Computing a maximal clique of graphs of cofinite submonoids

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Abstract: A graph $G_{\mathfrak{S}}$ is called an $\mathfrak{S}(\tau, \mathfrak{e})$-graph if there exists a numerical semigroup $\mathfrak{S}$ with multiplicity τ and embedding dimension $\mathfrak{e}$ such that $V(G_{\mathfrak{S}}) = \{v_\alpha : \alpha \in \mathbb{N}_0 \setminus \mathfrak{S}\}$ and $E(G_{\mathfrak{S}}) = \{v_\alpha v_\beta \Leftrightarrow \alpha + \beta \in \mathfrak{S}\}$. In this article, we give an algorithmic way to compute the clique number and the minimum degree of $\mathfrak{S}(\tau, 3)$-graphs, where $\mathfrak{S}$ is a class of symmetric numerical semigroups with arbitrary multiplicity and embedding dimension 3. On this basis, we give some bounds for the atom bond connectivity index of graphs $G_{\mathfrak{S}}$ in terms of Randić connectivity index, the first and second Zagreb indices, the maximum and minimum degrees, and the clique number.

Keywords: Symmetric numerical semigroup, Gaps, Multiplicity, Frobenius number, Embedding dimension, Complete graphs, Minimum and maximum degree, Clique number.

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1 Introduction

Graph theory is a significant branch of mathematics that explores the connections between pairs of objects through the representation of graphs. Presently, graph theory finds extensive applications in engineering, computer science, social networking, and various scientific disciplines [3, 26, 33]. The fusion of graph theory with algebraic structures has spurred fresh investigations in mathematics, delving into the interplay between the structures of algebraic objects and graphs. This concept involves examining how properties of both graphs and algebraic structures interact. Historically, researchers have linked graphs with diverse algebraic structures, contributing to a rich tapestry of mathematical exploration [21, 32, 35, 39]. The extensively explored concepts within this domain include the Cayley graph, nilpotent graphs, and the zero-divisor graph [2, 20, 25, 27]. Recently, Binyamin *et al.* [1, 6, 7, 30] introduced a graph associated with numerical semigroups and ideals of numerical semigroups, investigating various properties of this graph. In [28], Poonkuzhali *et al.* obtained the dominated chromatic number for various networks. Biswas *et al.* identified the rings R for which the graph $G(R)$ exhibits planarity in [8]. Additionally, they characterize the class of rings for which $G(R)$ is toroidal. Researchers have been linking graphs to a variety of algebraic structures over the past 20 years, which has opened up several avenues for studying algebraic structures using graph-theoretical features and vice versa. The term "clique" in graph theory is credited to the work of Duncan Luce and Albert D. Perry, as presented in their 1949 publication, "A method of matrix analysis of group structure". In this paper, they employed the term to refer to a complete subgraph within social networks, indicating a group of individuals who are all acquainted with one another (a "clique" in the conventional social context). Classical graph theoretical descriptors, such as the Atom-Bond Connectivity (ABC) index, have proven effective in correlating molecular structure with thermodynamic stability and chemical behavior. In this context, graphs of numerical semigroups represent a distinctive and structurally rich family of graphs with significant potential for chemical modeling.

A graph is an ordered pair of a vertex set $V(G)$ and an edge set $E(G)$; their cardinality is said to be the order and size of the graph, respectively. Two vertices are said to be neighbors of each other if they are connected by an edge, and the set of all neighbors of v_i is said to be the neighborhood of v_i. The number of edges incident on a vertex v_i is known as the degree of v_i, denoted by d_{v_i}. Moreover $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degree of a graph G. If there is an edge connecting every pair of unique vertices in a graph, then the graph is said to be complete. In a graph G, a clique $cl(G)$ is a complete subgraph; the maximum clique is the largest possible clique in G [31]. The clique number, $\omega(G)$, is the size of the largest clique in G. It holds significance across various fields due to its diverse applications in social network analysis, bioinformatics, combinatorial optimization, communication networks, cryptography, chemoinformatics, computer vision, and image processing.

Chemical graph theory is a field within mathematical chemistry that utilizes graph theory to mathematically model various chemical phenomena. Recent studies in graph theory and chemical graph analysis highlight structural parameters such as mean subtree order, subtrees with bounded diameter, and subpath numbers, which capture connectivity and branching patterns in trees and molecular graphs [11, 36, 38]. Research on subtree enumeration [37, 38] and subpath

numbers [23, 24] illustrates how local structures influence global graph invariants. Motivated by these developments, we study the graphs $G_\mathfrak{S}$ associated with symmetric numerical semigroups, analyzing clique numbers and connectivity indices, thereby linking algebraic structures to broader combinatorial and chemical graph-theoretic contexts. The graph invariants utilized for chemical investigations are referred to as topological indices [9, 10, 13]. A chemical compound can be conceptualized as a graph where atoms are represented by vertices and bonds by edges. These graphs are commonly referred to as chemical graphs.

In 1975, Randić [29] proposed the branching index

$$R(G) = \sum_{v_1 v_2 \in E(G)} \sqrt{\frac{1}{d_{v_1} d_{v_2}}},$$

which is also called connectivity index and Randić index of a graph G. Estrada *et al.* [16] defined the atom-bond connectivity index of a graph G as:

$$ABC(G) = \sum_{v_1 v_2 \in E(G)} \sqrt{\frac{d_{v_1} + d_{v_2} - 2}{d_{v_1} d_{v_2}}}.$$

Many authors work on atom-bond connectivity index [5, 12, 14, 15, 22]. Two type of Randić connectivity indices are defined as [34]

$$R_{-1}(G) = R_{-1} = \sum_{v_1 v_2 \in E(G)} \frac{1}{d_{v_1} d_{v_2}}.$$

and

$$R_{\frac{-1}{4}}(G) = R_{\frac{-1}{4}} = \sum_{v_1 v_2 \in E(G)} \frac{1}{(d_{v_1} d_{v_2})^{\frac{1}{4}}}.$$

The first Zagreb index is denoted by $M_1(G)$ and is defined as [19]

$$M_1(G) = M_1 = \sum_{v_1 v_2 \in E(G)} (d_{v_1} + d_{v_2}).$$

The second Zagreb index is denoted by $M_2(G)$ and is defined as [19]

$$M_2(G) = M_2 = \sum_{v_1 v_2 \in E(G)} d_{v_1} d_{v_2}.$$

A numerical semigroup $\mathfrak{S}$ is a cofinite submonoid of the additive monoid of nonnegative integers $\mathbb{N}$. The smallest positive integer belonging to set $\mathfrak{S}$ is termed the multiplicity of $\mathfrak{S}$, represented as τ. The complement set $\mathbb{N} \setminus \mathfrak{S}$ is termed the gap set of $\mathfrak{S}$, denoted as $g(\mathfrak{S})$. The numerical semigroup $\mathfrak{S}$ is symmetric if and only if the greatest element $\mathfrak{F}$ (Frobenius number) of $g(\mathfrak{S})$ satisfies the condition that for every $a \in g(\mathfrak{S})$, the element $\mathfrak{F} - a \in \mathfrak{S}$.

The remainder of this paper is organized as follows. In Section 2, we associate graphs with symmetric numerical semigroups and find the maximal possible clique of $\mathfrak{S}(m, 3)$-graphs. In Section 3, we compute the minimum degree of $\mathfrak{S}(\tau, \mathfrak{e})$-graphs and give some bounds for the atom bound connective index. The paper concludes in Section 4.

2 Maximal clique of graphs of symmetric cofinite submonoids

Throughout this section, we associate graphs with an arbitrary class of symmetric numerical semigroups (for details see [8]) and compute the clique number of $\mathfrak{S}(\tau, 3)$-graphs.

