

Explicit evaluation of some families of log-sine integrals via the hypergeometric mechanism and their applications

Shakir Hussain Malik¹  and Mohammad Idris Qureshi² 

¹ Department of Humanities and Basic Sciences, Sri Indu College of Engineering and Technology (An Autonomous Institution under UGC, New Delhi) Sheriguda (V), Ibrahimpatnam, Ranga Reddy District, Hyderabad, Telangana, 501510, India
e-mail: malikshakir774@gmail.com

² Department of Applied Sciences and Humanities, Faculty of Engineering and Technology, Jamia Millia Islamia (A Central University) New Delhi, 110025, India
e-mail: miqureshi_delhi@yahoo.co.in

Received: 29 October 2025

Revised: 23 November 2025

Accepted: 10 December 2025

Online First: 19 February 2026

Abstract: In this paper, we present explicit analytical expressions for certain families of log-sine definite integrals: $\int_0^{2\pi} x^m \{\ln(2 \sin \frac{x}{2})\}^n dx$ ($m \in \mathbb{N}_0, n \in \mathbb{N}$), expressed in terms of multiple hypergeometric functions of the Kampé de Fériet with arguments ± 1 and the Riemann zeta functions. As applications, we establish several mixed summation formulas (79), (81) and (83) involving the generalized hypergeometric functions ${}_3F_2(1)$, ${}_5F_4(1)$ and ${}_7F_6(1)$. Moreover, a collection of possibly new summation formulas (42), (52), (54), (56), (58), (62), (64), (66), (70), (72), (74) and (76) for multiple hypergeometric functions of the Kampé de Fériet are derived. In addition, mixed relations (80), (82) and (84) involving the Riemann zeta functions are also established.

Keywords: Log-sine integrals, Generalized hypergeometric functions, Riemann zeta function, Kampé de Fériet functions.

2020 Mathematics Subject Classification: 33C05, 33C20, 11M06, 26A42.



1 Introduction and preliminaries

The main aim of this work is to provide analytical expressions for the following families of log-sine integrals:

- $\int_0^{2\pi} \{\ln(2 \sin \frac{x}{2})\}^n dx; n \in \{1, 2, 3, 4, 5, 6, 7\}$
- $\int_0^{2\pi} x \{\ln(2 \sin \frac{x}{2})\}^n dx; n \in \{1, 2, 3, 4, 5, 6\}$
- $\int_0^{2\pi} x^2 \{\ln(2 \sin \frac{x}{2})\}^n dx; n \in \{1, 2, 3, 4, 5\}$
- $\int_0^{2\pi} x^3 \{\ln(2 \sin \frac{x}{2})\}^n dx; n \in \{1, 2, 3, 4\}$
- $\int_0^{2\pi} x^4 \{\ln(2 \sin \frac{x}{2})\}^n dx; n \in \{1, 2, 3\}$
- $\int_0^{2\pi} x^5 \{\ln(2 \sin \frac{x}{2})\}^n dx; n \in \{1, 2\}$
- $\int_0^{2\pi} x^6 \{\ln(2 \sin \frac{x}{2})\}^n dx; n = \{1\}$

in terms of multiple hypergeometric functions of Kampé de Fériet and the Riemann zeta functions.

For the definitions of the Pochhammer symbol and the generalized hypergeometric series (or function) ${}_pF_q(p, q \in \mathbb{N}_0)$ —the latter being a natural generalization of the classical Gaussian hypergeometric series ${}_2F_1$ —as well as for their convergence conditions, we refer to the standard monographs (see, [17, 19, 20]).

The generalized hypergeometric function plays a fundamental role in various branches of mathematical analysis and applied sciences. It provides a unifying framework for expressing a wide class of special functions and definite integrals. In particular, numerous integral representations and transformations formulas can be elegantly expressed using these functions, greatly simplifying their analytical treatment.

In the subsequent sections, we employ these standard definitions and notations to derive new results involving log-sine integrals and related hypergeometric expressions.

Here we remind the reader of the multiple hypergeometric functions of Kampé de Fériet. Let $(a_A) = a_1, a_2, \dots, a_A$ and $\left(b_{B^{(i)}}^{(i)}\right) = b_{B^{(1)}}^{(1)}, b_{B^{(2)}}^{(2)}, b_{B^{(3)}}^{(3)}, \dots, b_{B^{(i)}}^{(i)}$. We consider the corresponding problem concerning the convergence of the multiple hypergeometric series [18]:

$$\begin{aligned}
 & F_{C: D^{(1)}; \dots; D^{(n)}}^{A: B^{(1)}; \dots; B^{(n)}} \left[\begin{array}{c} (a_A) : \left(b_{B^{(1)}}^{(1)}\right); \dots; \left(b_{B^{(n)}}^{(n)}\right); \\ (c_C) : \left(d_{D^{(1)}}^{(1)}\right); \dots; \left(d_{D^{(n)}}^{(n)}\right); \end{array} \quad x_1, \dots, x_n \right] \\
 &= \sum_{m_1, m_2, \dots, m_n=0}^{\infty} \frac{\prod_{j=1}^A (a_j)_{m_1+m_2+\dots+m_n} \prod_{j=1}^{B^{(1)}} \left(b_j^{(1)}\right)_{m_1} \cdots \prod_{j=1}^{B^{(n)}} \left(b_j^{(n)}\right)_{m_n}}{\prod_{j=1}^C (c_j)_{m_1+m_2+\dots+m_n} \prod_{j=1}^{D^{(1)}} \left(d_j^{(1)}\right)_{m_1} \cdots \prod_{j=1}^{D^{(n)}} \left(d_j^{(n)}\right)_{m_n}} \prod_{i=1}^n \frac{x_i^{m_i}}{m_i!}, \quad (1)
 \end{aligned}$$

which unifies and extends the four Lauricella series $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$ in n variables. In fact, as already observed in the literature [19, pp. 37–38], the multiple hypergeometric series (1) is a special case of the generalized Lauricella series in several variables, which was introduced by Srivastava and Daoust in 1969.

Let us suppose $\Delta_k \equiv 1 + C + D^{(k)} - A - B^{(k)}$ that for every $k \in \{1, \dots, n\}$.

For the convergence [21, p. 1127, Eq. (4.3)–Eq.(4.5)] (see also [22]) of the multiple hypergeometric series (1), we have

$$(i) \quad \Delta_k > 0; \quad |x_1| < \infty, \quad |x_2| < \infty, \dots, \quad |x_n| < \infty, \quad (2)$$

$$(ii) \quad \Delta_k = 0; \quad A > C \quad \text{and} \quad |x_1|^{\frac{1}{(A-C)}} + \dots + |x_n|^{\frac{1}{(A-C)}} < 1, \quad (3)$$

$$(iii) \quad \Delta_k = 0; \quad A \leq C \quad \text{and} \quad \max\{|x_1|, \dots, |x_n|\} < 1. \quad (4)$$

In order to get the clear idea about the absolutely and conditionally convergence of (1), we summarize some results [12, pp. 113–114, Th. (4)–Th.(6)] as follows:

Remark 1.1. Let $\Delta_k = 0$ ($k = 1, \dots, n$), $A = C$ and $|x_1| = \dots = |x_n| = 1$. Then the series (1)

(i) converges absolutely if and only if

$$\rho_k = \operatorname{Re} \left(\sum_{j=1}^A a_j + \sum_{j=1}^{B^{(k)}} b_j^{(k)} - \sum_{j=1}^C c_j - \sum_{j=1}^{D^{(k)}} d_j^{(k)} \right) < 0; \quad (k = 1, \dots, n) \quad (5)$$

and

$$\sigma = \operatorname{Re} \left\{ \sum_{j=1}^A a_j + \sum_{k=1}^n \left(\sum_{j=1}^{B^{(k)}} b_j^{(k)} \right) - \sum_{j=1}^C c_j - \sum_{k=1}^n \left(\sum_{j=1}^{D^{(k)}} d_j^{(k)} \right) \right\} < 0; \quad (6)$$

(ii) converges conditionally when $x_k \neq 1$ ($k = 1, \dots, n$), if

$$\rho_k < 1 \quad (k = 1, \dots, n) \quad \text{and} \quad \sigma < n;$$

(iii) diverges if at least one of the following $n + 1$ conditions does not hold true:

$$\rho_k \leq 1 \quad (k = 1, \dots, n) \quad \text{and} \quad \sigma < n.$$

Remark 1.2. Let $\Delta_k = 0$ ($k = 1, \dots, n$), $A < C$ and $|x_1| = \dots = |x_n| = 1$. Then the series (1)

(i) converges absolutely if and only if $\rho_k < 0$ ($k = 1, \dots, n$),

(ii) converges conditionally when $x_k \neq 1$ ($k = 1, \dots, n$), if

$$\rho_k < 1 \quad (k = 1, \dots, n),$$

ρ_k being defined by (5).

Remark 1.3. Let $\Delta_k = 0$ ($k = 1, \dots, n$) and $A > C$. Then the series (1), converges absolutely when

$$|x_1|^{\frac{1}{(A-C)}} + \dots + |x_n|^{\frac{1}{(A-C)}} = 1 \quad (x_k \neq 0; k = 1, \dots, n),$$

if $\sigma + A - C < 1$, where σ is defined by (6).

For convenience, we adopt the following standard notations and definitions throughout this paper: $\mathbb{N} := \{1, 2, 3, \dots\}$, $\mathbb{N}_0 := \mathbb{N} \cup 0$, $\mathbb{Z}_0^- := \mathbb{Z}^- \cup 0 = \{0, -1, -2, -3, \dots\}$. The symbols $\mathbb{C}, \mathbb{R}, \mathbb{N}, \mathbb{Z}, \mathbb{R}^+$ and \mathbb{R}^- denote the sets of complex numbers, real numbers, natural numbers, integers, positive real number and negative real numbers, respectively.

