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Construction of generalized bicomplex Leonardo numbers

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Abstract: In this paper, we introduce a new class of bicomplex numbers whose components are expressed in terms of bicomplex Leonardo numbers. The motivation for this study arises from the growing interest in generalizations of well-known integer sequences within hypercomplex number systems, which reveal deeper algebraic and geometric properties. First, we define the bicomplex Leonardo numbers and establish their fundamental recurrence relation. Then, we derive a Binet-like formula, which serves as a powerful analytical tool for exploring further identities and relationships.

By employing this Binet-like representation, we obtain several new results, including summation formulas, d'Ocagne's identity, Catalan's identity, and Cassini's identity for bicomplex Leonardo numbers. These identities not only extend classical number-theoretic properties into the bicomplex domain but also demonstrate structural consistencies across related algebraic systems. Furthermore, we establish an important connection between the Catalan and Cassini identities, revealing an intrinsic relationship that enhances the understanding of their interdependence within the bicomplex setting.

Keywords: Leonardo numbers, Bicomplex numbers, Bicomplex Leonardo numbers.
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1 Introduction

Multicomplex numbers are a natural extension of the complex number system, and corresponding function theory has been studied by mathematicians for many centuries. Several different such extensions have been suggested, most notably the quaternions, which S.W. Hamilton described in 1843 [4], and the bicomplex numbers, which C. Segre described in 1892 with the aim of describing physical problems in a four-dimensional space [14]. However, bicomplex numbers and bicomplex quaternions are commutative, while quaternions are not. Another distinction is that, although bicomplex quaternions do not form a division algebra, quaternions do. In [12], Price created function theory and bicomplex algebra. Bicomplex algebra is a two-dimensional Clifford algebra that has significant applications in many domains, including digital image processing, geometry, and theoretical physics. It also satisfies the commutative rule of multiplication on $\mathbb{C}$ (see [12, 13]).

As it is well known, an element from the space of complex numbers $\mathbb{C}$ is represented as $x_1 + x_2 i$, such that $x_1, x_2 \in \mathbb{R}$, $i^2 = -1$. On the other hand, an element from the space of bicomplex numbers $\mathbb{C}_2$ is written by the basis $1, i, j, ij$ as follows:

$$x = x_1 + x_2 i + x_3 j + x_4 ij, \tag{1}$$

where x_s for $1 \leq s \leq 4$ are real numbers and for the imaginary units i and j it holds that $i^2 = -1$, $j^2 = -1$, $ij = ji$ with $(ij)^2 = 1$.

Table 1. Multiplication table

$\cdot$	1	i	j	ij
1	1	i	j	ij
i	i	-1	ij	$-j$
j	j	ij	-1	$-i$
ij	ij	$-j$	$-i$	1

Note that, it can be seen that $ij \in \mathbb{C}_2$, but $ij \notin \mathbb{C}$. From Eq. (1), it can also be seen that since $\mathbb{C}$ is of dimension two over $\mathbb{R}$, the space of bicomplex numbers $\mathbb{C}_2$ is an algebra over $\mathbb{R}$ with dimension four.

For any two bicomplex numbers $x = x_1 + x_2 i + x_3 j + x_4 ij$ and $y = y_1 + y_2 i + y_3 j + y_4 ij$, addition, multiplication, and scalar multiplication of an element in $\mathbb{C}_2$ by a real number c are given, respectively:

$$\begin{aligned} x + y &= (x_1 + y_1) + (x_2 + y_2)\, i + (x_3 + y_3)\, j + (x_4 + y_4)\, ij, \\ x \cdot y &= (x_1 y_1 - x_2 y_2 - x_3 y_3 + x_4 y_4) + (x_1 y_2 + x_2 y_1 - x_3 y_4 - x_4 y_3)\, i \\ &\quad + (x_1 y_3 + x_3 y_1 - x_2 y_4 - x_4 y_2)\, j + (x_1 y_4 + x_4 y_1 + x_2 y_3 + x_3 y_2)\, ij, \\ cx &= cx_1 + cx_2 i + cx_3 j + cx_4 ij. \end{aligned} \tag{2}$$

$\mathbb{C}$ and $\mathbb{C}_2$ differ significantly in that while the complex numbers form a field, the bicomplex numbers do not because they contain divisors of zero. Thus far, we have covered the fundamentals of bicomplex numbers and provided a synopsis of the bicomplex analysis literature. The reader is also referred to [9, 12], wherein basic research on bicomplex numbers, bicomplex analyses, and related topics have been documented.