Lemma 2.1 ([18]). *Let $\mathfrak{S} = \langle \tau, \tau + 1, \ldots, \tau + t \rangle$ be a numerical monoid with $1 \leq t < \tau$. Then $u \in \mathfrak{S}$ if and only if $u = k\tau + i$ with $k \in \mathbb{N}$ and $i \in \{0, \ldots, kt\}$.*

Proposition 2.2. *The induced subgraphs $G_{\mathcal{A}}, G_{\mathcal{B}}, G_{\mathcal{C}}$ of $G_{\mathfrak{S}}$ are cliques, where $\mathfrak{S} = \langle \tau, \tau + 1, h\tau + 2h + 2, \ldots, h\tau + (\tau - 1) \rangle$, with $\tau = 2h + 3, h \geq 1$ and,*

$$\mathcal{A} = \{\mathfrak{F} - p\tau : 0 \leq p \leq h\},$$

$$\mathcal{B} = \{\mathfrak{F} - (h\tau + 2h + 2)\},$$

$$\mathcal{C} = \{\mathfrak{F} - (x\tau + y(\tau + 1)) : 0 \leq x \leq \lceil \frac{h}{2} \rceil, 1 \leq y \leq h \text{ or } x = h, 1 \leq y \leq h - 1, \text{ or}$$

$$x = \lceil \frac{h}{2} \rceil + l, 1 \leq y \leq h - l, l = 1, \ldots, \lfloor \frac{h}{2} \rfloor - 1\}.$$

Proof. We have to show that the elements of $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are the gaps of $\mathfrak{S}$, and the sum of elements of $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$, respectively, is an element of $\mathfrak{S}$. For this, we have the following cases:

- Case 1: Note that for any $j \geq 0$, $j\tau \in \mathfrak{S}$, therefore for $0 \leq p \leq h$, $p\tau \in \mathfrak{S}$. Since $\mathfrak{S}$ is symmetric, therefore $\mathfrak{F} - p\tau \notin \mathfrak{S}$. Note that for maximum value of p,

$$\mathfrak{F} - p\tau = h\tau + 2h + 1 > 0,$$

therefore for any values of p, $\mathfrak{F} - p\tau > 0$. This implies $\mathcal{A} \subseteq V(G_{\mathfrak{S}})$, and $G_{\mathcal{A}}$ is induced subgraph of $G_{\mathfrak{S}}$. To prove that $G_{\mathcal{A}}$ is clique, we need to show that $r_1 + r_2 \in \mathfrak{S}$, for all $r_1, r_2 \in \mathcal{A}$.
Let $r_1 = \mathfrak{F} - p_1\tau$ and $r_2 = \mathfrak{F} - p_2\tau$, where $0 \leq p_1 \neq p_2 \leq h$ then

$$r_1 + r_2 = 2\mathfrak{F} - (p_1 + p_2)\tau = 4h\tau + 4h + 2 - (p_1 + p_2)\tau.$$

Now, $4h\tau + 4h + 2 - (p_1 + p_2)\tau - \mathfrak{F} = (2h - (p_1 + p_2))\tau + 2h + 1$. Since $1 \leq p_1 + p_2 \leq 2h - 1$, therefore $(2h - (p_1 + p_2))\tau + 2h + 1 > 0$. This implies $4h\tau + 4h + 2 - (p_1 + p_2)\tau > \mathfrak{F}$. So, $r_1 + r_2 \in \mathfrak{S}$. Hence $G_{\mathcal{A}}$ is a complete subgraph.

- Case 2: Note that for any $x, y \geq 0$, $x\tau + y(\tau + 1) \in \mathfrak{S}$, therefore for $0 \leq x \leq (\lceil \frac{h}{2} \rceil)$, $1 \leq y \leq h$ or $x = h, 1 \leq y \leq h - 1$, or $x = \lceil \frac{h}{2} \rceil + l, 1 \leq y \leq h - l, l = 1, \ldots, \lfloor \frac{h}{2} \rfloor - 1$, $x\tau + y(\tau + 1) \in \mathfrak{S}$. Since $\mathfrak{S}$ is symmetric, therefore $\mathfrak{F} - (x\tau + y(\tau + 1)) \notin \mathfrak{S}$. Note that for maximum values of x and y,

$$\mathfrak{F} - (x\tau + y(\tau + 1)) = \mathfrak{F} - ((\lceil \frac{h}{2} \rceil)\tau + (h)(\tau + 1)) = (h - \lceil \frac{h}{2} \rceil)\tau + h + 1 > 0,$$

$$\mathfrak{F} - ((h)\tau + (h - 1)(\tau + 1)) = \tau + h + 2 > 0,$$

$$\mathfrak{F} - ((\lceil \frac{h}{2} \rceil + l)\tau + (h - l)(\tau + 1)) = (h - \lceil \frac{h}{2} \rceil)\tau + h + l + 1 > 0.$$

Therefore for any values of x and y, $\mathfrak{F} - (x\tau + y(\tau + 1)) > 0$. This implies $\mathcal{C} \subseteq V(G_{\mathfrak{S}})$, and $G_{\mathcal{C}}$ is induced subgraph of $G_{\mathfrak{S}}$. To prove that $G_{\mathcal{C}}$ is clique, we need to show that $r_1 + r_2 \in \mathfrak{S}$, for all $r_1, r_2 \in \mathcal{C}$.

Let $r_1 = \mathfrak{F} - (x_1\tau + y_1(\tau + 1))$ and $r_2 = \mathfrak{F} - (x_2\tau + y_2(\tau + 1))$, then

$$r_1 + r_2 = 2\mathfrak{F} - (x_1 + x_2)\tau - (y_1 + y_2)(\tau + 1) = (3q - (x_1 + x_2 + y_1 + y_2))\tau$$

$$+ h\tau + 2h + 2 + 2h - (y_1 + y_2).$$

(1) **If $0 \leq x \leq \lceil \frac{h}{2} \rceil$, $1 \leq y \leq h$:**

Note that $3h - (x_1 + x_2 + y_1 + y_2) > 0$, because $3 \leq (x_1 + x_2 + y_1 + y_2) \leq 3h - 1$. Therefore $(3h - (x_1 + x_2 + y_1 + y_2))\tau \in \mathfrak{S}$. If $2h - (y_1 + y_2) \leq \tau - h - 1$, then consider

$$2h - (y_1 + y_2) = \tau - h - 1 - v, \text{ where } 0 \leq v \leq h + 1.$$

This implies

$$(h\tau + 2h + 2) + (2h - (y_1 + y_2)) = (h + 1)\tau + (h + 1 - v).$$

From Lemma 2.1, we get $(h + 1)\tau + (h + 1 - v) \in \mathfrak{S}$. So, $r_1 + r_2 \in \mathfrak{S}$. Now, if $2h - (y_1 + y_2) > \tau - h - 1$, then consider

$$2h - (y_1 + y_2) - (\tau - h - 1) = 3h + 1 - \tau - (y_1 + y_2).$$

We can write

$$r_1 + r_2 = [(3h - (x_1 + x_2 + y_1 + y_2))\tau + (3h + 1 - \tau - (y_1 + y_2))] + [(h\tau + 2h + 2) + (\tau - h - 1)].$$

Note that

$$3h - (x_1 + x_2 + y_1 + y_2) - (3h + 1 - \tau - (y_1 + y_2)) = \tau - (x_1 + x_2 + 1).$$

Since $4 \leq x_1 + x_2 + 1 \leq h + 1 < 2h + 3$, therefore $\tau - (x_1 + x_2 + 1) > 0$. By Lemma 2.1, $(3h - (x_1 + x_2 + y_1 + y_2))\tau + (3h + 1 - \tau - (y_1 + y_2)) \in \mathfrak{S}$. Also $(h\tau + 2h + 2) + (\tau - h - 1) = (h + 1)\tau + (h + 1) \in \mathfrak{S}$. Consequently, we get $r_1 + r_2 \in \mathfrak{S}$.

(2) **If $x = \lceil \frac{h}{2} \rceil + l$, $1 \leq y \leq h - l$, $l = 1, \ldots, \lfloor \frac{h}{2} \rfloor - 1$:**

Since $x_1 + x_2 + y_1 + y_2 \leq \frac{5h-1}{2}$, therefore

$$3h - (x_1 + x_2 + y_1 + y_2) \geq 0,$$

which implies $(3h - (x_1 + x_2 + y_1 + y_2))\tau \in \mathfrak{S}$, and $h\tau + 2h + 2 + 2h - (y_1 + y_2) \in \mathfrak{S}$ in a similar way used in (1). Consequently $r_1 + r_2 \in \mathfrak{S}$.