2 Relations in consideration

The following relations will be used throughout the paper:

$$\ln(1 - e^{ix}) = \ln\left(2\left|\sin\frac{x}{2}\right|\right) + i\left(\frac{x}{2} - \frac{\pi}{2}\right), x \in \mathbb{R}. \quad (7)$$

$$\ln(1 - z) = -z {}_2F_1(1, 1; 2; z), |z| < 1. \quad (8)$$

$$\ln(2) = {}_2F_1(1, 1; 2; -1). \quad (9)$$

A few evaluated log-sine integrals taken from Choi and Srivastava [6, pp. 771–778] are given below:

$$\int_0^{2\pi} x \{\ln(2 \sin \frac{x}{2})\}^3 dx = -3\pi^2 \zeta(3). \quad (10)$$

$$\int_0^{2\pi} x^2 \{\ln(2 \sin \frac{x}{2})\}^2 dx = \frac{13\pi^5}{45}. \quad (11)$$

$$\int_0^{2\pi} x^2 \{\ln(2 \sin \frac{x}{2})\}^3 dx = -12\pi \zeta(5) - 5\pi^3 \zeta(3). \quad (12)$$

$$\int_0^{2\pi} x^2 \{\ln(2 \sin \frac{x}{2})\}^4 dx = \frac{29\pi^7}{105} + 24\pi [\zeta(3)]^2. \quad (13)$$

$$\int_0^{2\pi} x^2 \{\ln(2 \sin \frac{x}{2})\}^5 dx = -90\pi \zeta(7) - 70\pi^3 \zeta(5) - \frac{71\pi^5}{12} \zeta(3). \quad (14)$$

$$\int_0^{2\pi} x^3 \{\ln(2 \sin \frac{x}{2})\}^2 dx = \frac{8\pi^6}{15}. \quad (15)$$

$$\int_0^{2\pi} x^3 \{\ln(2 \sin \frac{x}{2})\}^3 dx = -36\pi^2 \zeta(5) - 9\pi^4 \zeta(3). \quad (16)$$

$$\int_0^{2\pi} x^3 \{\ln(2 \sin \frac{x}{2})\}^4 dx = \frac{43\pi^8}{84} + 72\pi^2 [\zeta(3)]^2. \quad (17)$$

$$\int_0^{2\pi} x^3 \{\ln(2 \sin \frac{x}{2})\}^5 dx = \frac{296\pi^6}{315} + 48\pi [\zeta(3)]^2. \quad (18)$$

$$\int_0^{2\pi} x^4 \{\ln(2 \sin \frac{x}{2})\}^3 dx = 120\pi \zeta(7) - 84\pi^3 \zeta(5) - 20\pi^5 \zeta(3). \quad (19)$$

$$\int_0^{2\pi} x^4 \{\ln(2 \sin \frac{x}{2})\}^4 dx = 240\pi^2 \zeta(5) - 80\pi^4 \zeta(3). \quad (20)$$

$$\int_0^{2\pi} x^4 \{\ln(2 \sin \frac{x}{2})\}^5 dx = \frac{100\pi^8}{63} + 240\pi^2 [\zeta(3)]^2. \quad (21)$$

The Riemann zeta function is defined by [6, p.767, Eq.(1.1)]:

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \begin{cases} \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s}, & \text{for } \Re(s) > 1, \\ \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, & \text{for } \Re(s) > 0, s \neq 1. \end{cases} \quad (22)$$

For integers $s \geq 2$, (see [1]):

$$\zeta(s) = {}_{s+1}F_s \left[\begin{matrix} \overbrace{1, 1, \dots, 1, 1}^{s+1} \\ \underbrace{2, 2, \dots, 2}_s \end{matrix} \middle| 1 \right], \quad (23)$$

and

$${}_{s+1}F_s \left[\begin{matrix} \overbrace{1, 1, \dots, 1, 1}^{s+1} \\ \underbrace{2, 2, \dots, 2}_s \end{matrix} \middle| -1 \right] = (1-2^{1-s})\zeta(s). \quad (24)$$

Some typical values (see [23]):

$$\begin{aligned} \zeta(2) &= \frac{\pi^2}{6}, \quad \zeta(3) = \frac{\pi^3}{25.79436}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(5) = \frac{\pi^5}{295.1215}, \quad \zeta(6) = \frac{\pi^6}{945}, \\ \zeta(7) &= \frac{\pi^7}{2995.286}, \quad \zeta(8) = \frac{\pi^8}{9450}, \quad \zeta(9) = \frac{\pi^9}{29749.35}, \quad \zeta(10) = \frac{\pi^{10}}{93555}, \dots \end{aligned}$$

A generalized rule for successive integration by parts is:

$$\int [P(x)Q(x)]dx = + (P) \left(\int Qdx \right) - \left(\frac{dP}{dx} \right) \left(\iint Qdx dx \right) + \left(\frac{d^2P}{dx^2} \right) \left(\iiint Qdx dx dx \right) - \dots \quad (25)$$

The present article is organized as follows. Motivated by the work collected in several remarkable papers and monographs [2–5, 7–11, 13–16, 24], in Subsections 3.1 to 3.8, we present our main findings along with their related applications. In Section 4, we obtain the solution of a certain auxiliary integral $\int_0^{2\pi} x^p \{\ln(1 - e^{ix})\}^q dx = \int_0^{2\pi} x^p \{\ln(2 \sin \frac{x}{2}) + i(\frac{x}{2} - \frac{\pi}{2})\}^q dx$ ($p \in \mathbb{N}_0$ and $q \in \mathbb{N}$), separate real, imaginary parts and classify (i.e., arrange in systematic way) the proofs of the obtained results in Subsections 3.1 to 3.7. We also obtained in Subsection 3.8 some mixed summation formulas involving generalized hypergeometric functions ${}_3F_2$, ${}_5F_4$, ${}_7F_6$, other new summation formulas of multiple hypergeometric functions of Kampé de Fériet and mixed relations involving Riemann zeta functions.

3 Main results

3.1 Log-sine integrals with zeroth power of x

Here are the results we prove on the family of integrals $\int_0^{2\pi} \{\ln(2 \sin \frac{x}{2})\}^n dx$; $n \in \{1, \dots, 7\}$:

$$\int_0^{2\pi} \{\ln(2 \sin \frac{x}{2})\} dx = 0. \quad (26)$$

$$\int_0^{2\pi} \left(\frac{x}{2} - \frac{\pi}{2}\right) dx = 0. \quad (27)$$

$$\int_0^{2\pi} \{\ln(2 \sin \frac{x}{2})\}^2 dx = \frac{\pi^3}{6}. \quad (28)$$

$$\int_0^{2\pi} \{\ln(2 \sin \frac{x}{2})\}^3 dx = -3\pi {}_4F_3 \left[\begin{matrix} 1, 1, 1, 1; \\ 2, 2, 2; \\ 1 \end{matrix} \right]. \quad (29)$$

In a particular instance, the integral (29) reduces to:

$$\int_0^{2\pi} \{\ln(2 \sin \frac{x}{2})\}^3 dx = -3\pi \zeta(3). \quad (30)$$

$$\int_0^{2\pi} \{\ln(2 \sin \frac{x}{2})\}^4 dx = \frac{\pi^5}{8} + \frac{3\pi}{2} F_{2:1;1}^{2:2;2} \left[\begin{matrix} 2, 2:1, 1; 1, 1; \\ 3, 3: 2; 2; \\ 1, 1 \end{matrix} \right]. \quad (31)$$

In a particular instance, the integral (31) reduces to:

$$\int_0^{2\pi} \{\ln(2 \sin \frac{x}{2})\}^4 dx = \frac{19\pi^5}{120}. \quad (32)$$

$$\begin{aligned} \int_0^{2\pi} \{\ln(2 \sin \frac{x}{2})\}^5 dx = & -\frac{85\pi^3}{2} {}_4F_3 \left[\begin{matrix} 1, 1, 1, 1; \\ 2, 2, 2; \\ 1 \end{matrix} \right] + 75\pi {}_6F_5 \left[\begin{matrix} 1, 1, 1, 1, 1, 1; \\ 2, 2, 2, 2, 2; \\ 1 \end{matrix} \right] + \\ & + \frac{15\pi^3}{2} F_{1:1;1}^{1:2;2} \left[\begin{matrix} 2: 1, 1; 1, 1; \\ 3:2; 2; \\ 1, 1 \end{matrix} \right] - \frac{10\pi}{9} F_{2:1;1}^{2:2;2;2} \left[\begin{matrix} 3, 3: 1, 1; 1, 1; 1, 1; \\ 4, 4:2; 2; 2; \\ 1, 1, 1 \end{matrix} \right]. \quad (33) \end{aligned}$$

In a particular instance, the integral (33) reduces to:

$$\int_0^{2\pi} \{\ln(2 \sin \frac{x}{2})\}^5 dx = -45\pi\zeta(5) - \frac{5\pi^3}{2} \zeta(3). \quad (34)$$

$$\int_0^{2\pi} \{\ln(2 \sin \frac{x}{2})\}^6 dx = \frac{397\pi^7}{224} + \frac{255\pi^3}{8} F_{2:1;1}^{2:2;2} \left[\begin{matrix} 2, 2: 1, 1; 1, 1; \\ 3, 3: 2; 2; \\ 1, 1 \end{matrix} \right] -$$

$$\begin{aligned}
& -\frac{225\pi}{16} F_{4:1;1}^{4:2;2} \left[\begin{array}{c} 2, 2, 2, 2: 1, 1; 1, 1; \\ 3, 3, 3, 3: 2; 2; \end{array} \right. \left. \begin{array}{c} 1, 1 \\ 1, 1, 1 \end{array} \right] - 10\pi^3 F_{1:1;1}^{1:2;2;2} \left[\begin{array}{c} 3: 1, 1; 1, 1; 1, 1; \\ 4: 2; 2; 2; \end{array} \right. \left. \begin{array}{c} 1, 1, 1 \\ 1, 1, 1 \end{array} \right] + \\
& + \frac{15\pi}{16} F_{2:1;1;1}^{2:2;2;2;2} \left[\begin{array}{c} 4, 4: 1, 1; 1, 1; 1, 1; 1, 1; \\ 5, 5: 2; 2; 2; 2; \end{array} \right. \left. \begin{array}{c} 1, 1, 1, 1 \\ 1, 1, 1, 1 \end{array} \right]. \quad (35)
\end{aligned}$$

In a particular instance, the integral (35) reduces to:

$$\int_0^{2\pi} \{\ln(2 \sin \frac{x}{2})\}^6 dx = \frac{289\pi^7}{224} - \frac{37\pi^6}{42} + 45\pi[\zeta(3)]^2. \quad (36)$$

$$\begin{aligned}
& \int_0^{2\pi} \{\ln(2 \sin \frac{x}{2})\}^7 dx = -\frac{29001\pi^5}{16} {}_4F_3 \left[\begin{array}{c} 1, 1, 1, 1; \\ 2, 2, 2; \end{array} \right. \left. \begin{array}{c} 1 \\ 1 \end{array} \right] + \frac{11235\pi^3}{2} {}_6F_5 \left[\begin{array}{c} 1, 1, 1, 1, 1, 1; \\ 2, 2, 2, 2, 2; \end{array} \right. \left. \begin{array}{c} 1 \\ 1 \end{array} \right] - \\
& - \frac{19215\pi}{2} {}_8F_7 \left[\begin{array}{c} 1, 1, 1, 1, 1, 1, 1, 1; \\ 2, 2, 2, 2, 2, 2, 2; \end{array} \right. \left. \begin{array}{c} 1 \\ 1 \end{array} \right] - \frac{1575\pi^3}{16} F_{3:1;1}^{3:2;2} \left[\begin{array}{c} 2, 2, 2: 1, 1; 1, 1; \\ 3, 3, 3: 2; 2; \end{array} \right. \left. \begin{array}{c} 1, 1 \\ 1, 1 \end{array} \right] + \\
& + \frac{945\pi^5}{8} F_{1:1;1}^{1:2;2} \left[\begin{array}{c} 2: 1, 1; 1, 1; \\ 3: 2; 2; \end{array} \right. \left. \begin{array}{c} 1, 1 \\ 1, 1 \end{array} \right] + \frac{55\pi^9}{8} + \frac{525\pi^5}{8} F_{2:1;1}^{2:2;2} \left[\begin{array}{c} 2, 2: 1, 1; 1, 1; \\ 3, 3: 2; 2; \end{array} \right. \left. \begin{array}{c} 1, 1 \\ 1, 1 \end{array} \right] - \\
& - \frac{1575\pi^3}{64} F_{4:1;1}^{4:2;2} \left[\begin{array}{c} 2, 2, 2, 2: 1, 1; 1, 1; \\ 3, 3, 3, 3: 2; 2; \end{array} \right. \left. \begin{array}{c} 1, 1 \\ 1, 1, 1 \end{array} \right] - \frac{595\pi^3}{18} F_{2:1;1}^{2:2;2;2} \left[\begin{array}{c} 3, 3: 1, 1; 1, 1; 1, 1; \\ 4, 4: 2; 2; 2; \end{array} \right. \left. \begin{array}{c} 1, 1, 1 \\ 1, 1, 1 \end{array} \right] + \\
& + \frac{175\pi}{27} F_{4:1;1}^{4:2;2;2} \left[\begin{array}{c} \overbrace{3, \dots, 3}^4: 1, 1; 1, 1; 1, 1; \\ 4, \dots, 4: 2; 2; 2; \end{array} \right. \left. \begin{array}{c} 1, 1, 1 \\ 1, 1, 1 \end{array} \right] + \frac{105\pi^3}{8} F_{1:1;1;1}^{1:2;2;2;2} \left[\begin{array}{c} 4: 1, 1; 1, 1; 1, 1; 1, 1; \\ 5: 2; 2; 2; 2; \end{array} \right. \left. \begin{array}{c} 1, 1, 1, 1 \\ 1, 1, 1, 1 \end{array} \right] - \\
& - \frac{21\pi}{25} F_{2:1;1;1;1}^{2:2;2;2;2;2} \left[\begin{array}{c} 5, 5: 1, 1; 1, 1; 1, 1; 1, 1; 1, 1; \\ 6, 6: 2; 2; 2; 2; 2; \end{array} \right. \left. \begin{array}{c} 1, 1, 1, 1, 1 \\ 1, 1, 1, 1, 1 \end{array} \right]. \quad (37)
\end{aligned}$$