Numerous intriguing integer sequences exist. The Fibonacci sequence is the most researched numerical sequence. Numerous studies have been conducted on Fibonacci and Lucas numbers, including [5, 6, 8]. Additionally, quaternions, bicomplex numbers, and hybrid numbers have all been studied using them [1, 3, 7, 10, 18, 19].

The well-known Fibonacci and Lucas sequences are defined by the following recurrence relations: for $n \geq 0$,

$$F_{n+2} = F_{n+1} + F_n,$$
$$L_{n+2} = L_{n+1} + L_n,$$

where $F_0 = 0$, $F_1 = 1$, $L_0 = 2$ and $L_1 = 1$, respectively. The Binet-style formula for the Fibonacci numbers and Lucas numbers is given by

$$F_n = \frac{\phi^n - \psi^n}{\phi - \psi},$$
$$L_n = \phi^n + \psi^n, \tag{3}$$

respectively, where ϕ and ψ are the roots of the characteristic equation $x^2 - x - 1 = 0$.

In this paper, we examine the Leonardo sequence, which shares characteristics with the Fibonacci sequence. It is represented by the nth Leonardo numbers, L_{e_n}. In [2], Catarino and Borges introduced Leonardo numbers and listed some of their properties. The following recurrence relation defines the Leonardo sequence: considering $n \geq 2$,

$$L_{e_n} = L_{e_{n-1}} + L_{e_{n-2}} + 1, \tag{4}$$

with the initial conditions $L_{e_0} = L_{e_1} = 1$. There are large number of sequences indexed in *The Online Encyclopedia of Integer Sequences*, being in this case $\{L_{e_n}\} : A001595$ in [16].

Also, there is an equation following between Leonardo numbers for $n \geq 2$,

$$L_{e_{n+1}} = 2L_{e_n} - L_{e_{n-2}}. \tag{5}$$

The the Binet-like formula of the Leonardo numbers is

$$L_{e_n} = \frac{2\alpha^{n+1} - 2\beta^{n+1} - \alpha + \beta}{\alpha - \beta}, \tag{6}$$

where α and β are roots of characteristic equation $x^3 - 2x^2 + 1 = 0$.

From the Binet-like formula, the relationship between Leonardo and Fibonacci numbers is

$$L_{e_n} = 2F_{n+1} - 1, \tag{7}$$

where F_n is nth Fibonacci number.

For Leonardo numbers, Catarino *et al.* obtained the identities of Cassini, Catalan and d'Ocagne in [2]. Additionally, they demonstrated the matrix representation of Leonardo numbers and the two-dimensional recurrence relations. The generalised Leonardo numbers that Shannon defined in [15] are regarded as both Horadam's generalised sequence and Asveld's extension.

In [17], Turan, Özkaldı Karakuş, and Nurkan introduced bicomplex Leonardo numbers and they gave some important properties. Then, they obtained generating function, Binet-like formula, summation formulas, Catalan's identity, Cassini's identity, and other some identities.

For $n \geqslant 1$, the nth bicomplex Leonardo numbers are defined by

$$\mathbb{BL}_{e_n} = L_{e_n} + L_{e_{n+1}} i + L_{e_{n+2}} j + L_{e_{n+3}} ij. \tag{8}$$

Note that, throughout this paper, we denote nth bicomplex Leonardo numbers with $\mathbb{BL}_{e_n}$.

From the recurrence relation (4) and the definition of bicomplex Leonardo numbers (8), for $n \geq 2$ we get

$$\mathbb{BL}_{e_n} = \mathbb{BL}_{e_{n-1}} + \mathbb{BL}_{e_{n-2}} + C,$$

where $C = 1 + i + j + ij$. Thoughout this paper $1 + i + j + ij$ expression will be denoted by C. Also initial conditions are $\mathbb{BL}_{e_0} = 1 + i + 3j + 5ij$ and $\mathbb{BL}_{e_1} = 1 + 3i + 5j + 9ij$.

For any integer $n \geqslant 0$, the Binet-like formula for the bicomplex Leonardo numbers $\mathbb{BL}_{e_n}$ is

$$\mathbb{BL}_{e_n} = 2\left(\frac{\overline{\alpha}\alpha^{n+1} - \overline{\beta}\beta^{n+1}}{\alpha - \beta}\right) - C, \tag{9}$$

where α and β are roots of characteristic equation $x^3 - 2x^2 + 1 = 0$, $\overline{\alpha} = 1 + \alpha i + \alpha^2 j + \alpha^3 ij$ and $\overline{\beta} = 1 + \beta i + \beta^2 j + \beta^3 ij$.