(3) **If $x_1 = x_2 = h$, $y_1 \neq y_2 \in [1 \ h - 1]$:**

We can write

$$r_1 + r_2 = 2\mathfrak{F} - \tau(x_1 + x_2) - (\tau + 1)(y_1 + y_2) = (2h - (x_1 + x_2))\tau + (2h - 1 - (y_1 + y_2))\tau +$$

$$2(\tau + 1) + (2h - 3 - (y_1 + y_2)).$$

Since $x_1 + x_2 = 2h$ and $y_1 + y_2 \in [3\ 2h - 3]$, therefore, $2h - (x_1 + x_2) = 0$, $(2h - 1 - (y_1 + y_2)) > 0$. Now it is easy to see that $0 \leq 2h - 3 - (y_1 + y_2) \leq 2h - 1 - (y_1 + y_2)$. By Lemma 2.1, $(2h - 1 - (y_1 + y_2))\tau + (2h - 3 - (y_1 + y_2)) \in \mathfrak{S}$. Since $\tau + 1$ is a generator of $\mathfrak{S}$, therefore $2(\tau + 1) \in \mathfrak{S}$. This implies $r_1 + r_2 \in \mathfrak{S}$.

Consequently $G_{\mathcal{C}}$ is a clique.

- Case 3: Since $h\tau + 2h + 2 \in \mathfrak{S}$, and $\mathfrak{S}$ is symmetric, therefore $\mathfrak{F} - (h\tau + 2h + 2) \notin \mathfrak{S}$. Also

$$\mathfrak{F} - (h\tau + 2h + 2) = h\tau - 1 > 0.$$

This implies $\mathcal{B} \subseteq V(G_{\mathfrak{S}})$, and $G_{\mathcal{B}}$ is induced subgraph of $G_{\mathfrak{S}}$. Since $\mathcal{B}$ is a singleton set, therefore $G_{\mathcal{B}}$ is a clique.

This completes the proof. □

In Proposition 2.2, we prove that the graphs corresponding to the sets $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are complete graphs. The following example provides us with an intuitive explanation of Proposition 2.2.

Example 2.3. *Let $G_{\mathfrak{S}}$ be a graph of $\mathfrak{S} = \langle 7, 8, 20 \rangle$, then $V(G_{\mathfrak{S}}) = \{1, 2, 3, 5, 6, 9, 10, 11, 12, 13, 17, 18, 19, 25, 26, 33\}$. By Proposition 2.2, $\mathcal{A} = \{33, 26, 19\}$, $\mathcal{B} = \{13\}$ and $\mathcal{C} = \{25, 18, 17, 11, 10\}$. The corresponding graphs $G_{\mathcal{A}}, G_{\mathcal{B}}, G_{\mathcal{C}}$ are shown in Figure 1.*

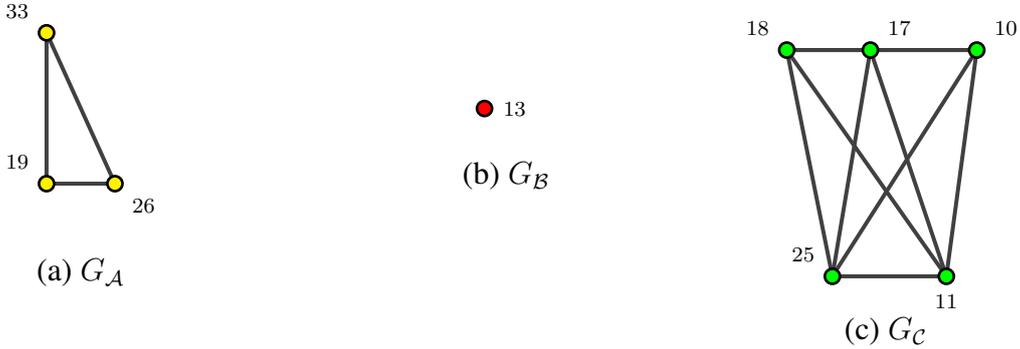Figure 1. Illustrations of the graphs $G_{\mathcal{A}}$, $G_{\mathcal{B}}$, and $G_{\mathcal{C}}$.

Proposition 2.4. *The induced subgraphs $G_{\mathcal{A} \cup \mathcal{B}}$, $G_{\mathcal{B} \cup \mathcal{C}}$, and $G_{\mathcal{A} \cup \mathcal{C}}$ are cliques of $G_{\mathfrak{S}}$, where $\mathfrak{S} = \langle \tau, \tau + 1, h\tau + 2h + 2, \ldots, h\tau + (\tau - 1) \rangle$, with $\tau = 2h + 3$, and $h \geq 1$.*

Proof. To prove that $G_{\mathcal{A} \cup \mathcal{B}}$, $G_{\mathcal{B} \cup \mathcal{C}}$, and $G_{\mathcal{A} \cup \mathcal{C}}$ are cliques, we need to show that the sum of elements of $\mathcal{A}$ with elements of $\mathcal{B}$ and $\mathcal{C}$ is an element of $\mathfrak{S}$. Similarly, the sum of elements of $\mathcal{B}$ with elements of $\mathcal{C}$ is an element of $\mathfrak{S}$. For this, we have the following cases:

- Case 1: If $t_1 \in \mathcal{A}$ and $t_2 \in \mathcal{B}$, then

$$t_1 + t_2 = 2\mathfrak{F} - p\tau - (h\tau + 2h + 2) = h(\tau + 1) + (2h - p)\tau + h.$$

Since $0 \leq h \leq 2h - p$, therefore by Lemma 2.1, $t_1 + t_2 \in \mathfrak{S}$. This implies $G_{\mathcal{A} \cup \mathcal{B}}$ is a clique.

- Case 2: If $t_1 \in \mathcal{A}$ and $t_2 \in \mathcal{C}$, then

$$t_1 + t_2 = 2\mathfrak{F} - (x + p)\tau - y(\tau + 1) = (3h - (x + y + p))\tau + h\tau + 2h + 2 + 2h - y.$$

Note that $3h - (x + y + p) \geq 0$, because $x + y + p \in [1 \ 3h]$.
If $2h - y \leq \tau - h - 1$, then consider

$$2h - y = \tau - h - 1 - v, \quad where \ \ 0 \leq v \leq h + 1.$$

This implies
$$(h\tau + 2h + 2) + (2h - y) = (h + 1)\tau + (h + 1 - v).$$

By Lemma 2.1, $(h + 1)\tau + (h + 1 - v) \in \mathfrak{S}$. So, we get $t_1 + t_2 \in \mathfrak{S}$.
Now if $2h - y > \tau - h - 1$, then consider

$$2h - y - (\tau - h - 1) = 3h + 1 - \tau - y.$$

We can write

$$t_1 + t_2 = [(3h - (x + y + p))\tau + (3h + 1 - \tau - y)] + [(h\tau + 2h + 2) + (\tau - h - 1)].$$

Note that

$$3h - (x + y + p) - (3h + 1 - \tau - y) = \tau - (x + p + 1) > 0,$$

because $1 \leq x + p + 1 \leq 2h + 1 < 2h + 3$. By Lemma 2.1,

$$(3h - (x + y + p))\tau + (3h + 1 - \tau - y) \in \mathfrak{S}.$$

Also $(h\tau + 2h + 2) + (\tau - h - 1) = (h + 1)\tau + h + 1 \in \mathfrak{S}$. Hence, we get $t_1 + t_2 \in \mathfrak{S}$. Consequently, $G_{\mathcal{A} \cup \mathcal{C}}$ is a clique.

- Case 3: If $t_1 \in \mathcal{C}$ and $t_2 \in \mathcal{B}$, then

$$t_1 + t_2 = 2\mathfrak{F} - (x\tau + y(\tau + 1)) - (h\tau + 2h + 2) = (2h - y)(\tau + 1) + (h - x)\tau.$$

Note that $2h - y, h - x \geq 0$. This implies $t_1 + t_2 \in \mathfrak{S}$. Consequently, $G_{\mathcal{B} \cup \mathcal{C}}$ is a clique. $\qquad \square$

Corollary 2.5. *If the induced subgraphs $G_{\mathcal{A} \cup \mathcal{B}}$, $G_{\mathcal{B} \cup \mathcal{C}}$, and $G_{\mathcal{A} \cup \mathcal{C}}$ are cliques of $G_{\mathfrak{S}}$, then $G_{\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}}$ is a clique of $G_{\mathfrak{S}}$, where $\mathfrak{S} = \langle \tau, \tau + 1, h\tau + 2h + 2 \rangle$, with $\tau = 2h + 3$, and $h \geq 1$.*

Proof. The proof of this corollary follows from Propositions 2.2 and 2.4. $\qquad \square$

An intuitive explanation of Proposition 2.4 can be found in the following example.

