a-1}{4} \right) + a^2b^2 \left(s(a, b) + \frac{b-1}{4} \right) = \\ \frac{ab}{12}((1+a^2)(1+b^2) - ab(a-3)(b-3)), \end{aligned}$$

⁵ Not to be mistaken with the Quadratic reciprocity law.

from where we get the reciprocity law for Dedekind sums:

$$s(b, a) + s(a, b) = -\frac{1}{4} + \frac{1}{12} \left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a} \right).$$

4.2 A formula for the classical Dedekind sum

From Equation (26) we have

$$s(b, a) = \frac{\mathbf{W}_a(b)}{a^2} - \frac{a-1}{4}.$$

This equation together with Equation (22) gives a closed-form formula for $s(b, a)$ as a function of the remainders obtained with the Euclidean algorithm applied to $u_0 = b$ and $u_1 = a$ (see Section 3.3).

5 Expression of $\mathbf{S}_r(b, a)$ as a function of generalized Dedekind sums

Proposition 5.1. *For positive coprime integers a and b and for a positive integer r we have the following expression for $\mathbf{S}_r(b, a)$*

$$\mathbf{S}_r(b, a) = \frac{b}{2^r} \sum_{\substack{u+v+w=r \\ u, v, w \geq 0}} \binom{r}{u, v, w} \left(\frac{a}{b}\right)^u \left(\frac{-1}{b}\right)^v (a-1)^w \delta(b; \underbrace{1, \dots, 1}_{u \text{ times}}, \underbrace{a, \dots, a}_{v \text{ times}}). \quad (27)$$

Proof. We transform $\mathbf{S}_r(b, a)$ using the trinomial expansion and recognize generalized Dedekind sums (Equation (5)) in the expression:

$$\begin{aligned} \mathbf{S}_r(b, a) &= \sum_{k=1}^{b-1} \left(\frac{ka}{b} - \frac{a}{2} - \left(\left\{ \frac{ka}{b} \right\} - \frac{1}{2} \right) + \frac{a-1}{2} \right)^r \\ &= \sum_{k=1}^{b-1} \sum_{\substack{u+v+w=r \\ u, v, w \geq 0}} \binom{r}{u, v, w} a^u \left(\frac{k}{b} - \frac{1}{2} \right)^u (-1)^v \left(\left\{ \frac{ka}{b} \right\} - \frac{1}{2} \right)^v \left(\frac{a-1}{2} \right)^w \\ &= \sum_{\substack{u+v+w=r \\ u, v, w \geq 0}} \binom{r}{u, v, w} a^u (-1)^v \left(\frac{a-1}{2} \right)^w \frac{2^{u+v} b^{u+v-1}}{2^{u+v} b^{u+v-1}} \sum_{k=1}^{b-1} \left(\frac{k}{b} \right)^u \left(\frac{ka}{b} \right)^v. \quad \square \end{aligned}$$

5.1 Application of Equation (27)

In this section, we use Equation (27) to prove Equations (10), (14) and (16). In the following proofs, we use the fact that the generalized Dedekind sum (5) is zero when n is odd. The sum in (27) is on all nonnegative integers u, v and w such that $u + v + w = r$. The previous argument means that we can discard the triplets (u, v, w) where $u + v$ is even since they contribute to zero to the formula because of the factor $\delta(b; \underbrace{1, \dots, 1}_{u \text{ times}}, \underbrace{a, \dots, a}_{v \text{ times}})$ which is equal to zero in that case.

5.1.1 Another proof of Equation (10)

To calculate $S_1(b, a)$ with (27) the only possibility is $(u, v, w) = (0, 0, 1)$ hence

$$S_1(b, a) = \frac{b}{2}(a-1)\delta(b; \emptyset).$$

But

$$\begin{aligned} \delta(b; \emptyset) &= 2^0 b^{-1} \sum_{k=1}^{b-1} \prod_{a \in \emptyset} \left(\binom{ka}{b} \right) \\ &= \frac{(b-1)}{b}. \end{aligned} \quad (28)$$

Hence

$$S_1(b, a) = \frac{(a-1)(b-1)}{2},$$

which is Equation (10).

5.1.2 Another proof of Equation (14)

To calculate $S_2(b, a)$ with (27) the only possibilities for (u, v, w) are $(0, 0, 2)$, $(2, 0, 0)$, $(1, 1, 0)$, and $(0, 2, 0)$, therefore,

$$S_2(b, a) = \frac{(a-1)^2 b}{4} \delta(b; \emptyset) + \frac{a^2}{4b} \delta(b; 1, 1) - \frac{a}{2b} \delta(b; 1, a) + \frac{1}{4b} \delta(b; a, a). \quad (29)$$

From Equations (9) in [6, p. 151], we have

$$\delta(b; a, a) = \delta(b; 1, 1). \quad (30)$$

A straightforward calculation leads to

$$\delta(b; 1, 1) = \frac{1}{3}(b-1)(b-2). \quad (31)$$

We now calculate $\delta(b; 1, a)$ by means of Equation (26).

$$\begin{aligned} \delta(b; 1, a) &= 4bs(a, b) \\ &= \frac{4}{b} \mathbf{W}_b(a) - b(b-1). \end{aligned} \quad (32)$$

Finally, we get Equation (14) after using Equations (28), (30), (31) and (32) in Equation (29).

5.1.3 Another proof of Equation (16)

Similarly, in order to calculate $S_3(b, a)$ with (27) the only possibilities for (u, v, w) are $(0, 0, 3)$, $(2, 0, 1)$, $(1, 1, 1)$, and $(0, 2, 1)$, which leads to

$$\begin{aligned} S_3(b, a) &= \frac{(a-1)^3 b}{8} \delta(b; \emptyset) + \frac{3a^2(a-1)}{8b} \delta(b; 1, 1) \\ &\quad - \frac{3a(a-1)}{4b} \delta(b; 1, a) + \frac{3(a-1)}{8b} \delta(b; a, a). \end{aligned} \quad (33)$$

We get Equation (16) after using Equations (28), (30), (31) and (32) in Equation (33).

6 Conclusion

In this paper, we have given closed-form formulas for S_2 , S_3 , W and the classical Dedekind sums. In addition, we have shown the reciprocity laws that these expressions satisfy. In the last section, we have shown how S_r depend on generalized Dedekind sums through Equation (27). For $r \geq 4$ there is more than one Dedekind sum involved in the formula of S_r making the analysis more difficult than it is for S_2 and S_3 where only one Dedekind sum is involved, however that could probably be investigated further.

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