Finally, Polatlı examined hybrid numbers Fibonacci and Lucas hybrid coefficients in [11]. Then, he gave the Binet-like formulas, generating functions, exponential generating functions for these numbers. After that he defined an associate matrix for these numbers. In addition, using this matrix, he presented two different versions of Cassini identity of these numbers.

2 Bicomplex numbers with bicomplex Leonardo number coefficients

In this section, we first define bicomplex numbers with bicomplex Leonardo number coefficients. After that, we get Binet-like formulas, exponential generating function, some summation formulas Catalan identity, Cassini identity, d'Ocagne's identity and Honsberger identity for these numbers.

Definition 2.1. *For $n \geq 0$, the nth term of bicomplex numbers with bicomplex Leonardo number coefficients is given as following*

$$\mathbb{L}_{e_n} = \mathbb{BL}_{e_n} + \mathbb{BL}_{e_{n+1}} i + \mathbb{BL}_{e_{n+2}} j + \mathbb{BL}_{e_{n+3}} ij. \tag{10}$$

Remark 2.1. *If we expand the definition of $\mathbb{L}_{e_n}$, we obtain:*

$$\mathbb{L}_{e_n} = L_{e_n} - L_{e_{n+2}} - L_{e_{n+4}} + L_{e_{n+6}} + 2(L_{e_{n+1}} - L_{e_{n+5}})i + 2(L_{e_{n+2}} - L_{e_{n+4}})j + 4L_{e_{n+3}} ij,$$

respectively.

For $n \geq 0$, we can see that:

$$\mathbb{L}_{e_{n+2}} - C^2 = \mathbb{L}_{e_{n+1}} + \mathbb{L}_{e_n}, \tag{11}$$

where $C = 1 + i + j + ij$.

Theorem 2.1. *For $n \geq 0$, the Binet-like formula of bicomplex numbers with bicomplex Leonardo numbers coefficients are given as following*

$$\mathbb{L}_{e_n} = 2\left(\frac{(\overline{\alpha})^2\alpha^{n+1} - (\overline{\beta})^2\beta^{n+1}}{\alpha - \beta}\right) - C^2, \tag{12}$$

respectively, where $\overline{\alpha} = 1 + \alpha i + \alpha^2 j + \alpha^3 k$, $\overline{\beta} = 1 + \beta i + \beta^2 j + \beta^3 k$ and $C = 1 + i + j + ij$.

Proof. Using the definition of bicomplex numbers with bicomplex Leonardo number coefficients (10) and the Binet-like formula for bicomplex Leonardo numbers (9), we get

$$\begin{aligned}
\mathbb{L}_{e_n} &= BL_{e_n} + BL_{e_{n+1}}i + BL_{e_{n+2}}j + BL_{e_{n+3}}ij, \\
&= \left(2\left(\frac{\overline{\alpha}\alpha^{n+1} - \overline{\beta}\beta^{n+1}}{\alpha - \beta}\right) - C\right) + i\left(2\left(\frac{\overline{\alpha}\alpha^{n+2} - \overline{\beta}\beta^{n+2}}{\alpha - \beta}\right) - C\right), \\
&\quad + j\left(2\left(\frac{\overline{\alpha}\alpha^{n+3} - \overline{\beta}\beta^{n+3}}{\alpha - \beta}\right) - C\right) + ij\left(2\left(\frac{\overline{\alpha}\alpha^{n+1} - \overline{\beta}\beta^{n+1}}{\alpha - \beta}\right) - C\right), \\
&= 2\left(\frac{(\overline{\alpha})^2\alpha^{n+1} - (\overline{\beta})^2\beta^{n+1}}{\alpha - \beta}\right) - C^2,
\end{aligned}$$

where $\overline{\alpha} = 1 + \alpha i + \alpha^2 j + \alpha^3 k$, $\overline{\beta} = 1 + \beta i + \beta^2 j + \beta^3 k$ and $C = 1 + i + j + ij$. □

Theorem 2.2. *Exponential generating functions of $\mathbb{L}_{e_n}$ are given as following*

$$\sum_{n\geq 0} \mathbb{L}_{e_n} \frac{x^n}{n!} = \frac{2\overline{\alpha}^2\alpha e^{\alpha x} - 2\overline{\beta}^2\beta e^{\beta x}}{\alpha - \beta} - C^2 e^x. \tag{13}$$