In a particular instance, integral (37) reduces to:

$$\int_0^{2\pi} \{\ln(2 \sin \frac{x}{2})\}^7 dx = -\frac{133\pi^5}{16} \zeta(3) - \frac{315\pi^3}{4} \zeta(5) - \frac{1785\pi}{2} \zeta(7) - \frac{37\pi^8}{4}. \quad (38)$$

3.2 Log-sine integrals with first power of x

Here are the results we prove on the family of integrals $\int_0^{2\pi} x \{\ln(2 \sin \frac{x}{2})\}^n dx$; $n \in \{1, \dots, 6\}$.

$$\int_0^{2\pi} x \{\ln(2 \sin \frac{x}{2})\} dx = 0. \quad (39)$$

$$\int_0^{2\pi} x \left\{ \ln \left(2 \sin \frac{x}{2} \right) \right\}^2 dx = \frac{\pi^4}{6}. \quad (40)$$

$$\int_0^{2\pi} x \left\{ \ln \left(2 \sin \frac{x}{2} \right) \right\}^3 dx = -\pi^2 {}_4F_3 \left[\begin{matrix} 1, 1, 1, 1; \\ 2, 2, 2; \end{matrix} \middle| 1 \right] - \frac{\pi^2}{2} F_{1:1;1}^{1:2;2} \left[\begin{matrix} 2: 1, 1; 1, 1; \\ 3: 2; 2; \end{matrix} \middle| 1, 1 \right]. \quad (41)$$

In a particular instance, the integral (41) reduces to:

$$F_{1:1;1}^{1:2;2} \left[\begin{matrix} 2: 1, 1; 1, 1; \\ 3: 2; 2; \end{matrix} \middle| 1, 1 \right] = 4\zeta(3). \quad (42)$$

$$\begin{aligned} \int_0^{2\pi} x \left\{ \ln \left(2 \sin \frac{x}{2} \right) \right\}^4 dx &= \frac{7\pi^6}{120} + \frac{\pi^2}{2} F_{2:1;1}^{2:2;2} \left[\begin{matrix} 2, 2: 1, 1; 1, 1; \\ 3, 3: 2; 2; \end{matrix} \middle| 1, 1 \right] + \\ &+ \frac{4\pi^2}{9} F_{1:1;1;1}^{1:2;2;2} \left[\begin{matrix} 3: 1, 1; 1, 1; 1, 1; \\ 4: 2; 2; 2; \end{matrix} \middle| 1, 1, 1 \right]. \end{aligned} \quad (43)$$

In a particular instance, the integral (43) reduces to:

$$\int_0^{2\pi} x \left\{ \ln \left(2 \sin \frac{x}{2} \right) \right\}^4 dx = \frac{19\pi^6}{120}. \quad (44)$$

$$\begin{aligned} \int_0^{2\pi} x \left\{ \ln \left(2 \sin \frac{x}{2} \right) \right\}^5 dx &= \frac{11\pi^4}{2} {}_4F_3 \left[\begin{matrix} 1, 1, 1, 1; \\ 2, 2, 2; \end{matrix} \middle| 1 \right] + 7\pi^2 {}_6F_5 \left[\begin{matrix} 1, 1, 1, 1, 1, 1; \\ 2, 2, 2, 2, 2; \end{matrix} \middle| 1 \right] + \\ &+ \frac{\pi^4}{12} F_{1:1;1}^{1:2;2} \left[\begin{matrix} 2: 1, 1; 1, 1; \\ 3: 2; 2; \end{matrix} \middle| 1, 1 \right] - \frac{10\pi^2}{27} F_{2:1;1;1}^{2:2;2;2} \left[\begin{matrix} 3, 3: 1, 1; 1, 1; 1, 1; \\ 4, 4: 2; 2; 2; \end{matrix} \middle| 1, 1, 1 \right] + \\ &+ \frac{7\pi^2}{8} F_{3:1;1}^{3:2;2} \left[\begin{matrix} 2, 2, 2: 1, 1; 1, 1; \\ 3, 3, 3: 2; 2; \end{matrix} \middle| 1, 1 \right] - \frac{5\pi^2}{12} F_{1:1;1;1;1}^{1:2;2;2;2} \left[\begin{matrix} 4: 1, 1; 1, 1; 1, 1; 1, 1; \\ 5: 2; 2; 2; 2; \end{matrix} \middle| 1, 1, 1, 1 \right]. \end{aligned} \quad (45)$$

In a particular instance, the integral (45) reduces to:

$$\int_0^{2\pi} x \left\{ \ln \left(2 \sin \frac{x}{2} \right) \right\}^5 dx = -45\pi^2 \zeta(5) - \frac{5\pi^4}{2} \zeta(3). \quad (46)$$

$$\begin{aligned} \int_0^{2\pi} x \left\{ \ln \left(2 \sin \frac{x}{2} \right) \right\}^6 dx &= \frac{201\pi^8}{1120} + \frac{23\pi^4}{8} F_{2:1;1}^{2:2;2} \left[\begin{matrix} 2, 2: 1, 1; 1, 1; \\ 3, 3: 2; 2; \end{matrix} \middle| 1, 1 \right] - \\ &- \frac{21\pi^2}{16} F_{4:1;1}^{4:2;2} \left[\begin{matrix} 2, 2, 2, 2: 1, 1; 1, 1; \\ 3, 3, 3, 3: 2; 2; \end{matrix} \middle| 1, 1 \right] - \frac{\pi^4}{9} F_{1:1;1;1}^{1:2;2;2} \left[\begin{matrix} 3: 1, 1; 1, 1; 1, 1; \\ 4: 2; 2; 2; \end{matrix} \middle| 1, 1, 1 \right] + \end{aligned}$$

$$\begin{aligned}
& + \frac{5\pi^2}{16} F_{2:1;1;1}^{2:2;2;2;2} \left[\begin{array}{c} 4, 4:1, 1; 1, 1; 1, 1; 1, 1; \\ 5, 5:2; 2; 2; 2; \end{array} \right] - \\
& - \frac{14\pi^2}{27} F_{3:1;1}^{3:2;2;2} \left[\begin{array}{c} 3, 3, 3:1, 1; 1, 1; 1, 1; \\ 4, 4, 4:2; 2; 2; \end{array} \right] + \\
& + \frac{2\pi^2}{5} F_{1:1;1;1;1}^{1:2;2;2;2;2} \left[\begin{array}{c} 5: 1, 1; 1, 1; 1, 1; 1, 1; 1, 1; \\ 6: 2; 2; 2; 2; 2; \end{array} \right] 1, 1, 1, 1, 1 \right]. \quad (47)
\end{aligned}$$

In a particular instance, the integral (47) reduces to:

$$\int_0^{2\pi} x \left\{ \ln\left(2 \sin \frac{x}{2}\right) \right\}^6 dx = \frac{2477\pi^8}{2016} + 45\pi^2 [\zeta(3)]^2. \quad (48)$$

3.3 Log-sine integrals with second power of x

Here are the results we prove on the family of integrals $\int_0^{2\pi} x^2 \left\{ \ln\left(2 \sin \frac{x}{2}\right) \right\}^n dx$; $n \in \{1, 2, 3, 4, 5\}$.

$$\int_0^{2\pi} x^2 \left\{ \ln\left(2 \sin \frac{x}{2}\right) \right\} dx = -4\pi {}_4F_3 \left[\begin{array}{c} 1, 1, 1, 1; \\ 2, 2, 2; \end{array} \right] 1. \quad (49)$$

In a particular instance, the integral (49) reduces to:

$$\int_0^{2\pi} x^2 \left\{ \ln\left(2 \sin \frac{x}{2}\right) \right\} dx = -4\pi \zeta(3). \quad (50)$$

$$\int_0^{2\pi} x^2 \left\{ \ln\left(2 \sin \frac{x}{2}\right) \right\}^2 dx = \frac{4\pi^5}{15} + \pi F_{2:1;1}^{2:2;2;2} \left[\begin{array}{c} 2, 2: 1, 1; 1, 1; \\ 3, 3: 2; 2; \end{array} \right] 1, 1. \quad (51)$$

In a particular instance, the integral (51) reduces to:

$$F_{2:1;1}^{2:2;2;2} \left[\begin{array}{c} 2, 2:1, 1; 1, 1; \\ 3, 3: 2; 2; \end{array} \right] 1, 1 = \frac{\pi^4}{45}. \quad (52)$$

$$\begin{aligned}
\int_0^{2\pi} x^2 \left\{ \ln\left(2 \sin \frac{x}{2}\right) \right\}^3 dx & = -21\pi^3 {}_4F_3 \left[\begin{array}{c} 1, 1, 1, 1; \\ 2, 2, 2; \end{array} \right] 1 + 36\pi {}_6F_5 \left[\begin{array}{c} 1, 1, 1, 1, 1, 1; \\ 2, 2, 2, 2, 2; \end{array} \right] 1 + \\
& + 3\pi^3 F_{1:1;1}^{1:2;2;2} \left[\begin{array}{c} 2: 1, 1; 1, 1; \\ 3: 2; 2; \end{array} \right] 1, 1 - \frac{4\pi}{9} F_{2:1;1}^{2:2;2;2} \left[\begin{array}{c} 3, 3: 1, 1; 1, 1; 1, 1; \\ 4, 4: 2; 2; 2; \end{array} \right] 1, 1, 1. \quad (53)
\end{aligned}$$