Example 2.6. *Let $G_{\mathfrak{S}}$ be a graph of $\mathfrak{S} = \langle 7, 8, 20 \rangle$, then*

$$V(G_{\mathfrak{S}}) = \{1, 2, 3, 5, 6, 9, 10, 11, 12, 13, 17, 18, 19, 25, 26, 33\}.$$

By Proposition 2.2, $\mathcal{A} \cup \mathcal{B} = \{33, 26, 19, 13\}$, $\mathcal{B} \cup \mathcal{C} = \{25, 18, 17, 13, 11, 10\}$, and $\mathcal{A} \cup \mathcal{C} = \{33, 26, 25, 19, 18, 17, 11, 10\}$. The corresponding graphs $G_{\mathcal{A} \cup \mathcal{B}}$, $G_{\mathcal{B} \cup \mathcal{C}}$, $G_{\mathcal{A} \cup \mathcal{C}}$ are shown in Figure 2, respectively.
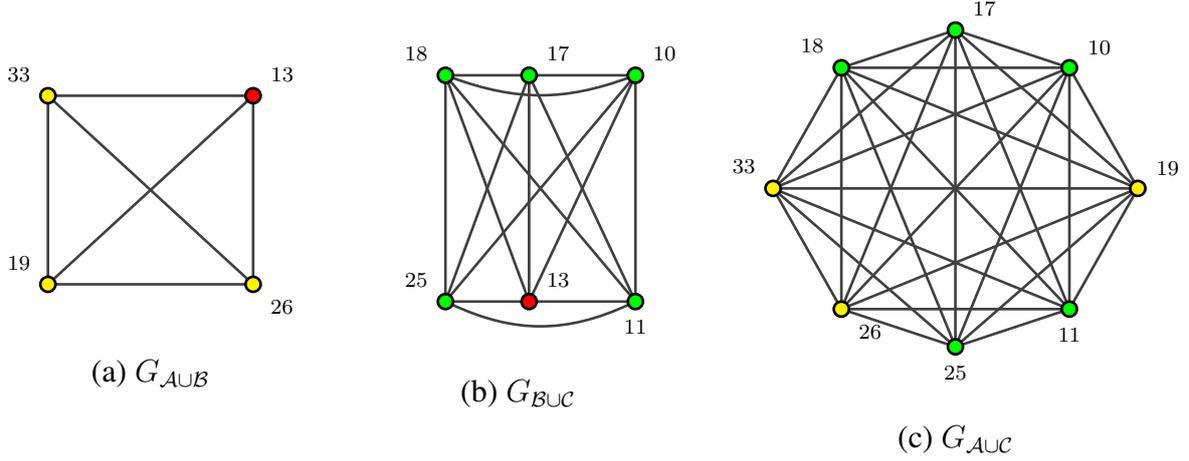
(a) $G_{\mathcal{A} \cup \mathcal{B}}$

(b) $G_{\mathcal{B} \cup \mathcal{C}}$

(c) $G_{\mathcal{A} \cup \mathcal{C}}$

Figure 2. The graphs $G_{\mathcal{A} \cup \mathcal{B}}$, $G_{\mathcal{B} \cup \mathcal{C}}$, $G_{\mathcal{A} \cup \mathcal{C}}$

Proposition 2.7. *Let $G_{\mathfrak{S}}$ be a graph of the cofinite submonoid $\mathfrak{S}$. Then $G_{\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}}$ is a maximal clique of $G_{\mathfrak{S}}$.*

Proof. From Corollary 2.5, it follows that $G_{\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}}$ is a clique of $G_{\mathfrak{S}}$. To prove that this is a maximal clique, we need to show that for every gap g outside the $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ there exists some $g_1 \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ such that $g + g_1 \notin \mathfrak{S}$. Let $\mathfrak{F} - (x\tau + y(\tau + 1) + ct) \in g(\mathfrak{S}) \setminus (\mathcal{A} \cup \mathcal{B} \cup \mathcal{C})$, where $x, y \geq 0$.

- <u>Case 1:</u> If $x \neq 0$, $y = 0$, $c = 0$, then $x = h + z$, $z \in [1 \ h]$ and for $\mathfrak{F} - (h\tau + 2h + 2) \in (\mathcal{A} \cup \mathcal{B} \cup \mathcal{C})$, we have

$$\mathfrak{F} - ((h + z)\tau) + \mathfrak{F} - (h\tau + 2h + 2) = (h - z)\tau + h(\tau + 1) + h \notin \mathfrak{S},$$

  because $\mathfrak{F} - ((h - z)\tau + h(\tau + 1) + h) = \tau z + 1 \in \mathfrak{S}$ (by Lemma 2.1).

- <u>Case 2:</u> If $x \neq 0$, $y \neq 0$, $c = 0$, then (1) $x = h - z$, $z \in [0 \ h]$, and $y = h + r$, $r \in [1 \ h]$, and for $\mathfrak{F} - h(\tau + 1) \in (\mathcal{A} \cup \mathcal{B} \cup \mathcal{C})$, we have

$$\mathfrak{F} - ((h + z)\tau + (h + r)(\tau + 1)) + \mathfrak{F} - h(\tau + 1) = (h + z - r)\tau + 2h + 2 - r \notin \mathfrak{S},$$

  because by Lemma 2.1, $\mathfrak{F} - ((h + z - r)\tau + 2h + 2 - r) = (h + r - z)\tau + (r - 1) \in \mathfrak{S}$. (2) $x = h + z$, $z \in [1 \ h - 1]$, $y = h - r$, $r \in [0 \ h - 1]$, then for $\mathfrak{F} - (h\tau + 2h + 2) \in (\mathcal{A} \cup \mathcal{B} \cup \mathcal{C})$, we have

$$\mathfrak{F} - ((h + z)\tau + (\tau + 1)(h - r) + 0) + \mathfrak{F} - (h\tau + 2h + 2) = (h + r)(\tau + 1) - \tau z \notin \mathfrak{S},$$

  because by Lemma 2.1, $\mathfrak{F} - ((h + r)(\tau + 1) - \tau z) = (h + z - r)\tau + (h + 1 - r) \in \mathfrak{S}$.

- <u>Case 3:</u> If $x \neq 0$, $y \neq 0$, and $c = 1$, then $x = h - z$, $z \in [0 \ h]$, $y = h - r$, $r \in [0 \ h - 1]$, and $t = h\tau + 2h + 2 + u$, $u \in [0 \ h - 1]$, then for $\mathfrak{F} - h\tau \in (\mathcal{A} \cup \mathcal{B} \cup \mathcal{C})$, we have

$$\mathfrak{F} - ((h - z)\tau + (h - r)(\tau + 1) + h\tau + 2h + 2 + u) + \mathfrak{F} - h\tau = \tau z + r(\tau + 1) + 2h - u \notin \mathfrak{S},$$

  because by Lemma 2.1, $\mathfrak{F} - (\tau z + r(\tau + 1) + 2h - u) = (2h - z - r)\tau + (u + 1 - r) \in \mathfrak{S}$.