Proof. By using the Binet-like formula for bicomplex numbers with bicomplex Leonardo number coefficients (12), we can obtain

$$\begin{aligned}
\sum_{n\geq 0} \mathbb{L}_{e_n} \frac{x^n}{n!} &= \sum_{n\geq 0} \left(2\left(\frac{(\overline{\alpha})^2\alpha^{n+1} - (\overline{\beta})^2\beta^{n+1}}{\alpha - \beta}\right) - C^2\right)\frac{x^n}{n!}, \\
&= \frac{2\overline{\alpha}^2\alpha e^{\alpha x} - 2\overline{\beta}^2\beta e^{\beta x}}{\alpha - \beta} - C^2 e^x,
\end{aligned}$$

where $C = 1 + i + j + ij$. □

Theorem 2.3. *The following formulas containing the n-th term of bicomplex numbers with bicomplex Leonardo number coefficients are valid:*

(i)

$$\sum_{k=0}^{n} \mathbb{L}_{e_k} = \mathbb{L}_{e_{n+2}} + (59 + 42i + 20j - 36ij) - C^2(n+1),$$

where $C = 1 + i + j + ij$.

(ii)

$$\sum_{k=0}^{n} \binom{n}{k} \mathbb{L}_{e_k} = \mathbb{L}_{e_{2n}} - C^2(2^n - 1).$$

Proof. We obtain the proofs of (i) and (ii).

(i) From Equation (11), we can write as the following

$$\begin{aligned}
\mathbb{L}_{e_0} &= \mathbb{L}_{e_2} - \mathbb{L}_{e_1} - C^2, \\
\mathbb{L}_{e_1} &= \mathbb{L}_{e_3} - \mathbb{L}_{e_2} - C^2, \\
\mathbb{L}_{e_2} &= \mathbb{L}_{e_4} - \mathbb{L}_{e_3} - C^2, \\
&\vdots \\
\mathbb{L}_{e_{n-1}} &= \mathbb{L}_{e_{n+1}} - \mathbb{L}_{e_n} - C^2, \\
\mathbb{L}_{e_n} &= \mathbb{L}_{e_{n+2}} - \mathbb{L}_{e_{n+1}} - C^2.
\end{aligned}$$

If we add above equations side by side, then we get

$$\sum_{k=0}^{n} \mathbb{L}_{e_k} = \mathbb{L}_{e_{n+2}} + (59 + 42i + 20j - 36ij) - C^2(n+1),$$

where $C = 1 + i + j + ij$.

(ii) By using the Binet-like formula for bicomplex numbers with bicomplex Leonardo number coefficients (12), we can obtain

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} \mathbb{L}_{e_k} &= \sum_{k=0}^{n} \binom{n}{k} \left(2\left(\frac{(\overline{\alpha})^2 \alpha^{k+1} - (\overline{\beta})^2 \beta^{k+1}}{\alpha - \beta} \right) - C^2 \right), \\
&= \frac{2(\overline{\alpha})^2 \alpha}{\alpha - \beta} \sum_{k=0}^{n} \binom{n}{k} \alpha^k - \frac{2(\overline{\beta})^2 \beta}{\alpha - \beta} \sum_{k=0}^{n} \binom{n}{k} \beta^k - \sum_{k=0}^{n} \binom{n}{k} C^2, \\
&= \left(\frac{2(\overline{\alpha})^2 \alpha^{2n+1} - 2(\overline{\beta})^2 \beta^{2n+1}}{\alpha - \beta} \right) - C^2 2^n, \\
&= \mathbb{L}_{e_{2n}} - C^2(2^n - 1),
\end{aligned}$$

where $C = 1 + i + j + ij$. □

Theorem 2.4. ***(Catalan Identity)*** *For positive integers n and r with $n \geq r$; let $\mathbb{L}_{e_n}$ be n-th bicomplex numbers with bicomplex Leonardo number coefficients. Then, Catalan identity is as follows:*

$$\begin{aligned}
\mathbb{L}_{e_n}^2 - \mathbb{L}_{e_{n+r}} \mathbb{L}_{e_{n-r}} &= \frac{(-144i - 108)(-1)^{n+1}}{5} \left(\alpha^r \beta^{-r} + \alpha^{-r} \beta^r - 2 \right) \\
&+ \quad C^2 \left(-2\mathbb{L}_{e_n} + \mathbb{L}_{e_{n+r}} - \mathbb{L}_{e_{n-r}} \right), \qquad (14)
\end{aligned}$$

where $\overline{\alpha} = 1 + \alpha i + \alpha^2 j + \alpha^3 k$, $\overline{\beta} = 1 + \beta i + \beta^2 j + \beta^3 k$, $\overline{\alpha}\overline{\beta} = \overline{\beta}\overline{\alpha}$ and $C = 1 + i + j + ij$.