In a particular instance, the integral (53) reduces to:

$$F_{2:1;1}^{2:2;2;2} \left[\begin{array}{c} 3, 3:1, 1; 1, 1; 1, 1; \\ 1, 1, 1 \end{array} \right] = -9\pi^2\zeta(3) + 108\zeta(5). \quad (54)$$

$$\begin{aligned} \int_0^{2\pi} x^2 \left\{ \ln\left(2 \sin \frac{x}{2}\right) \right\}^4 dx &= \frac{43\pi^7}{70} + \frac{21\pi^3}{2} F_{2:1;1}^{2:2;2;2} \left[\begin{array}{c} 2, 2: 1, 1 ; 1, 1 ; \\ 3, 3: 2 ; 2 ; \end{array} \right] - \\ &- \frac{9\pi}{2} F_{4:1;1}^{4:2;2} \left[\begin{array}{c} 2, 2, 2, 2:1, 1; 1, 1; \\ 3, 3, 3, 3: 2 ; 2; \end{array} \right] - \frac{8\pi^3}{3} F_{1:1;1;1}^{1:2;2;2} \left[\begin{array}{c} 3:1, 1; 1, 1; 1, 1; \\ 4: 2 ; 2; 2; \end{array} \right] + \\ &+ \frac{\pi}{4} F_{2:1;1;1;1}^{2:2;2;2;2} \left[\begin{array}{c} 4, 4:1, 1; 1, 1; 1, 1; 1, 1; \\ 5, 5: 2 ; 2; 2; 2; \end{array} \right]. \end{aligned} \quad (55)$$

In a particular instance, the integral (55) reduces to:

$$F_{2:1;1;1;1}^{2:2;2;2;2} \left[\begin{array}{c} 4, 4:1, 1; 1, 1; 1, 1; 1, 1; \\ 5, 5: 2 ; 2; 2; 2; \end{array} \right] = \frac{208\pi^6}{35} - \frac{592\pi^5}{105} - 192[\zeta(3)]^2. \quad (56)$$

$$\begin{aligned} \int_0^{2\pi} x^2 \left\{ \ln\left(2 \sin \frac{x}{2}\right) \right\}^5 dx &= -\frac{1685\pi^5}{4} {}_4F_3 \left[\begin{array}{c} 1, 1, 1, 1; \\ 2, 2, 2; \end{array} \right] + 1320\pi^3 {}_6F_5 \left[\begin{array}{c} 1, 1, 1, 1, 1, 1; \\ 2, 2, 2, 2, 2; \end{array} \right] - \\ &- 2250\pi {}_8F_7 \left[\begin{array}{c} 1, 1, 1, 1, 1, 1, 1, 1; \\ 2, 2, 2, 2, 2, 2, 2; \end{array} \right] - \frac{45\pi^3}{2} F_{3:1;1}^{3:2;2} \left[\begin{array}{c} 2, 2, 2:1, 1; 1, 1; \\ 3, 3, 3: 2 ; 2; \end{array} \right] + \\ &+ \frac{55\pi^5}{2} F_{1:1;1}^{1:2;2} \left[\begin{array}{c} 2:1, 1; 1, 1; \\ 3: 2 ; 2; \end{array} \right] + \frac{11\pi^9}{7} + 15\pi^5 F_{2:1;1}^{2:2;2;2} \left[\begin{array}{c} 2, 2:1, 1; 1, 1; \\ 3, 3: 2 ; 2; \end{array} \right] - \\ &- \frac{45\pi^3}{8} F_{4:1;1}^{4:2;2} \left[\begin{array}{c} 2, 2, 2, 2:1, 1; 1, 1; \\ 3, 3, 3, 3: 2 ; 2; \end{array} \right] - \frac{70\pi^3}{9} F_{2:1;1;1}^{2:2;2;2} \left[\begin{array}{c} 3, 3:1, 1; 1, 1; 1, 1; \\ 4, 4: 2 ; 2; 2; \end{array} \right] + \\ &+ \frac{40\pi}{27} F_{4:1;1;1}^{4:2;2;2} \left[\begin{array}{c} 3, 3, 3, 3:1, 1; 1, 1; 1, 1; \\ 4, 4, 4, 4: 2 ; 2; 2; \end{array} \right] + \frac{5\pi^3}{2} F_{1:1;1;1;1}^{1:2;2;2;2} \left[\begin{array}{c} 4:1, 1; 1, 1; 1, 1; 1, 1; \\ 5: 2 ; 2; 2; 2; \end{array} \right] - \\ &- \frac{4\pi}{25} F_{2:1;1;1;1;1}^{2:2;2;2;2;2} \left[\begin{array}{c} 5, 5:1, 1; 1, 1; 1, 1; 1, 1; 1, 1; \\ 6, 6: 2 ; 2; 2; 2; 2; \end{array} \right]. \end{aligned} \quad (57)$$

In a particular instance, the integral (57) reduces to:

$$F_{2:1;1;1;1}^{2:2;2;2;2} \left[\begin{array}{c} 5, 5:1, 1; 1, 1; 1, 1; 1, 1; 1, 1; \\ 1, 1, 1, 1, 1 \\ 6, 6: 2; 2; 2; 2; 2; \end{array} \right] = 5250\zeta(7) - 500\pi^2\zeta(5) - \frac{100\pi^4}{3}\zeta(3) + \frac{925\pi^7}{84}. \quad (58)$$

3.4 Log-sine integrals with third power of x

Here are the results we prove on the family of integrals $\int_0^{2\pi} x^3 \{\ln(2 \sin \frac{x}{2})\}^n dx$; $n \in \{1, 2, 3, 4\}$.

$$\int_0^{2\pi} x^3 \{\ln(2 \sin \frac{x}{2})\} dx = -4\pi^2 {}_4F_3 \left[\begin{array}{c} 1, 1, 1, 1; \\ 2, 2, 2; \end{array} \right] 1 - 2\pi^2 F_{1:1;1}^{1:2;2} \left[\begin{array}{c} 2:1, 1; 1, 1; \\ 3: 2; 2; \end{array} \right] 1, 1. \quad (59)$$

In a particular instance, the integral (59) reduces to:

$$\int_0^{2\pi} x^3 \{\ln(2 \sin \frac{x}{2})\} dx = -12\pi^2\zeta(3). \quad (60)$$

$$\begin{aligned} \int_0^{2\pi} x^3 \{\ln(2 \sin \frac{x}{2})\}^2 dx &= \frac{\pi^6}{3} + \pi^2 F_{2:1;1}^{2:2;2} \left[\begin{array}{c} 2, 2:1, 1; 1, 1; \\ 3, 3: 2; 2; \end{array} \right] 1, 1 + \\ &+ \frac{8\pi^2}{9} F_{1:1;1;1}^{1:2;2;2} \left[\begin{array}{c} 3:1, 1; 1, 1; 1, 1; \\ 4: 2; 2; 2; \end{array} \right] 1, 1, 1. \end{aligned} \quad (61)$$

In a particular instance, the integral (61) reduces to:

$$F_{1:1;1;1}^{1:2;2;2} \left[\begin{array}{c} 3:1, 1; 1, 1; 1, 1; \\ 4: 2; 2; 2; \end{array} \right] 1, 1, 1 = \frac{\pi^4}{5}. \quad (62)$$

$$\begin{aligned} \int_0^{2\pi} x^3 \{\ln(2 \sin \frac{x}{2})\}^3 dx &= 9\pi^4 {}_4F_3 \left[\begin{array}{c} 1, 1, 1, 1; \\ 2, 2, 2; \end{array} \right] 1 + 12\pi^2 {}_6F_5 \left[\begin{array}{c} 1, 1, 1, 1, 1, 1; \\ 2, 2, 2, 2, 2; \end{array} \right] 1 - \\ &- \frac{\pi^4}{2} F_{1:1;1}^{1:2;2} \left[\begin{array}{c} 2:1, 1; 1, 1; \\ 3: 2; 2; \end{array} \right] 1, 1 - \frac{4\pi^2}{9} F_{2:1;1;1}^{2:2;2;2} \left[\begin{array}{c} 3, 3:1, 1; 1, 1; 1, 1; \\ 4, 4: 2; 2; 2; \end{array} \right] 1, 1, 1 + \\ &+ \frac{3\pi^2}{2} F_{3:1;1}^{3:2;2} \left[\begin{array}{c} 2, 2, 2:1, 1; 1, 1; \\ 3, 3, 3: 2; 2; \end{array} \right] 1, 1 - \frac{\pi^2}{2} F_{1:1;1;1;1}^{1:2;2;2;2} \left[\begin{array}{c} 4:1, 1; 1, 1; 1, 1; 1, 1; \\ 5: 2; 2; 2; 2; \end{array} \right] 1, 1, 1, 1. \end{aligned} \quad (63)$$

In a particular instance, the integral (63) reduces to:

$$F_{1:1;1;1;1}^{1:2;2;2;2} \left[\begin{array}{c} 4:1, 1; 1, 1; 1, 1; 1, 1; \\ 5: 2; 2; 2; 2; \end{array} \quad 1, 1, 1, 1 \right] = 96\zeta(5). \quad (64)$$

$$\begin{aligned} \int_0^{2\pi} x^3 \left\{ \ln\left(2 \sin \frac{x}{2}\right) \right\}^4 dx &= \frac{\pi^8}{4} + \frac{7\pi^4}{2} F_{2:1;1}^{2:2;2} \left[\begin{array}{c} 2, 2:1, 1; 1, 1; \\ 3, 3: 2; 2; \end{array} \quad 1, 1 \right] - \\ &- \frac{3\pi^2}{2} F_{4:1;1}^{4:2;2} \left[\begin{array}{c} 2, 2, 2, 2:1, 1; 1, 1; \\ 3, 3, 3, 3: 2; 2; \end{array} \quad 1, 1 \right] + \frac{4\pi^4}{9} F_{1:1;1;1}^{1:2;2;2} \left[\begin{array}{c} 3:1, 1; 1, 1; 1, 1; \\ 4: 2; 2; 2; \end{array} \quad 1, 1, 1 \right] + \\ &+ \frac{\pi^2}{4} F_{2:1;1;1;1}^{2:2;2;2;2} \left[\begin{array}{c} 4, 4:1, 1; 1, 1; 1, 1; 1, 1; \\ 5, 5: 2; 2; 2; 2; \end{array} \quad 1, 1, 1, 1 \right] - \frac{16\pi^2}{27} F_{3:1;1;1}^{3:2;2;2} \left[\begin{array}{c} 3, 3, 3:1, 1; 1, 1; 1, 1; \\ 4, 4, 4: 2; 2; 2; \end{array} \quad 1, 1, 1 \right] + \\ &+ \frac{8\pi^2}{25} F_{1:1;1;1;1;1}^{1:2;2;2;2;2} \left[\begin{array}{c} 5:1, 1; 1, 1; 1, 1; 1, 1; 1, 1; \\ 6: 2; 2; 2; 2; 2; \end{array} \quad 1, 1, 1, 1, 1 \right]. \end{aligned} \quad (65)$$

In a particular instance, the integral (65) reduces to:

$$F_{1:1;1;1;1;1}^{1:2;2;2;2;2} \left[\begin{array}{c} 5:1, 1; 1, 1; 1, 1; 1, 1; 1, 1; \\ 6: 2; 2; 2; 2; 2; \end{array} \quad 1, 1, 1, 1, 1 \right] = -\frac{1735\pi^6}{1008} + \frac{185\pi^5}{42}. \quad (66)$$