This completes the proof. $\qquad \square$

Theorem 2.8. *Let $G_\mathfrak{S}$ be a graph of the cofinite monoid $\mathfrak{S}$.*

1. *If h is even, then $\omega(G_\mathfrak{S}) = \frac{h^2+6h+2}{2} + \Sigma_{l=1}^{\frac{h-2}{2}}(h-l)$.*

2. *If h is odd, then $\omega(G_\mathfrak{S}) = \frac{h^2+7h+2}{2} + \Sigma_{l=1}^{\frac{h-3}{2}}(h-l)$.*

Proof. To compute the clique number, we have to show that $\mathcal{A}, \mathcal{B}, \mathcal{C}$ sets are all disjoint. For this we suppose that $\mathcal{A} \cap \mathcal{C} \neq \phi, \mathcal{A} \cap \mathcal{B} \neq \phi$, and $\mathcal{B} \cap \mathcal{C} \neq \phi$. If $\mathcal{A} \cap \mathcal{C} \neq \phi$, then $\mathfrak{F} - p\tau = \mathfrak{F} - (x\tau + y(\tau+1))$, this implies $(p-x)\tau = y(\tau+1)$. If $p \leq x$, then $(x-p)\tau + y(\tau+1) = 0$, this gives $x - p = 0$ and $y = 0$, which is a contradiction. If $p > x$, then $(p-x)\tau = y(\tau+1)$. Since $\gcd(\tau, \tau+1) = 1$, therefore, $(\tau+1)|(p-x)$. This gives $\tau+1 \leq (p-x)$, which is a contradiction because $0 \leq (p-x) \leq h < 2h+4$. If $\mathcal{B} \cap \mathcal{C} \neq \phi$, then $\mathfrak{F} - (h\tau + 2h + 2) = \mathfrak{F} - (x\tau + y(\tau+1))$, this implies $h\tau + 2h + 2 = \tau x + y(\tau+1)$, which is not possible because $h\tau + 2h + 2$ is a generator. If $\mathcal{A} \cap \mathcal{B} \neq \phi$, then $\mathfrak{F} - p\tau = \mathfrak{F} - (h\tau + 2h + 2)$, this implies $p\tau = (h\tau + 2h + 2)$, which is a contradiction. Hence all sets are disjoint. so,

$$|\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}| = |\mathcal{A}| + |\mathcal{B}| + |\mathcal{C}| = h + 1 + 1 + (\lceil\frac{h}{2}\rceil + 1)(h) + h - 1 + \sum_{l=1}^{\lfloor\frac{h}{2}\rfloor-1}(h-l).$$

If h is even, then

$$|\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}| = 2h + 1 + (\frac{h}{2} + 1)(h) + \sum_{l=1}^{\frac{h}{2}-1}(h-l) = \frac{h^2+6h+2}{2} + \sum_{l=1}^{\frac{h-2}{2}}(h-l).$$

If h is odd, then

$$|\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}| = 2h + 1 + (\frac{h+1}{2} + 1)(h) + \sum_{l=1}^{\frac{h-1}{2}-1}(h-l) = \frac{h^2+7h+2}{2} + \sum_{l=1}^{\frac{h-3}{2}}(h-l).$$

This completes the proof. $\square$

Example 2.9. *Let $G_\mathfrak{S}$ be a graph of $\mathfrak{S} = \langle 7, 8, 20 \rangle$, then*

$$V(G_\mathfrak{S}) = \{1, 2, 3, 5, 6, 9, 10, 11, 12, 13, 17, 18, 19, 25, 26, 33\}.$$

Proposition 2.2 and Theorem 2.8 implies

$$cl(G_\mathfrak{S}) = \{10, 11, 13, 17, 18, 19, 25, 26, 33\}$$

and so,

$$\omega(G_\mathfrak{S}) = 9.$$

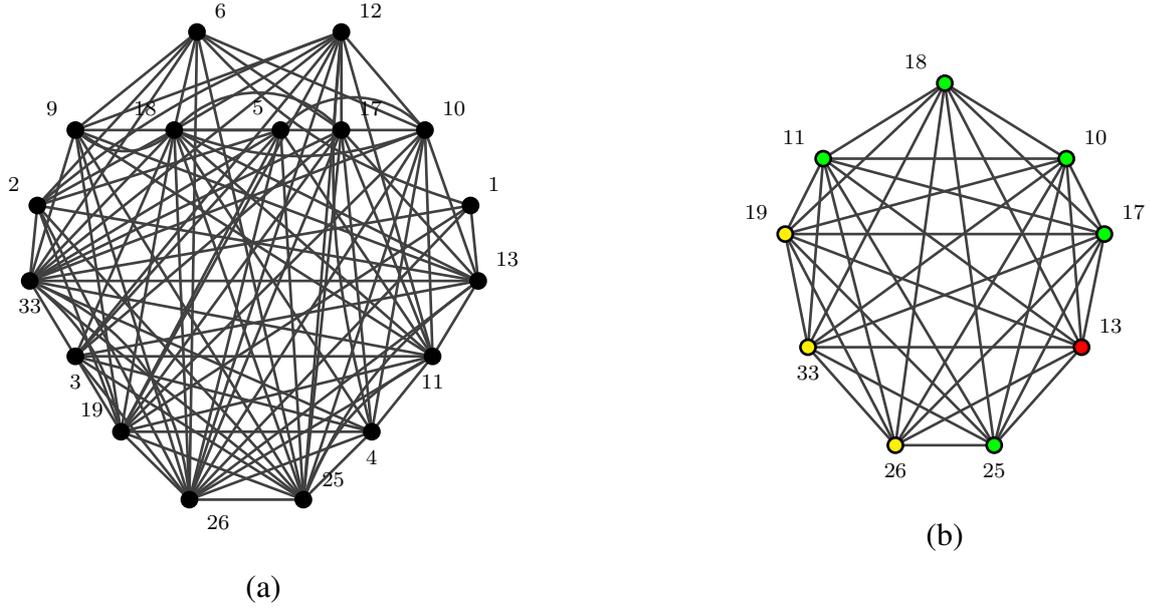*The graph $G_\mathfrak{S}$ and its maximal clique are given in Figure 3(a) and Figure 3(b), respectively.*

Figure 3. The graph $G_\mathfrak{S}$ (a) and its maximal clique (b)

3 Atom bond connectivity index of graphs of symmetric cofinite submonoids

In this section, we compute the minimum degree of $\mathfrak{S}(\tau, 3)$-graphs. Moreover, we give some bounds for atom bond connectivity index of $\mathfrak{S}(\tau, 3)$-graphs in term of different graph invariants.

Lemma 3.1. *Let $G_\mathfrak{S}$ be a graph of cofinite submonoid $\mathfrak{S}$. Then $d_1 = 2h + 1$. Moreover, $\delta(G_\mathfrak{S}) = d_1$.*

Proof. It is easy to see that $V(G_\mathfrak{S})$ of the graph $G_\mathfrak{S}$ is the partition of the following sets:

$$A = \{1, 2, 3, \ldots, \tau - 1\},$$

$$B = \{\{(\tau + 1) + 1, (\tau + 1) + 2, \ldots, (\tau + 1) + (\tau - 2)\}\{2(\tau + 1) + 1, 2(\tau + 1) + 2, \ldots,$$
$$2(\tau + 1) + (\tau - 3)\}\{3(\tau + 1) + 1, 3(\tau + 1) + 2, \ldots, 3(\tau + 1) + (\tau - 4)\}, \ldots,$$
$$\{(h - 1)(\tau + 1) + 1, (h - 1)(\tau + 1) + 2, \ldots, (h - 1)(\tau + 1) + (\frac{\tau + 1}{2} + 1)\}\},$$

$$C = \{\{h(\tau + 1) + 1, h(\tau + 1) + 2, h(\tau + 1) + 3, \ldots, h(\tau + 1) + (\frac{\tau + 1}{2} - 1)\},$$
$$\{(h + 1)(\tau + 1) + 1, (h + 1)(\tau + 1) + 2, \ldots, (h + 1)(\tau + 1) + (\frac{\tau + 1}{2} - 2)\},$$
$$\{(h + 2)(\tau + 1) + 1, (h + 2)(\tau + 1) + 2, \ldots, (h + 2)(\tau + 1) + (\frac{\tau + 1}{2} - 3)\}, \ldots,$$
$$\{2(h - 1)(\tau + 1) + 1, 2(h - 1)(\tau + 1) + 2, 2(h - 1)(\tau + 1) + 3, \ldots,$$
$$2(h - 1)(\tau + 1) + (\frac{\tau + 1}{2} - (h - 1))\}\},$$

$$D = \{(\tau - 4)(\tau + 1) + 1, (\tau - 4)(\tau + 1) + 2, (\tau - 3)(\tau + 1) + 1\}.$$