Proof. If we use the Binet-like formula for bicomplex numbers with bicomplex Leonardo Number coefficients (12), we have

$$\begin{aligned}
LHS = &\left(2\left(\frac{(\overline{\alpha})^2 \alpha^{n+1} - (\overline{\beta})^2 \beta^{n+1}}{\alpha - \beta} \right) - C^2 \right) \left(2\left(\frac{(\overline{\alpha})^2 \alpha^{n+1} - (\overline{\beta})^2 \beta^{n+1}}{\alpha - \beta} \right) - C^2 \right), \\
&- \left(2\left(\frac{(\overline{\alpha})^2 \alpha^{n+r+1} - (\overline{\beta})^2 \beta^{n+r+1}}{\alpha - \beta} \right) - C^2 \right) \left(2\left(\frac{(\overline{\alpha})^2 \alpha^{n-r+1} - (\overline{\beta})^2 \beta^{n-r+1}}{\alpha - \beta} \right) - C^2 \right), \\
= &\frac{(-144i - 108)(-1)^{n+1}}{5} \left(\alpha^r \beta^{-r} + \alpha^{-r} \beta^r - 2 \right) + C^2 \left(-2\mathbb{L}_{e_n} + \mathbb{L}_{e_{n+r}} - \mathbb{L}_{e_{n-r}} \right),
\end{aligned}$$

where $\overline{\alpha} = 1 + \alpha i + \alpha^2 j + \alpha^3 k$, $\overline{\beta} = 1 + \beta i + \beta^2 j + \beta^3 k$, $\overline{\alpha}\overline{\beta} = \overline{\beta}\overline{\alpha}$ and $C = 1 + i + j + ij$. □

Remark that in the case $r = 1$ in (14), it reduces to Cassini's identity of the bicomplex numbers with bicomplex Leonardo number coefficients.

Theorem 2.5. ***(Cassini Identity)*** *Let $\mathbb{L}_{e_n}$ be n-th bicomplex numbers with bicomplex Leonardo Number coefficients. Then, Cassini identity is as follows: For $n \geq 1$;*

$$\mathbb{L}_{e_n}^2 - \mathbb{L}_{e_{n+1}}\mathbb{L}_{e_{n-1}} = (144i + 108)(-1)^{n+1} + C^2\left(-2\mathbb{L}_{e_n} + \mathbb{L}_{e_{n+r}} - \mathbb{L}_{e_{n-r}}\right),$$

where $\overline{\alpha} = 1 + \alpha i + \alpha^2 j + \alpha^3 k$, $\overline{\beta} = 1 + \beta i + \beta^2 j + \beta^3 k$, $\overline{\alpha}\overline{\beta} = \overline{\beta}\overline{\alpha}$ and $C = 1 + i + j + ij$.

Theorem 2.6. ***(D'Ocagne's Identity)*** *For m≥1 and n≥1; let $\mathbb{L}_{e_n}$ be n-th bicomplex numbers with bicomplex Leonardo number coefficients. For this number D'Ocagne's identity is as follows:*

$$\mathbb{L}_{e_n}\mathbb{L}_{e_{m+1}} - \mathbb{L}_{e_{n+1}}\mathbb{L}_{e_m} = \frac{\sqrt{5}}{5}(-144i - 108)(\alpha^{n+1}\beta^{m+1} - \alpha^{m+1}\beta^{n+1}) + C^2(\mathbb{L}_{e_{n-1}} - \mathbb{L}_{e_{m-1}}),$$

where $\overline{\alpha} = 1 + \alpha i + \alpha^2 j + \alpha^3 k$, $\overline{\beta} = 1 + \beta i + \beta^2 j + \beta^3 k$, $\overline{\alpha}\overline{\beta} = \overline{\beta}\overline{\alpha}$ and $C = 1 + i + j + ij$.