3.5 Log-sine integrals with fourth power of x

Here are the results we prove on the family of integrals $\int_0^{2\pi} x^4 \left\{ \ln\left(2 \sin \frac{x}{2}\right) \right\}^n dx$; $n \in \{1, 2, 3\}$.

$$\int_0^{2\pi} x^4 \left\{ \ln\left(2 \sin \frac{x}{2}\right) \right\} dx = -32\pi^3 {}_4F_3 \left[\begin{array}{c} 1, 1, 1, 1; \\ 2, 2, 2; \end{array} \quad 1 \right] + 48\pi {}_6F_5 \left[\begin{array}{c} 1, 1, 1, 1, 1, 1; \\ 2, 2, 2, 2, 2; \end{array} \quad 1 \right]. \quad (67)$$

In a particular instance, the integral (67) reduces to:

$$\int_0^{2\pi} x^4 \left\{ \ln\left(2 \sin \frac{x}{2}\right) \right\} dx = -32\pi^3 \zeta(3) + 48\pi \zeta(5). \quad (68)$$

$$\begin{aligned} \int_0^{2\pi} x^4 \left\{ \ln\left(2 \sin \frac{x}{2}\right) \right\}^2 dx &= \frac{88\pi^7}{105} + 8\pi^3 F_{2:1;1}^{2:2;2} \left[\begin{array}{c} 2, 2:1, 1; 1, 1; \\ 3, 3: 2; 2; \end{array} \quad 1, 1 \right] - \\ &- 3\pi F_{4:1;1}^{4:2;2} \left[\begin{array}{c} 2, 2, 2, 2:1, 1; 1, 1; \\ 3, 3, 3, 3: 2; 2; \end{array} \quad 1, 1 \right]. \end{aligned} \quad (69)$$

In a particular instance, the integral (69) reduces to:

$$F_{4:1;1}^{4:2;2} \left[\begin{array}{c} 2, 2, 2, 2:1, 1; 1, 1; \\ 3, 3, 3, 3: 2; 2; \end{array} \right]_{1,1} = \frac{64\pi^6}{189} - \frac{296\pi^5}{945} - 16[\zeta(3)]^2. \quad (70)$$

$$\begin{aligned} \int_0^{2\pi} x^4 \left\{ \ln\left(2 \sin \frac{x}{2}\right) \right\}^3 dx = & -192\pi^5 {}_4F_3 \left[\begin{array}{c} 1, 1, 1, 1; \\ 2, 2, 2; \end{array} \right]_1 + 648\pi^3 {}_6F_5 \left[\begin{array}{c} 1, 1, 1, 1, 1, 1; \\ 2, 2, 2, 2, 2; \end{array} \right]_1 - \\ & -1080\pi {}_8F_7 \left[\begin{array}{c} 1, 1, 1, 1, 1, 1, 1, 1; \\ 2, 2, 2, 2, 2, 2, 2; \end{array} \right]_1 - 9\pi^3 F_{3:1;1}^{3:2;2} \left[\begin{array}{c} 2, 2, 2:1, 1; 1, 1; \\ 3, 3, 3: 2; 2; \end{array} \right]_{1,1} + \\ & + 12\pi^5 F_{1:1;1}^{1:2;2} \left[\begin{array}{c} 2:1, 1; 1, 1; \\ 3: 2; 2; \end{array} \right]_{1,1} + \frac{22\pi^9}{35} + 6\pi^5 F_{2:1;1}^{2:2;2} \left[\begin{array}{c} 2, 2:1, 1; 1, 1; \\ 3, 3: 2; 2; \end{array} \right]_{1,1} - \\ & - \frac{9\pi^3}{4} F_{4:1;1}^{4:2;2} \left[\begin{array}{c} 2, 2, 2, 2:1, 1; 1, 1; \\ 3, 3, 3, 3: 2; 2; \end{array} \right]_{1,1} - \frac{32\pi^3}{9} F_{2:1;1}^{2:2;2;2} \left[\begin{array}{c} 3, 3:1, 1; 1, 1; 1, 1; \\ 4, 4: 2; 2; 2; \end{array} \right]_{1,1,1} + \\ & + \frac{16\pi}{27} F_{4:1;1;1}^{4:2;2;2} \left[\begin{array}{c} 3, 3, 3, 3:1, 1; 1, 1; 1, 1; \\ 4, 4, 4, 4: 2; 2; 2; \end{array} \right]_{1,1,1}. \end{aligned} \quad (71)$$

In a particular instance, the integral (71) reduces to:

$$\begin{aligned} F_{4:1;1;1}^{4:2;2;2} \left[\begin{array}{c} 3, 3, 3, 3:1, 1; 1, 1; 1, 1; \\ 4, 4, 4, 4: 2; 2; 2; \end{array} \right]_{1,1,1} = & 2025\zeta(7) - \frac{405\pi^2}{4}\zeta(5) - \frac{189\pi^4}{4}\zeta(3) - \frac{333\pi^7}{280} - \\ & - \frac{243\pi^2}{4}[\zeta(3)]^2. \end{aligned} \quad (72)$$

3.6 Log-sine integrals with fifth power of x

Here are the results we prove on the family of integrals $\int_0^{2\pi} x^5 \left\{ \ln\left(2 \sin \frac{x}{2}\right) \right\}^n dx$; $n \in \{1, 2\}$.

$$\begin{aligned} \int_0^{2\pi} x^5 \left\{ \ln\left(2 \sin \frac{x}{2}\right) \right\} dx = & 32\pi^4 {}_4F_3 \left[\begin{array}{c} 1, 1, 1, 1; \\ 2, 2, 2; \end{array} \right]_1 + 48\pi^2 {}_6F_5 \left[\begin{array}{c} 1, 1, 1, 1, 1, 1; \\ 2, 2, 2, 2, 2; \end{array} \right]_1 + \\ & + 6\pi^2 F_{3:1;1}^{3:2;2} \left[\begin{array}{c} 2, 2, 2:1, 1; 1, 1; \\ 3, 3, 3: 2; 2; \end{array} \right]_{1,1} - 8\pi^4 F_{1:1;1}^{1:2;2} \left[\begin{array}{c} 2:1, 1; 1, 1; \\ 3: 2; 2; \end{array} \right]_{1,1}. \end{aligned} \quad (73)$$

In a particular instance, the integral (73) reduces to:

$$F_{3:1;1}^{3:2;2} \left[\begin{array}{c} 2, 2, 2:1, 1; 1, 1; \\ 3, 3, 3: 2; 2; \end{array} \quad 1, 1 \right] = 32\zeta(5) - \frac{40\pi^2}{3}\zeta(3). \quad (74)$$

$$\begin{aligned} \int_0^{2\pi} x^5 \left\{ \ln\left(2 \sin \frac{x}{2}\right) \right\}^2 dx &= \frac{112\pi^8}{105} + 8\pi^4 F_{2:1;1}^{2:2;2} \left[\begin{array}{c} 2, 2:1, 1; 1, 1; \\ 3, 3: 2; 2; \end{array} \quad 1, 1 \right] - \\ -3\pi^2 F_{4:1;1}^{4:2;2} \left[\begin{array}{c} 2, 2, 2, 2:1, 1; 1, 1; \\ 3, 3, 3, 3: 2; 2; \end{array} \quad 1, 1 \right] &- \frac{32\pi^2}{27} F_{3:1;1}^{3:2;2;2} \left[\begin{array}{c} 3, 3, 3:1, 1; 1, 1; 1, 1; \\ 4, 4, 4: 2; 2; 2; \end{array} \quad 1, 1, 1 \right] + \\ + \frac{32\pi^4}{9} F_{1:1;1;1}^{1:2;2;2} \left[\begin{array}{c} 3:1, 1; 1, 1; 1, 1; \\ 4: 2; 2; 2; \end{array} \quad 1, 1, 1 \right]. & \end{aligned} \quad (75)$$

In a particular instance, the integral (75) reduces to:

$$F_{3:1;1;1}^{3:2;2;2} \left[\begin{array}{c} 3, 3, 3:1, 1; 1, 1; 1, 1; \\ 4, 4, 4: 2; 2; 2; \end{array} \quad 1, 1, 1 \right] = -\frac{153\pi^6}{280} + \frac{111\pi^5}{140} - 162[\zeta(3)]^2. \quad (76)$$

3.7 Log-sine integrals with sixth power of x

Here are the results we prove on the family of integrals $\int_0^{2\pi} x^6 \left\{ \ln\left(2 \sin \frac{x}{2}\right) \right\}^n dx$; $n = \{1\}$.

$$\begin{aligned} \int_0^{2\pi} x^6 \left\{ \ln\left(2 \sin \frac{x}{2}\right) \right\} dx &= -192\pi^5 {}_4F_3 \left[\begin{array}{c} 1, 1, 1, 1; \\ 2, 2, 2; \end{array} \quad 1 \right] + 960\pi^3 {}_6F_5 \left[\begin{array}{c} 1, 1, 1, 1, 1, 1; \\ 2, 2, 2, 2, 2; \end{array} \quad 1 \right] - \\ -1440\pi {}_8F_7 \left[\begin{array}{c} 1, 1, 1, 1, 1, 1, 1, 1; \\ 2, 2, 2, 2, 2, 2, 2; \end{array} \quad 1 \right]. & \end{aligned} \quad (77)$$

In a particular instance, the integral (77) reduces to:

$$\int_0^{2\pi} x^6 \left\{ \ln\left(2 \sin \frac{x}{2}\right) \right\} dx = -192\pi^5 \zeta(3) + 960\pi^3 \zeta(5) - 1440\pi \zeta(7). \quad (78)$$

3.8 Mixed summation formulas and relations

Here are the results we prove on mixed summation formulas and relations:

$$\frac{\pi^4}{3} = 2\pi^2 {}_3F_2 \left[\begin{array}{c} 1, 1, 1; \\ 2, 2; \end{array} \quad 1 \right]. \quad (79)$$

From equation (79), we get a **mixed relation**

$$\frac{\pi^4}{3} = 2\pi^2 \zeta(2). \quad (80)$$

$$\frac{\pi^4}{15} = \frac{\pi^2}{2} {}_3F_2 \left[\begin{matrix} 1, 1, 1; \\ 2, 2 \end{matrix}; 1 \right] - \frac{3}{2} {}_5F_4 \left[\begin{matrix} 1, 1, 1, 1, 1; \\ 2, 2, 2, 2; \end{matrix}; 1 \right]. \quad (81)$$

From equation (81), we get a **mixed relation**

$$\frac{\pi^4}{15} = \frac{\pi^2}{2} \zeta(2) - \frac{3}{2} \zeta(4). \quad (82)$$

$$\frac{48\pi^6}{7} = 64\pi^4 {}_3F_2 \left[\begin{matrix} 1, 1, 1; \\ 2, 2 \end{matrix}; 1 \right] - 480\pi^2 {}_5F_4 \left[\begin{matrix} 1, 1, 1, 1, 1; \\ 2, 2, 2, 2; \end{matrix}; 1 \right] + 1440 {}_7F_6 \left[\begin{matrix} 1, 1, 1, 1, 1, 1, 1; \\ 2, 2, 2, 2, 2, 2; \end{matrix}; 1 \right]. \quad (83)$$