Let
$$\mathbb{T} = \{t \in G_{\mathfrak{S}} : t + 1 \in g(\mathfrak{S})\}.$$

Note that
$$d_1 = \Delta(G_{\mathfrak{S}}) - |\mathbb{T}|.$$

From the sets A, B, C and D it is clear that,

$$|\mathbb{T}| = \tau - 3 + \{(\tau - 3) + (\tau - 4) + \cdots + \frac{\tau + 1}{2}\} + \{(\frac{\tau + 1}{2} - 2) + (\frac{\tau + 1}{2} - 3) + \cdots$$

$$+ (\frac{\tau + 1}{2} - h)\} + 1.$$

Moreover,

$$\{(\tau - 3) + (\tau - 4) + \cdots + \frac{\tau + 1}{2}\} = \frac{h - 1}{2}(\tau - 1 + \frac{\tau + 1}{2}) = \frac{(h - 1)(3h + 2)}{2}.$$

and

$$\{(\frac{\tau + 1}{2} - 2) + (\frac{\tau + 1}{2} - 3) + \cdots + (\frac{\tau + 1}{2} - h)\} = \frac{h - 1}{2}(\frac{\tau + 1}{2} - 2 + \frac{\tau + 1}{2} - h)$$
$$= \frac{(h - 1)(h + 2)}{2}.$$

Therefore,

$$|\mathbb{T}| = 2h + \frac{(h - 1)(3h + 2)}{2} + \frac{(h - 1)(h + 2)}{2} + 1 = 2h^2 - 1 + 2h.$$

Hence
$$d_1 = 2h^2 + 4h - (2h^2 - 1 + 2h) = 2h + 1.$$

From the partition of $G(\mathfrak{S})$ it is clear that

$$\delta(G_{\mathfrak{S}}) = d_1. \qquad \square$$

Lemma 3.2 ([35]). *Let G be a connected $K_{\alpha+1}$-free graph of order $|V(G)|$ and size $|E(G)|$. Then*

$$|E(G)| \leq \frac{(\alpha - 1).|V(G)|^2}{2\alpha}.$$

Lemma 3.3. *Let $G_{\mathfrak{S}}$ be a graph associated with numerical semigroup $\mathfrak{S}$.*

1. *If h is even, then*

$$\frac{(h^2 + 6h + 2 + 2(\sum_{l=1}^{\frac{h-2}{2}}(h - l))(h^2 + 6h + 2(\sum_{l=1}^{\frac{h-2}{2}}(h - l))}{8}$$

$$\leq \text{size}(G_{\mathfrak{S}}) \leq \frac{(2h^2 + 4h + 1)^2(h^2 + 6h + 2(\sum_{l=1}^{\frac{h-2}{2}}(h - l)))}{2(h^2 + 6h + 2 + 2(\sum_{l=1}^{\frac{h-2}{2}}(h - l)))}.$$

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2. *If h is odd, then*

$$\frac{(h^2 + 7h + 2 + 2(\sum_{l=1}^{\frac{h-3}{2}} (h-l)))(h^2 + 7h + 2(\sum_{l=1}^{\frac{h-3}{2}} (h-l)))}{8}$$

$$\leq \text{size}(G_{\mathfrak{S}}) \leq \frac{(2h^2 + 4h + 1)^2(h^2 + 7h + 2(\sum_{l=1}^{\frac{h-3}{2}} (h-l)))}{2(h^2 + 7h + 2 + 2(\sum_{l=1}^{\frac{h-3}{2}} (h-l)))}.$$

Proof. This Lemma's proof follows from Lemma 3.2 and Theorem 2.8. □

Example 3.4. *Let $G_{\mathfrak{S}}$ be a graph of $\mathfrak{S} = \langle 7, 8, 20 \rangle$, then*

$$V(G_{\mathfrak{S}}) = \{1, 2, 3, 4, 5, 6\} \cup \{9, 10, 11, 12, 13\} \cup \{17, 18, 19\} \cup \{25, 26, 33\}.$$

By Lemma 3.1, $T = \{2, 3, 4, 5, 9, 10, 11, 12, 17, 18, 25\}$. Since $\triangle = 2h^2 + 4h$, therefore $\delta(G_{\mathfrak{S}}) = d_1 = \triangle - |T| = 16 - 11 = 5$. From Theorem 2.8, $\omega(G_{\mathfrak{S}}) = 9$. By Lemmas 3.2 and 3.3, $36 \leq \text{size}(G_{\mathfrak{S}}) \leq \frac{1156}{9}$. The graph $G_{\mathfrak{S}}$ is shown in Figure 4.
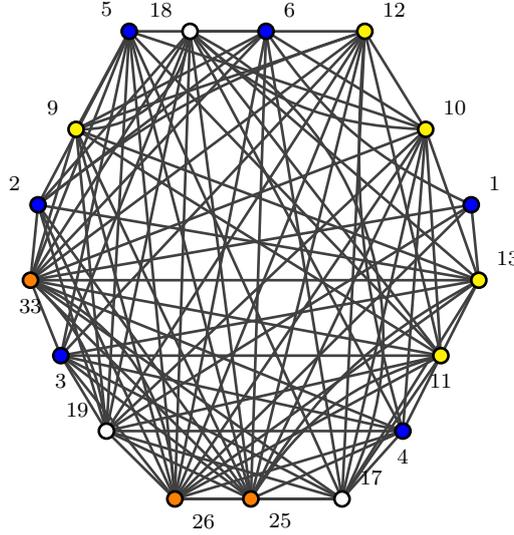

Figure 4. The graph associated with the numerical semigroup $\mathfrak{S}$.

Lemma 3.5 ([28]). *Let $f(i,j) = \sqrt{\frac{i+j-2}{ij}} = \sqrt{\frac{1}{i} + \frac{1}{j} - \frac{2}{ij}}$ with $i, j \geq 1$. Moreover if $j \geq 2$ is fixed, then $f(i,j)$ is decreasing for i.*

Theorem 3.6. *Let $G_{\mathfrak{S}}$ be a graph of cofinite submonoid $\mathfrak{S}$. Then one of the following inequalities hold:*

1. *If h is even, then*

$$ABC(G_{\mathfrak{S}}) \leq \frac{2\sqrt{h}}{2h+1} \left[\frac{(2h^2 + 4h + 1)^2 \left(\frac{h^2 + 6h + 2}{2} + \sum_{l=1}^{\frac{h-2}{2}})(h-l) - 1 \right)}{2 \left(\frac{h^2 + 6h + 2}{2} + \sum_{l=1}^{\frac{h-2}{2}} (h-l) \right)} - (2h^2 + 4h) \right]$$

$$+ \sqrt{\frac{2h(2h^3 + 10h^2 + 11h - 2)}{2h+1}}.$$

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2. *If h is odd, then*

$$ABC(G_{\mathfrak{S}}) \leq \frac{2\sqrt{h}}{2h+1} \left[\frac{(2h^2 + 4h + 1)^2 \left(\frac{h^2 + 7h + 2}{2} + \sum_{l=1}^{\frac{h-3}{2}} (h-l) - 1 \right)}{2 \left(\frac{h^2 + 7h + 2}{2} + \sum_{l=1}^{\frac{h-3}{2}} (h-l) \right)} - (2h^2 + 4h) \right]$$
$$+ \sqrt{\frac{2h(2h^3 + 10h^2 + 11h - 2)}{2h+1}}.$$

Proof. If x_k is the maximum degree vertex in $G_{\mathfrak{S}}$, then $d_k = 2h^2 + 4h$. Also if $d_\alpha \geq 2h + 1$, we obtain

$$\sum_{x_\alpha x_k \in E(G_{\mathfrak{S}})} \sqrt{\frac{d_k + d_\alpha - 2}{d_k d_\alpha}} = \sum_{x_\alpha x_k \in E(G_{\mathfrak{S}})} \sqrt{\frac{1}{2h^2 + 4h} + \frac{1}{d_\alpha}\left(1 - \frac{2}{2h^2 + 4h}\right)}$$
$$\leq \sqrt{\frac{2h(2h^3 + 10h^2 + 11h - 2)}{2h+1}}. \tag{1}$$