Proof. If we use bicomplex numbers with bicomplex Leonardo Number coefficients (12) and bicomplex numbers with bicomplex Leonardo Number coefficients recurrence relation (11), we have

$$\begin{aligned} LHS &= \left(2\left(\frac{(\overline{\alpha})^2\alpha^{n+1} - (\overline{\beta})^2\beta^{n+1}}{\alpha - \beta}\right) - C^2\right)\left(2\left(\frac{(\overline{\alpha})^2\alpha^{m+2} - (\overline{\beta})^2\beta^{m+2}}{\alpha - \beta}\right) - C^2\right), \\ &- \left(2\left(\frac{(\overline{\alpha})^2\alpha^{n+2} - (\overline{\beta})^2\beta^{n+2}}{\alpha - \beta}\right) - C^2\right)\left(2\left(\frac{(\overline{\alpha})^2\alpha^{m+1} - (\overline{\beta})^2\beta^{m+1}}{\alpha - \beta}\right) - C^2\right), \\ &= \frac{\sqrt{5}}{5}(-144i - 108)(\alpha^{n+1}\beta^{m+1} - \alpha^{m+1}\beta^{n+1}) + C^2(\mathbb{L}_{e_{n-1}} - \mathbb{L}_{e_{m-1}}), \end{aligned}$$

where $\overline{\alpha} = 1 + \alpha i + \alpha^2 j + \alpha^3 k$, $\overline{\beta} = 1 + \beta i + \beta^2 j + \beta^3 k$, $\overline{\alpha}\overline{\beta} = \overline{\beta}\overline{\alpha}$ and $C = 1 + i + j + ij$. □

Theorem 2.7. ***(Honsberger Identity)*** *Let $\mathbb{L}_{e_n}$ be n-th bicomplex numbers with bicomplex Leonardo Number coefficients. For this number Honsberger identity is as follows: For $n, m \geq 0$,*

$$\begin{aligned} \mathbb{L}_{e_n}\mathbb{L}_{e_m} + \mathbb{L}_{e_{n+1}}\mathbb{L}_{e_{m+1}} = &\frac{4(\overline{\alpha})^2\alpha^{n+m+2}(1 + \alpha^2) + 4(\overline{\beta})^4\beta^{n+m+2}(1 + \beta^2)}{5} \\ &- C^2(\mathbb{L}_{e_{n+2}} + \mathbb{L}_{e_{m+2}}), \end{aligned}$$

where $\overline{\alpha}\overline{\beta} = \overline{\beta}\overline{\alpha}$, $\overline{\alpha} = 1 + \alpha i + \alpha^2 j + \alpha^3 k$, $\overline{\beta} = 1 + \beta i + \beta^2 j + \beta^3 k$ and $C = 1 + i + j + ij$.

Proof. If we use the Binet-like formula for bicomplex numbers with bicomplex Leonardo number coefficients (12)and bicomplex numbers with bicomplex Leonardo number coefficients recurrence relation (11), we have

$$\begin{aligned} LHS &= \left(2\left(\frac{(\overline{\alpha})^2\alpha^{n+1} - (\overline{\beta})^2\beta^{n+1}}{\alpha - \beta}\right) - C^2\right)\left(2\left(\frac{(\overline{\alpha})^2\alpha^{m+1} - (\overline{\beta})^2\beta^{m+1}}{\alpha - \beta}\right) - C^2\right), \\ &+ \left(2\left(\frac{(\overline{\alpha})^2\alpha^{n+2} - (\overline{\beta})^2\beta^{n+2}}{\alpha - \beta}\right) - C^2\right)\left(2\left(\frac{(\overline{\alpha})^2\alpha^{m+1} - (\overline{\beta})^2\beta^{n+2}}{\alpha - \beta}\right) - C^2\right), \\ &= \frac{4(\overline{\alpha})^2\alpha^{n+m+2}(1 + \alpha^2) + 4(\overline{\beta})^4\beta^{n+m+2}(1 + \beta^2)}{5} - C^2(\mathbb{L}_{e_{n+2}} + \mathbb{L}_{e_{m+2}}), \end{aligned}$$

where $\overline{\alpha}\overline{\beta} = \overline{\beta}\overline{\alpha}$, $\overline{\alpha} = 1 + \alpha i + \alpha^2 j + \alpha^3 k$ and $\overline{\beta} = 1 + \beta i + \beta^2 j + \beta^3 k$. □

3 Conclusion

In the present paper, bicomplex numbers with bicomplex Leonardo number coefficients have been introduces. First of all the recurrence relation, generating function and exponential generating function for these numbers have been obtained. Then summation formulas for these numbers have been provided. Furthermore, Catalan identity, Cassini identity, d'Ocagne's identity, Honsberger identity and some interesting properties have been given.

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