From equation (83), we get a **mixed relation**

$$\frac{48\pi^6}{7} = 64\pi^4 \zeta(2) - 480\pi^2 \zeta(4) + 1440 \zeta(6). \quad (84)$$

4 Proofs of the results in Subsections 3.1–3.8

Proof of the results (77), (78), (83) and (84). Consider,

$$\int_0^{2\pi} x^6 \{\ln(1 - e^{ix})\} dx.$$

Applying the result (7), we derive

$$\int_0^{2\pi} x^6 \{\ln(1 - e^{ix})\} dx = \int_0^{2\pi} x^6 \{\ln(2 \sin \frac{x}{2})\} dx + \frac{i}{2} \int_0^{2\pi} x^6 (x - \pi) dx. \quad (85)$$

Also consider,

$$\int_0^{2\pi} x^6 \{\ln(1 - e^{ix})\} dx.$$

Using the result (8), we obtain

$$\int_0^{2\pi} x^6 \{\ln(1 - e^{ix})\} dx = - \sum_{m=0}^{\infty} \frac{(1)_m (1)_m}{(2)_m m!} \int_0^{2\pi} x^6 e^{i(1+m)x} dx$$

Evaluating the inner integral yields

$$\begin{aligned} \int_0^{2\pi} x^6 \{\ln(1 - e^{ix})\} dx &= - \sum_{m=0}^{\infty} \frac{(1)_m (1)_m}{(2)_m m!} \times \\ &\times \left[\frac{192\pi^5}{(1+m)^2} - \frac{960\pi^3}{(1+m)^4} + \frac{1440\pi}{(1+m)^6} + i \left(\frac{480\pi^4}{(1+m)^3} - \frac{64\pi^6}{1+m} - \frac{1440\pi^2}{(1+m)^5} \right) \right]. \quad (86) \end{aligned}$$

Now, by comparing the real and imaginary components of equations (85) and (86), we obtain

$$\begin{aligned} \int_0^{2\pi} x^6 \{\ln(2 \sin \frac{x}{2})\} dx &= - \sum_{m=0}^{\infty} \frac{(1)_m (1)_m}{(2)_m m!} \left[\left\{ \frac{192\pi^5}{\{1+m\}^2} - \frac{960\pi^3}{\{1+m\}^4} + \frac{1440\pi}{\{1+m\}^6} \right\} \right] \\ &= -192\pi^5 \sum_{m=0}^{\infty} \frac{(1)_m (1)_m \{(1)_m\}^2}{(2)_m m! \{(2)_m\}^2} + 960\pi^3 \sum_{m=0}^{\infty} \frac{(1)_m (1)_m \{(1)_m\}^4}{(2)_m m! \{(2)_m\}^4} - 1440\pi \sum_{m=0}^{\infty} \frac{(1)_m (1)_m \{(1)_m\}^6}{(2)_m m! \{(2)_m\}^6}, \end{aligned}$$

and applying the definition of the generalized hypergeometric function of one variable, we arrive at the desired result (77).

Further, substituting the result (23) into the right-hand side of equation (77), we find the result (78).

$$\begin{aligned} \frac{1}{2} \int_0^{2\pi} x^6(x - \pi)dx &= - \sum_{m=0}^{\infty} \frac{(1)_m(1)_m}{(2)_m m!} \left[\left\{ \frac{480\pi^4}{\{1+m\}^3} - \frac{64\pi^6}{\{1+m\}} - \frac{1440\pi^2}{\{1+m\}^5} \right\} \right] \\ \frac{48\pi^8}{7} &= -480\pi^4 \sum_{m=0}^{\infty} \frac{(1)_m(1)_m \{(1)_m\}^3}{(2)_m m! \{(2)_m\}^3} + 64\pi^6 \sum_{m=0}^{\infty} \frac{(1)_m(1)_m \{(1)_m\}}{(2)_m m! \{(2)_m\}} + 1440\pi^2 \sum_{m=0}^{\infty} \frac{(1)_m(1)_m \{(1)_m\}^5}{(2)_m m! \{(2)_m\}^5} \\ \frac{48\pi^6}{7} &= -480\pi^2 \sum_{m=0}^{\infty} \frac{(1)_m(1)_m \{(1)_m\}^3}{(2)_m m! \{(2)_m\}^3} + 64\pi^4 \sum_{m=0}^{\infty} \frac{(1)_m(1)_m \{(1)_m\}}{(2)_m m! \{(2)_m\}} + 1440 \sum_{m=0}^{\infty} \frac{(1)_m(1)_m \{(1)_m\}^5}{(2)_m m! \{(2)_m\}^5}, \end{aligned}$$

and by employing once again the definition of the generalized hypergeometric function of one variable, we reach the required result (83).

Finally, substituting result (23) into the right-hand side of equation (83), we derive a mixed relation corresponding to (84). \square

Proof of the results (67), (68), (81) and (82). Consider,

$$\int_0^{2\pi} x^4 \{\ln(1 - e^{ix})\} dx.$$

Applying the result (7), we derive

$$\int_0^{2\pi} x^4 \{\ln(1 - e^{ix})\} dx = \int_0^{2\pi} x^4 \{\ln(2 \sin \frac{x}{2})\} dx + \frac{i}{2} \int_0^{2\pi} x^4(x - \pi) dx. \quad (87)$$

Also consider,

$$\int_0^{2\pi} x^4 \{\ln(1 - e^{ix})\} dx.$$

Using the result (8), we obtain

$$\int_0^{2\pi} x^4 \{\ln(1 - e^{ix})\} dx = - \sum_{m=0}^{\infty} \frac{(1)_m(1)_m}{(2)_m m!} \int_0^{2\pi} x^4 e^{i(1+m)x} dx$$

Evaluating the inner integral yields

$$\begin{aligned} \int_0^{2\pi} x^4 \{\ln(1 - e^{ix})\} dx &= - \sum_{m=0}^{\infty} \frac{(1)_m(1)_m}{(2)_m m!} \times \\ &\times \left[\frac{32\pi^3}{(1+m)^2} - \frac{48\pi}{(1+m)^4} + i \left(\frac{48\pi^2}{(1+m)^3} - \frac{16\pi^4}{1+m} \right) \right]. \end{aligned} \quad (88)$$

Now, by comparing the real and imaginary components of equations (87) and (88), we obtain

$$\begin{aligned} \int_0^{2\pi} x^4 \{\ln(2 \sin \frac{x}{2})\} dx &= - \sum_{m=0}^{\infty} \frac{(1)_m(1)_m}{(2)_m m!} \left[\left\{ \frac{32\pi^3}{\{1+m\}^2} - \frac{48\pi}{\{1+m\}^4} \right\} \right] \\ &= -32\pi^3 \sum_{m=0}^{\infty} \frac{(1)_m(1)_m \{(1)_m\}^2}{(2)_m m! \{(2)_m\}^2} + 48\pi \sum_{m=0}^{\infty} \frac{(1)_m(1)_m \{(1)_m\}^4}{(2)_m m! \{(2)_m\}^4}, \end{aligned}$$

and applying the definition of generalized hypergeometric function of one variable, we arrive at the desired result (67).

Further, substituting the result (23) into the right-hand side of equation (67), we obtain the result (68).

$$\begin{aligned} \frac{1}{2} \int_0^{2\pi} x^4(x - \pi)dx &= - \sum_{m=0}^{\infty} \frac{(1)_m(1)_m}{(2)_m m!} \left[\left\{ \frac{48\pi^2}{\{1+m\}^3} - \frac{16\pi^4}{\{1+m\}} \right\} \right] \\ \frac{32\pi^6}{15} &= -48\pi^2 \sum_{m=0}^{\infty} \frac{(1)_m(1)_m\{(1)_m\}^3}{(2)_m m!\{(2)_m\}^3} + 16\pi^4 \sum_{m=0}^{\infty} \frac{(1)_m(1)_m\{(1)_m\}}{(2)_m m!\{(2)_m\}} \\ \frac{\pi^4}{15} &= -\frac{3}{2} \sum_{m=0}^{\infty} \frac{(1)_m(1)_m\{(1)_m\}^3}{(2)_m m!\{(2)_m\}^3} + \frac{\pi^2}{2} \sum_{m=0}^{\infty} \frac{(1)_m(1)_m\{(1)_m\}}{(2)_m m!\{(2)_m\}}, \end{aligned}$$

and by employing once again the definition of generalized hypergeometric function of one variable, we have the required result (81).

Finally, substituting result (23) into the right-hand side of equation (81), we derive a mixed relation (82). \square

Proof of the results (69), (70), (73) and (74). Consider,

$$\int_0^{2\pi} x^4 \{\ln(1 - e^{ix})\}^2 dx.$$

Applying the result (7), we derive

$$\begin{aligned} \int_0^{2\pi} x^4 \{\ln(1 - e^{ix})\}^2 dx &= \int_0^{2\pi} x^4 \{\ln(2 \sin \frac{x}{2})\}^2 dx - \frac{1}{4} \int_0^{2\pi} x^4 (x - \pi)^2 dx + \\ &+ i \left\{ \int_0^{2\pi} x^5 \{\ln(2 \sin \frac{x}{2})\} dx - \pi \int_0^{2\pi} x^4 \{\ln(2 \sin \frac{x}{2})\} dx \right\}. \end{aligned} \quad (89)$$

Also consider,

$$\int_0^{2\pi} x^4 \{\ln(1 - e^{ix})\}^2 dx.$$

Using the result (8), we obtain

$$\int_0^{2\pi} x^4 \{\ln(1 - e^{ix})\}^2 dx = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(1)_m(1)_m(1)_n(1)_n}{(2)_m m! (2)_n n!} \int_0^{2\pi} x^4 e^{i(2+m+n)x} dx$$

Evaluating the inner integral yields

$$\begin{aligned} \int_0^{2\pi} x^4 \{\ln(1 - e^{ix})\}^2 dx &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(1)_m(1)_m(1)_n(1)_n}{(2)_m m! (2)_n n!} \times \\ &\times \left[\frac{32\pi^3}{\{2+(m+n)\}^2} - \frac{48\pi}{\{2+(m+n)\}^4} + i \left(\frac{48\pi^2}{\{2+(m+n)\}^3} - \frac{16\pi^4}{2+(m+n)} \right) \right]. \end{aligned} \quad (90)$$

Now, by separating the real and imaginary components from equations (89) and (90), we obtain

$$\begin{aligned} \int_0^{2\pi} x^4 \left\{ \ln\left(2 \sin \frac{x}{2}\right) \right\}^2 dx - \frac{1}{4} \int_0^{2\pi} x^4 (x - \pi)^2 dx &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(1)_m (1)_m (1)_n (1)_n}{(2)_m m! (2)_n n!} \times \\ &\times \left[\left\{ \frac{32\pi^3}{\{2 + (m+n)\}^2} - \frac{48\pi}{\{2 + (m+n)\}^4} \right\} \right] \\ \int_0^{2\pi} x^4 \left\{ \ln\left(2 \sin \frac{x}{2}\right) \right\}^2 dx - \frac{88\pi^7}{105} &= 32\pi^3 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(1)_m (1)_m (1)_n (1)_n \{(2)_{m+n}\}^2}{(2)_m m! (2)_n n! \{2(3)_{m+n}\}^2} - \\ &- 48\pi \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(1)_m (1)_m (1)_n (1)_n \{(2)_{m+n}\}^4}{(2)_m m! (2)_n n! \{2(3)_{m+n}\}^4} \\ \int_0^{2\pi} x^4 \left\{ \ln\left(2 \sin \frac{x}{2}\right) \right\}^2 dx &= \frac{88\pi^7}{105} + 8\pi^3 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(1)_m (1)_m (1)_n (1)_n \{(2)_{m+n}\}^2}{(2)_m m! (2)_n n! \{(3)_{m+n}\}^2} - \\ &- 3\pi \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(1)_m (1)_m (1)_n (1)_n \{(2)_{m+n}\}^4}{(2)_m m! (2)_n n! \{(3)_{m+n}\}^4}, \end{aligned}$$

and using the definition (1) of the multiple hypergeometric functions of Kampé de Fériet type, the desired result (69) is established.