Since $2h + 1 > 2$, therefore by Lemma 3.5, we have $f(d_\alpha, d_\beta) \leq f(2h+1, 2h+1)$ for each edge in $G_{\mathfrak{S}}$. Hence

$$\sqrt{\frac{d_\beta + d_\alpha - 2}{d_\beta d_\alpha}} \leq \frac{2\sqrt{h}}{2h+1}$$

for any edge. Therefore, we have

$$\sum_{x_\alpha x_\beta \in E(G_{\mathfrak{S}})} \sqrt{\frac{d_\beta + d_\alpha - 2}{d_\beta d_\alpha}} \leq (\text{size}(G_{\mathfrak{S}}) - (2h^2 + 4h))\frac{2\sqrt{h}}{2h+1} \tag{2}$$

with $\alpha \neq \beta \neq k$. Since $G_{\mathfrak{S}}$ is a $K_{\omega(G_{\mathfrak{S}})+1}$ free graph, therefore by Lemma 3.2, we have that if h is even, then

$$\frac{(2h^2 + 4h + 1)^2(h^2 + 6h + 2(\sum_{l=1}^{\frac{h-2}{2}}(h-l)))}{2(h^2 + 6h + 2 + 2(\sum_{l=1}^{\frac{h-2}{2}}(h-l)))} \leq \frac{(2h^2 + 4h + 1)^2(\omega(G_{\mathfrak{S}}) - 1)}{2\omega(G_{\mathfrak{S}})}. \tag{3}$$

If h is odd, then

$$\frac{(2h^2 + 4h + 1)^2(h^2 + 7h + 2(\sum_{l=1}^{\frac{h-3}{2}}(h-l)))}{2(h^2 + 7h + 2 + 2(\sum_{l=1}^{\frac{h-3}{2}}(h-l)))} \leq \frac{(2h^2 + 4h + 1)^2(\omega(G_{\mathfrak{S}}) - 1)}{2\omega(G_{\mathfrak{S}})}. \tag{4}$$

Since

$$ABC(G_{\mathfrak{S}}) = \sum_{v_1 v_2 \in E(G(\mathfrak{S}))} \sqrt{\frac{d_{v_1} + d_{v_2} - 2}{d_{v_1} d_{v_2}}}$$
$$= \sum_{x_\alpha x_k \in E(G_{\mathfrak{S}})} \sqrt{\frac{d_k + d_\alpha - 2}{d_k d_\alpha}} + \sum_{x_\alpha x_\beta \in E(G_{\mathfrak{S}})} \sqrt{\frac{d_\beta + d_\alpha - 2}{d_\beta d_\alpha}}$$

Now by using the above results in (1) and (2), we obtain

$$ABC(G_{\mathfrak{S}}) \leq \sqrt{\frac{2h(2h^3 + 10h^2 + 11h - 2)}{2h+1}} + (\text{size}(G_{\mathfrak{S}}) - (2h^2 + 4h))\frac{2\sqrt{h}}{2h+1}$$

The rest of the proof follows from the above expressions in (3), (4), by Lemma 3.3 and from Theorem 2.8. □

Corollary 3.7. *Let $G_\mathfrak{S}$ be a graph of cofinite submonoid $\mathfrak{S}$. Then one of the following inequalities hold:*

1. *If h is even, then*

$$ABC(G_\mathfrak{S}) \leq \frac{2\sqrt{h}}{2h+1}\left[\frac{(2h^2+4h+1)^2\left(\frac{h^2+6h+2}{2}+\sum_{l=1}^{\frac{h-2}{2}}(h-l)-1\right)}{2\left(\frac{h^2+6h+2}{2}+\sum_{l=1}^{\frac{h-2}{2}}(h-l)\right)}\right].$$

2. *If h is odd, then*

$$ABC(G_\mathfrak{S}) \leq \frac{2\sqrt{h}}{2h+1}\left[\frac{(2h^2+4h+1)^2\left(\frac{h^2+7h+2}{2}+\sum_{l=1}^{\frac{h-3}{2}}(h-l)-1\right)}{2\left(\frac{h^2+7h+2}{2}+\sum_{l=1}^{\frac{h-3}{2}}(h-l)\right)}\right].$$

Proposition 3.8. *Let $G_\mathfrak{S}$ be a graph of cofinite submonoid of $\mathfrak{S}$. Then*

1. *if h is even, then*

$$ABC(G_\mathfrak{S}) \leq \sqrt{\frac{(2h^2+4h+1)^2(h^2+6h+2(\sum_{l=1}^{\frac{h-2}{2}}(h-l)))}{2(h^2+6h+2+2(\sum_{l=1}^{\frac{h-2}{2}}(h-l)))}(2h^2+4h+1-2R_{-1})}.$$

2. *If h is odd, then*

$$ABC(G_\mathfrak{S}) \leq \sqrt{\frac{(2h^2+4h+1)^2(h^2+7h+2(\sum_{l=1}^{\frac{h-3}{2}}(h-l)))}{2(h^2+7h+2+2(\sum_{l=1}^{\frac{h-3}{2}}(h-l)))}(2h^2+4h+1-2R_{-1})}.$$

Proof. Since

$$ABC(G_\mathfrak{S}) = \sum_{v_1 v_2 \in E(G_\mathfrak{S})} \sqrt{\frac{d_{v_1}+d_{v_2}-2}{d_{v_1}d_{v_2}}}.$$

Therefore by the Cauchy–Schwarz inequality and by Lemma 3.3, if h is even, then

$$ABC(G_\mathfrak{S}) \leq \sqrt{\frac{(2h^2+4h+1)^2(h^2+6h+2(\sum_{l=1}^{\frac{h-2}{2}}(h-l)))}{2(h^2+6h+2+2(\sum_{l=1}^{\frac{h-2}{2}}(h-l)))} \sum_{v_1 v_2 \in E(G_\mathfrak{S})}\frac{d_{v_1}+d_{v_2}-2}{d_{v_1}d_{v_2}}}.$$

If h is odd, then

$$ABC(G_\mathfrak{S}) \leq \sqrt{\frac{(2h^2+4h+1)^2(h^2+7h+2(\sum_{l=1}^{\frac{h-3}{2}}(h-l)))}{2(h^2+7h+2+2(\sum_{l=1}^{\frac{h-3}{2}}(h-l)))} \sum_{v_1 v_2 \in E(G_\mathfrak{S})}\frac{d_{v_1}+d_{v_2}-2}{d_{v_1}d_{v_2}}}.$$

Moreover,

$$\sum_{v_1 v_2 \in E(G(\mathfrak{S}))} \frac{d_{v_1}+d_{v_2}-2}{d_{v_1}d_{v_2}} = \sum_{v_1 v_2 \in E(G_\mathfrak{S})}\left(\frac{1}{v_1}+\frac{1}{v_2}-\frac{2}{d_{v_1}d_{v_2}}\right)$$

$$= \sum_{v_1 \in V(G_\mathfrak{S}, d_{v_1}\geq 1)} \frac{1}{d_{v_1}}d_{v_1} - 2R_{-1} = 2h^2+4h+1-2R_{-1}.$$

which completes our proof. $\qquad\square$

Theorem 3.9. *Let $G_\mathfrak{S}$ be a graph of cofinite submonoid $\mathfrak{S}$. Then*

1. *If h is even, then*

$$ABC(G_\mathfrak{S}) \leq \sqrt{(2h^2 + 4h + 1 - \frac{((2h^2 + 4h + 1)^2(h^2 + 6h + 2(\sum_{l=1}^{\frac{h-2}{2}}(h-l))))^2}{2(h^2 + 6h + 2 + 2(\sum_{l=1}^{\frac{h-2}{2}}(h-l)))^2 M_2(G_\mathfrak{S})}) \times}$$

$$\sqrt{\frac{(2h^2 + 4h + 1)^2(h^2 + 6h + 2(\sum_{l=1}^{\frac{h-2}{2}}(h-l)))}{2(h^2 + 6h + 2 + 2(\sum_{l=1}^{\frac{h-2}{2}}(h-l)))}}.$$

2. *If h is odd, then*

$$ABC(G_\mathfrak{S}) \leq \sqrt{(2h^2 + 4h + 1 - \frac{((2h^2 + 4h + 1)^2(h^2 + 7h + 2(\sum_{l=1}^{\frac{h-3}{2}}(h-l))))^2}{2(h^2 + 7h + 2 + 2(\sum_{l=1}^{\frac{h-3}{2}}(h-l)))^2 M_2(G_\mathfrak{S})}) \times}$$

$$\sqrt{\frac{(2h^2 + 4h + 1)^2(h^2 + 7h + 2(\sum_{l=1}^{\frac{h-3}{2}}(h-l)))}{2(h^2 + 7h + 2 + 2(\sum_{l=1}^{\frac{h-3}{2}}(h-l)))}}.$$