Moreover, by applying the value of the integral (18) of Choi and Srivastava on the left-hand side and substituting the new summation relation (52) in the right-hand side of equation (69), we derive a potentially new summation identity (70) involving double hypergeometric functions of the Kampé de Fériet.

$$\begin{aligned} \int_0^{2\pi} x^5 \left\{ \ln\left(2 \sin \frac{x}{2}\right) \right\} dx - \pi \int_0^{2\pi} x^4 \left\{ \ln\left(2 \sin \frac{x}{2}\right) \right\} dx &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(1)_m (1)_m (1)_n (1)_n}{(2)_m m! (2)_n n!} \times \\ &\times \left[\left\{ \frac{48\pi^2}{\{2 + (m+n)\}^3} - \frac{16\pi^4}{\{2 + (m+n)\}} \right\} \right] \\ \int_0^{2\pi} x^5 \left\{ \ln\left(2 \sin \frac{x}{2}\right) \right\} dx &= \pi \int_0^{2\pi} x^4 \left\{ \ln\left(2 \sin \frac{x}{2}\right) \right\} dx + 48\pi^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(1)_m (1)_m (1)_n (1)_n \{(2)_{m+n}\}^3}{(2)_m m! (2)_n n! \{2(3)_{m+n}\}^3} - \\ &- 16\pi^4 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(1)_m (1)_m (1)_n (1)_n \{(2)_{m+n}\}}{(2)_m m! (2)_n n! \{2(3)_{m+n}\}}, \end{aligned}$$

and by employing the integral (67) together with the definition (1) of the multiple hypergeometric functions of the Kampé de Fériet, the desired expression (73) is achieved.

Furthermore, utilizing the result (23), the newly derived summation formula (42) in the right-hand side and the value of integral (20) of Choi and Srivastava in the left-hand side of equation (73), we can establish a new summation formula (74) concerning the double hypergeometric functions of Kampé de Fériet. \square

Proof of the results (71), (72), (75) and (76). Consider,

$$\int_0^{2\pi} x^4 \left\{ \ln(1 - e^{ix}) \right\}^3 dx.$$

Applying the result (7), we derive

$$\begin{aligned} \int_0^{2\pi} x^4 \{\ln(1 - e^{ix})\}^3 dx &= \int_0^{2\pi} x^4 \{\ln(2 \sin \frac{x}{2})\}^3 dx - \frac{3}{4} \int_0^{2\pi} x^6 \{\ln(2 \sin \frac{x}{2})\} dx + \\ &+ \frac{3\pi}{2} \int_0^{2\pi} x^5 \{\ln(2 \sin \frac{x}{2})\} dx - \frac{3\pi^2}{4} \int_0^{2\pi} x^4 \{\ln(2 \sin \frac{x}{2})\} dx + \\ &+ i \left\{ \frac{3}{2} \int_0^{2\pi} x^5 \{\ln(2 \sin \frac{x}{2})\}^2 dx - \frac{3\pi}{2} \int_0^{2\pi} x^4 \{\ln(2 \sin \frac{x}{2})\}^2 dx - \frac{1}{8} \int_0^{2\pi} x^4 (x - \pi)^3 dx \right\}. \end{aligned} \quad (91)$$

Also consider,

$$\int_0^{2\pi} x^4 \{\ln(1 - e^{ix})\}^3 dx.$$

Using result (8), we obtain

$$\int_0^{2\pi} x^4 \{\ln(1 - e^{ix})\}^3 dx = - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(1)_m (1)_m (1)_n (1)_n (1)_p (1)_p}{(2)_m m! (2)_n n! (2)_p p!} \int_0^{2\pi} x^4 e^{i(3+m+n+p)x} dx$$

Evaluating the inner integral yields

$$\begin{aligned} \int_0^{2\pi} x^4 \{\ln(1 - e^{ix})\}^3 dx &= - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(1)_m (1)_m (1)_n (1)_n (1)_p (1)_p}{(2)_m m! (2)_n n! (2)_p p!} \times \\ &\times \left[\frac{32\pi^3}{\{3 + (m + n + p)\}^2} - \frac{48\pi}{\{3 + (m + n + p)\}^4} + i \left(\frac{48\pi^2}{\{3 + (m + n + p)\}^3} - \frac{16\pi^4}{3 + (m + n + p)} \right) \right]. \end{aligned} \quad (92)$$

Now, by separating the real and imaginary components from equations (91) and (92), we obtain

$$\begin{aligned} \int_0^{2\pi} x^4 \{\ln(2 \sin \frac{x}{2})\}^3 dx - \frac{3}{4} \int_0^{2\pi} x^6 \{\ln(2 \sin \frac{x}{2})\} dx + \frac{3\pi}{2} \int_0^{2\pi} x^5 \{\ln(2 \sin \frac{x}{2})\} dx - \\ - \frac{3\pi^2}{4} \int_0^{2\pi} x^4 \{\ln(2 \sin \frac{x}{2})\} dx = - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(1)_m (1)_m (1)_n (1)_n (1)_p (1)_p}{(2)_m m! (2)_n n! (2)_p p!} \times \\ \times \left[\left\{ \frac{32\pi^3}{\{3 + (m + n + p)\}^2} - \frac{48\pi}{\{3 + (m + n + p)\}^4} \right\} \right]. \end{aligned} \quad (93)$$

$$\begin{aligned} \int_0^{2\pi} x^4 \{\ln(2 \sin \frac{x}{2})\}^3 dx &= \frac{3}{4} \int_0^{2\pi} x^6 \{\ln(2 \sin \frac{x}{2})\} dx - \frac{3\pi}{2} \int_0^{2\pi} x^5 \{\ln(2 \sin \frac{x}{2})\} dx + \\ &+ \frac{3\pi^2}{4} \int_0^{2\pi} x^4 \{\ln(2 \sin \frac{x}{2})\} dx - 32\pi^3 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(1)_m (1)_m (1)_n (1)_n (1)_p (1)_p \{(3)_{m+n+p}\}^2}{(2)_m m! (2)_n n! (2)_p p! \{3(4)_{m+n+p}\}^2} + \\ &+ 48\pi \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(1)_m (1)_m (1)_n (1)_n (1)_p (1)_p \{(3)_{m+n+p}\}^4}{(2)_m m! (2)_n n! (2)_p p! \{3(4)_{m+n+p}\}^4}, \end{aligned}$$

and employing the relations (77), (73) and (67), along with the definition (1) of the multiple hypergeometric functions of the Kampé de Fériet, the desired relation (71) is obtained.

Furthermore, by applying the result (23) together with our newly established summation formulas (74), (42), (52), (70) and (54) on the right-hand side and by substituting the integral

(19) of Choi and Srivastava into the left-hand side of equation (71), we derive a potentially new summation identity (72) involving triple hypergeometric functions of the Kampé de Fériet.

$$\begin{aligned} & \frac{3}{2} \int_0^{2\pi} x^5 \left\{ \ln\left(2 \sin \frac{x}{2}\right) \right\}^2 dx - \frac{3\pi}{2} \int_0^{2\pi} x^4 \left\{ \ln\left(2 \sin \frac{x}{2}\right) \right\}^2 dx - \frac{1}{8} \int_0^{2\pi} x^4 (x - \pi)^3 dx \\ &= - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(1)_m (1)_m (1)_n (1)_n (1)_p (1)_p}{(2)_m m! (2)_n n! (2)_p p!} \left[\left\{ \frac{48\pi^2}{\{3 + (m + n + p)\}^3} - \frac{16\pi^4}{\{3 + (m + n + p)\}} \right\} \right], \end{aligned}$$

hence,

$$\begin{aligned} & \frac{3}{2} \int_0^{2\pi} x^5 \left\{ \ln\left(2 \sin \frac{x}{2}\right) \right\}^2 dx = \frac{3\pi}{2} \int_0^{2\pi} x^4 \left\{ \ln\left(2 \sin \frac{x}{2}\right) \right\}^2 dx + \frac{1}{8} \left[\frac{96\pi^8}{35} \right] - \\ & - 48\pi^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(1)_m (1)_m (1)_n (1)_n (1)_p (1)_p \{(3)_{m+n+p}\}^3}{(2)_m m! (2)_n n! (2)_p p! \{3(4)_{m+n+p}\}^3} + \\ & + 16\pi^4 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(1)_m (1)_m (1)_n (1)_n (1)_p (1)_p \{(3)_{m+n+p}\}}{(2)_m m! (2)_n n! (2)_p p! \{3(4)_{m+n+p}\}}, \end{aligned}$$

by substituting the relation (69) and again invoking the definition (1) of multiple hypergeometric functions of Kampé de Fériet, we arrive at the required identity (75).

Moreover, applying the summation relations (52), (70) and (62) on the right-hand side and incorporating the integral (21) of Choi and Srivastava into the left-hand side of equation (75), we derive another new summation formula (76) corresponding to triple hypergeometric functions of Kampé de Fériet. \square

Proof of the results (49), (50) and (79): Consider the auxiliary integral $\int_0^{2\pi} x^2 \{\ln(1 - e^{ix})\} dx$. Employing the results (7) and (8) and equating the real and imaginary components, we obtain the desired expression. Using the definition of the generalized hypergeometric function of one variable, we derive the required result (49). Furthermore, substituting the result (23) into the right-hand side of equation (49), we obtain the result (50).

Next, using the definition of generalized hypergeometric function of one variable, we get the required result (79). \square

Proof of the results (51), (52), (59) and (60): Consider the auxiliary integral $\int_0^{2\pi} x^2 \{\ln(1 - e^{ix})\}^2 dx$. Employing the results (7) and (8) and equating the real and imaginary components, we obtain the desired expression. Using the definition (1) of multiple hypergeometric functions of Kampé de Fériet, we have the required result (51). Again using the result (11) in the left-hand side of equation (51), we obtain the result (52).

Next, using equation (49) and the definition (1) of multiple hypergeometric functions of Kampé de Fériet, we obtain the required result (59). Furthermore, by substitute the equation (23) and our new summation formula (42) in the right-hand side of equation (59), we get the required result (60). \square

Proof of the results (53), (54), (61) and (62): Consider the auxiliary integral $\int_0^{2\pi} x^2 \{\ln(1 - e^{ix})\}^3 dx$. Employing the results (7) and (8) and equating the real and imaginary components, we obtain the desired expression. Using the equations (67), (49) and (59) and also using the definition (1) of

multiple hypergeometric functions of Kampé de Fériet, we obtain the required result (53). Again using the equation (23), our new summation formula (42) and equation (12) in the equation (53), we obtain the result (54).