Proof. We know that the Arithmetic Mean is greater than or equal to the Harmonic Mean, therefore,

$$\sum_{v_1 v_2 \in E(G_\mathfrak{S})} \frac{1}{d_{v_1} d_{v_2}} \geq \frac{(\text{size}(G_\mathfrak{S}))^2}{\sum_{v_1 v_2} \in E(G_\mathfrak{S}) d_{v_1} d_{v_2}}$$

The rest of the proof follows from Proposition 3.8 and by Lemma 3.3. $\qquad\square$

Lemma 3.10 ([20])**.** *Let α and β are real number such that $\alpha \geq \beta \geq 0$. Then $\sqrt{\alpha - \beta} \geq \sqrt{\alpha} - \sqrt{\beta}$, the equality holds if and only if $\alpha = \beta$ or $\beta = 0$.*

Theorem 3.11. *Let $G_\mathfrak{S}$ be a graph of cofinite submonoid $\mathfrak{S}$. Then*

$$ABC(G_\mathfrak{S}) \geq \sqrt{2} \left(\frac{\sqrt{2h+1} - 1}{\sqrt{2h+1}} \right) R_{\frac{-1}{4}}(G_\mathfrak{S}).$$

Proof. We know that

$$ABC(G_\mathfrak{S}) = \sum_{v_1 v_2 \in E(G_\mathfrak{S})} \sqrt{\frac{1}{d_{v_1} d_{v_2}}} \sqrt{d_{v_1} + d_{v_2} - 2}$$

By Lemma 3.10, we obtain

$$ABC(G_\mathfrak{S}) \geq \sum_{v_1 v_2 \in E(G_\mathfrak{S})} \sqrt{\frac{1}{d_{v_1} d_{v_2}}} \left(\sqrt{d_{v_1} + d_{v_2}} - \sqrt{2} \right)$$

$$= \sum_{v_1 v_2 \in E(G_\mathfrak{S})} \sqrt{\left(\frac{1}{d_{v_1}} - \frac{1}{d_{v_2}} \right)^2 + \frac{2}{\sqrt{d_{v_2} d_{v_1}}}} - \sum_{v_1 v_2 \in E(G_\mathfrak{S})} \frac{\sqrt{2}}{\sqrt{d_{v_1} d_{v_2}}}$$

$$\geq \sum_{v_1 v_2 \in E(G_\mathfrak{S})} \frac{\sqrt{2}}{(d_{v_1} d_{v_2})^{\frac{1}{4}}} \left(1 - \frac{1}{(d_{v_1} d_{v_2})^{\frac{1}{4}}} \right)$$

$$\geq \sqrt{2} \left(\frac{\sqrt{2h+1} - 1}{\sqrt{2h+1}} \right) R_{\frac{-1}{4}}(G_\mathfrak{S}). \qquad\square$$

Theorem 3.12. *Let $G_\mathfrak{S}$ be a graph of cofinite submonoid $\mathfrak{S}$. Then*

1. *If h is even, then*

$$ABC(G_\mathfrak{S}) \leq \sqrt{\left(M_1 - \frac{(2h^2 + 4h + 1)^2(h^2 + 6h + 2(\sum_{l=1}^{\frac{h-2}{2}}(h-l)))}{(h^2 + 6h + 2 + 2(\sum_{l=1}^{\frac{h-2}{2}}(h-l)))}\right) R_{-1}}.$$

2. *If h is odd, then*

$$ABC(G_\mathfrak{S}) \leq \sqrt{\left(M_1 - \frac{(2h^2 + 4h + 1)^2(h^2 + 7h + 2(\sum_{l=1}^{\frac{h-3}{2}}(h-l)))}{(h^2 + 7h + 2 + 2(\sum_{l=1}^{\frac{h-3}{2}}(h-l)))}\right) R_{-1}}.$$

Proof. We know that

$$ABC(G_\mathfrak{S}) = \sum_{v_1 v_2 \in E(G_\mathfrak{S})} \sqrt{\frac{d_{v_1} + d_{v_2} - 2}{d_{v_1} d_{v_2}}}.$$

Using the Cauchy–Schwarz inequality we have,

$$ABC(G_\mathfrak{S}) \leq \sqrt{\sum_{v_1 v_2 \in E(G_\mathfrak{S})} (d_{v_1} + d_{v_2} - 2) \sum_{v_1 v_2 \in E(G_\mathfrak{S})} \frac{1}{d_{v_1} d_{v_2}}}$$

$$= \sqrt{\left[\sum_{v_1 v_2 \in E(G_\mathfrak{S})} (d_{v_1} + d_{v_2}) - 2 \sum_{v_1 v_2 \in E(G_\mathfrak{S})} (1)\right] \sum_{v_1 v_2 \in E(G_\mathfrak{S})} \frac{1}{d_{v_1} d_{v_2}}}$$

$$= \sqrt{\left[M_1(G_\mathfrak{S}) - 2 \sum_{v_1 v_2 \in E(G_\mathfrak{S})} (1)\right] R_{-1}}$$

The rest of the proof follows from Lemma 3.3. $\qquad\square$

A numerical and graphical comparison of different bounds is given in Table 1 and Figure 5.

Table 1. Comparison of ABC index bounds for $G_\mathfrak{S}$

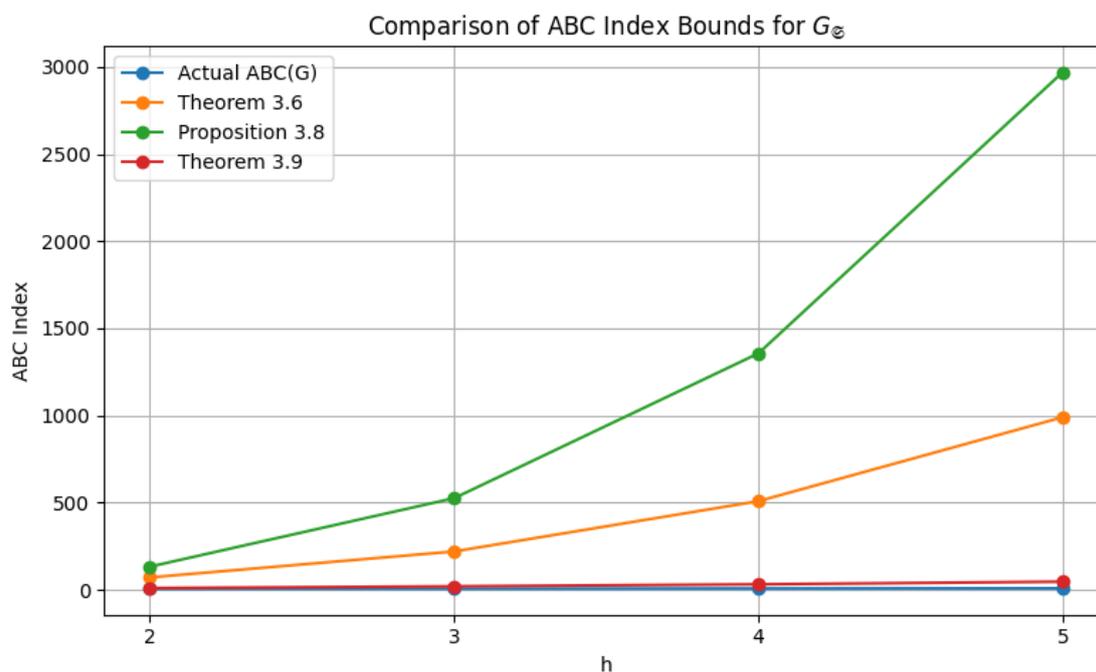| h	Exact	Theorem 3.6	Proposition 3.8	Theorem 3.9
2	5.77	71.38	133.05	10.11
3	6.5	220.32	526.08	20.21
4	7.5	508.83	1356.8	31.82
5	8.5	990.39	2969.4	46.95

Figure 5. Comparision of different bounds of $G_{\mathfrak{S}}$.

4   Conclusion

In this paper, we find the minimum degree, maximal clique, and bounds for the clique number of graphs associated with a symmetric cofinite submonoid. Also, we find upper and lower bounds for the atom-bond-connectivity index of graphs related to symmetric numerical semigroups in terms of different graph invariants. In the future, we can extend these results to broader classes of numerical semigroups, develop sharper bounds for topological indices, and explore additional graph invariants, such as spectral properties and metric dimension, to better understand the structural behavior of these algebraically defined graphs.

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