Next, using equation (51) and the definition (1) of multiple hypergeometric functions of Kampé de Fériet, we have the required result (61). Furthermore, by applying our new summation formula (52) and equation (15) in the equation (61), we derive result (62). \square

Proof of the results (55), (56), (63) and (64): Consider the auxiliary integral $\int_0^{2\pi} x^2 \{\ln(1-e^{ix})\}^4 dx$. Employing the results (7) and (8) and equating the real and imaginary components, we obtain the desired expression. Using equations (69), (61) and (51), along with the definition (1) of multiple hypergeometric functions of Kampé de Fériet, we obtain the required result (55). Furthermore, by applying the new summation formulas (52), (70) and (62) together with equation (13) in equation (55), we obtain the result (56).

Next, using equations (53), (73), (67), (59) and (49), along with the definition (1) of multiple hypergeometric functions of Kampé de Fériet, we obtain the required result (63). Finally, by substituting equation (23) and the new summation formulas (42), (54), and (74) together with equation (16) in equation (63), we obtain the result (64). \square

Proof of the results (57), (58), (65) and (66): Consider the auxiliary integral $\int_0^{2\pi} x^2 \{\ln(1-e^{ix})\}^5 dx$. Employing the results (7) and (8) and equating the real and imaginary components, we obtain the desired expression. Using the equations (71), (63), (53), (77), (73), (67), (59), (49), along with the definition (1) of multiple hypergeometric functions of Kampé de Fériet, we obtain the required result (57). Furthermore, by using the equation (23) and new summation formulas (74), (42), (52), (70), (54), (72) and (64) together with equation (14) in the equation (57), we obtain the result (58).

Next, using equations (55), (75), (69), (61), (51), along with the definition (1) of multiple hypergeometric functions of Kampé de Fériet, we obtain the required result (65). Finally, by applying our new summation formulas (52), (70), (62), (56) and (76) together with equation (17) in equation (65), we obtain the result (66). \square

Proof of the results (26) and (27): Consider the auxiliary integral $\int_0^{2\pi} \{\ln(1-e^{ix})\} dx$. Employing the results (7) and (8) and equating the real and imaginary components, we obtain the required result (26) and (27). \square

Proof of the results (28) and (39): Consider the auxiliary integral $\int_0^{2\pi} \{\ln(1-e^{ix})\}^2 dx$. Employing the results (7) and (8) and equating the real and imaginary components, we obtain the required result (28).

Next, using equation (26), we obtain the result (39). \square

Proof of the results (29), (30) and (40): Consider the auxiliary integral $\int_0^{2\pi} \{\ln(1-e^{ix})\}^3 dx$. Employing the results (7) and (8) and equating the real and imaginary components, we obtain the desired expression. Using the equations (49), (26) and (39), we get the required result (29). Further, by using the equation (23) in the equation (29), we get the required result (30).

Next, using equation (28), we obtain result (40). \square

Proof of the results (31), (32), (41) and (42): Consider the auxiliary integral $\int_0^{2\pi} \{\ln(1-e^{ix})\}^4 dx$. Employing the results (7) and (8) and equating the real and imaginary components, we obtain the desired expression. Using the equations (51), (40), (28), we get the required result (31). Furthermore, by applying the new summation formula (52) in the equation (31), we obtain the result (32).

Next, using equations (29), (59), (49), (39) and (26), we get the required result (41). Finally, using the equations (23) and (10) in the equation (41), we obtain the result (42). \square

Proof of the results (33), (34), (43) and (44): Consider the auxiliary integral $\int_0^{2\pi} \{\ln(1-e^{ix})\}^5 dx$. Employing the results (7) and (8) and equating the real and imaginary components, we obtain the desired expression. Using the equations (53), (41), (29), (67), (59), (49), (39), (26), we get the required result (33). Again using the equation (23) and our new summation formula (42) and (54) in the equation (33), we obtain the result (34).

Next, using equations (31), (61), (51), (40) and (28), we get the required result (43). Finally, using the summation formula (52) and (62) in the equation (43), we obtain the required result (44). \square

Proof of the results (35), (36), (45) and (46): Consider the auxiliary integral $\int_0^{2\pi} \{\ln(1-e^{ix})\}^6 dx$. Employing the results (7) and (8) and equating the real and imaginary components, we obtain the desired expression. Using the equations (55), (43), (31), (69), (61), (51), (40), (28), we get the required result (35). Furthermore, by substituting the summation formulas (52), (70), (62) and (56) in the equation (35), we obtain the result (36).

Next, using equations (33), (63), (53), (41), (29), (73), (67), (59), (49), (39) and (26), we get the required result (45). Finally, using the equation (23) and our new summation formulas (42), (54), (74) and (64) in the equation (45), we obtain the result (46). \square

Proof of the results (37), (38), (47) and (48): Consider the auxiliary integral $\int_0^{2\pi} \{\ln(1-e^{ix})\}^7 dx$. Employing the results (7) and (8) and equating the real and imaginary components, we obtain the desired expression. Using the equations (57), (45), (33), (71), (63), (53), (41), (29), (77), (73), (67), (59), (49), (39) and (26), we get the required result (37). Further, using the equation (23) and our summation formulas (74), (42), (52), (70), (54), (72), (64) and (58) in the equation (37), we obtain the result (38).

Next, using equations (35), (65), (55), (43), (31), (75), (69), (61), (51), (40) and (28), we get the required result (47). Finally, using the summation formulas (52), (70), (62), (56), (76) and (66) in the equation (47), we obtain the result (48). \square

5 Conclusion

In this work, explicit analytical evaluations of several families of log-sine integrals of the form $\int_0^{2\pi} x^m \{\ln(2 \sin \frac{x}{2})\}^n dx$, ($m \in \mathbb{N}_0$ and $n \in \mathbb{N}$), the study presents new closed-form expressions, several previously unreported summation formulas and mixed relations linking generalized hypergeometric functions ${}_3F_2(1)$, ${}_5F_4(1)$ and ${}_7F_6(1)$ with the zeta values $\zeta(3)$, $\zeta(5)$ and $\zeta(7)$. An auxiliary integral discussed in Section 2 (see last paragraph) provides a unified framework

connecting these results and demonstrating the deep interrelation between log-sine integrals, hypergeometric functions and number-theoretic constants. The method developed here can be further extended to other classes of special integrals arising in analytic number theory and mathematical analysis.

Acknowledgements

The authors are grateful to the reviewers for their valuable comments, suggestions and critical remarks which have contributed significantly to improving the paper in its present revised form.

References

- [1] Baboo, M. S. (2017). *Exact Solutions of Outstanding Problems and Novel Proofs Through Hypergeometric Approach*. Ph.D. Thesis, Jamia Millia Islamia (A Central University), New Delhi, India.
- [2] Beumer, M. G. (1961). Some special integrals. *The American Mathematical Monthly*, 68(7), 645–647.
- [3] Borwein, D., & Borwein, J. M. (1995). On an intriguing integral and some series related to $\zeta(4)$. *Proceedings of the American Mathematical Society*, 123(4), 1191–1198.
- [4] Borwein, J. M., Bradley, D. M., Broadhurst, D. J., & Lisoněk, P. (2001). Special values of multiple polylogarithms. *Transactions of the American Mathematical Society*, 353(3), 907–941.
- [5] Borwein, J. M., Broadhurst, D. J., & Kamnitzer, J. (2001). Central binomial sums, multiple Clausen values and zeta values. *Experimental Mathematics*, 10(1), 25–34.
- [6] Choi, J., & Srivastava, H. M. (2011). Explicit evaluations of some families of log-sine and log-cosine integrals. *Integral Transforms and Special Functions*, 22(10), 767–783.
- [7] Choi, J., Cho, Y. J., & Srivastava, H. M. (2009). Log-sine integrals involving series associated with the Zeta function and polylogarithms. *Mathematica Scandinavica*, 105(2), 199–217.
- [8] Cho, Y. J., Jung, M., Choi, J., & Srivastava, H. M. (2006). Closed-form evaluations of definite integrals and associated infinite series involving the Riemann Zeta function. *International Journal of Computer Mathematics*, 83(5–6), 461–472.
- [9] Coffey, M. W. (2003). On some log-cosine integrals related to $\zeta(3)$, $\zeta(4)$ and $\zeta(6)$. *Journal of Computational and Applied Mathematics*, 159(2), 205–215.
- [10] De Doelder, P. J. (1991). On some series containing $\psi(x) - \psi(y)$ and $(\psi(x) - \psi(y))^2$ for certain values of x and y . *Journal of Computational and Applied Mathematics*, 37, 125–141.

- [11] Edwards, H. M. (1974). *Riemann's Zeta Function*. Academic Press, New York.
- [12] Hàì, N. T., Marichev, O. I., & Srivastava, H. M. (1992). A note on the convergence of certain families of multiple hypergeometric series. *Journal of Mathematical Analysis and Applications*, 164(1), 104–115.
- [13] Karatsuba, A. A., & Voronin, S. M. (1992). *The Riemann Zeta-Function*. Walter de Gruyter, New York.
- [14] Lee, T. Y., & Lin, W. (2014). An elementary evaluation of an intriguing integral via Fourier series. *Analysis Mathematica*, 40, 63–67.
- [15] Lewin, L. (1981). *Polylogarithms and Associated Functions*. North-Holland, New York, Amsterdam.
- [16] Qureshi, M. I., & Malik, S. H. (2024). Evaluation of certain families of log-cosine integrals using hypergeometric function approach and applications. *Notes on Number Theory and Discrete Mathematics*, 30(3), 499–515.
- [17] Rainville, E. D. (1971). *Special Functions*. The Macmillan Co. Inc., New York 1960, Reprinted by Chelsea Publ. Co., Bronx, New York.
- [18] Srivastava, H. M., & Daoust, M. C. (1969). Certain generalized Neumann expansions associated with the Kampé de Fériet's function. *Koninklijke Nederlandse Akademie van Wetenschappen Proceedings. Series A. Mathematical Sciences*, 72= *Indagationes Mathematicae*, 31, 449–457.
- [19] Srivastava, H. M., & Karlsson, P. W. (1985). *Multiple Gaussian Hypergeometric Series*. Halsted Press (Ellis Horwood Limited, Chichester), John Wiley, New York / Chichester / Brisbane / Toronto.
- [20] Srivastava, H. M., & Manocha, H. L. (1984). *A Treatise on Generating Functions*. Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York / Chichester / Brisbane / Toronto.
- [21] Srivastava, H. M., & Panda, R. (1975). Some analytic or asymptotic confluent expansions for functions of several variables. *Mathematics of Computation*, 29(132), 1115–1128.
- [22] Srivastava, H. M., & Panda, R. (1976). Some bilateral generating functions for a class of generalized hypergeometric polynomials. *Journal für die Reine und Angewandte Mathematik*, 283/284, 265–274.
- [23] Titchmarsh, E. C. (1986). *The Theory of the Riemann Zeta-Function*. 2nd Edition, Oxford University Press, Oxford.
- [24] Van der Poorten, A. J. (1980). Some wonderful formulas... An introduction to polylogarithms. *Queen's Papers in Pure and Applied Mathematics*, 54, 269–286.