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Characterizations of L-additive functions via generalized arithmetic convolutions

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Abstract: This paper investigates the properties of L-additive functions within the algebraic frameworks of two generalized arithmetic convolutions: the K-convolution and Narkiewicz's A-convolution. We establish the foundational algebraic context for these operations by citing the established conditions for their associativity and commutativity. Our main results provide rigorous characterization theorems for completely additive and L-additive functions, which manifest as Leibniz-type rules that these functions satisfy with respect to the convolutions. Furthermore, we provide insightful, non-trivial examples using classical arithmetic functions to illustrate the mechanics of these characterizations, thereby demonstrating the utility of the generalized convolution framework in the study of arithmetic derivatives and their generalizations.
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1 Introduction

The set of arithmetic functions, equipped with pointwise addition and the Dirichlet convolution, forms a unique factorization domain that is central to the study of number theory. Within this structure, certain classes of functions exhibit properties analogous to operators in classical calculus. The arithmetic derivative D, introduced by Barbeau [1], is one such function, defined by its action on primes, $D(p) = 1$, and its adherence to the Leibniz rule: $D(mn) = D(m)n + mD(n)$ for all $m, n \in \mathbb{N}$.

This concept was generalized by Haukkanen *et al.* [3] to the class of Leibniz-additive (or L-additive) functions. An arithmetic function f is L-additive if there exists a completely multiplicative function h_f such that for all $m, n \in \mathbb{N}$,

$$f(mn) = f(m)h_f(n) + f(n)h_f(m). \tag{1}$$

The arithmetic derivative D is the special case where $h_D(n) = n$, while completely additive functions correspond to the case where $h_f(n) = 1$ for all n. The notion of Leibniz-type behavior for arithmetic functions originates in the work of Haukkanen [3], while the term Leibniz-additive was explicitly introduced later in Haukkanen, Merikoski, and Tossavainen [4]. Further structural properties and examples of Leibniz-additive functions were developed in subsequent works; see, for instance, Merikoski, Haukkanen, and Tossavainen [5, 6].

However, the Dirichlet convolution is itself a special case of broader classes of convolutions. This paper extends the study of L-additive functions to two such generalizations: the K-convolution and Narkiewicz's A-convolution.

The K-convolution, studied by Gioia [2], modifies the Dirichlet product with a kernel function K, while Narkiewicz's A-convolution restricts the sum to a pre-defined set of divisors A_n [7]. These frameworks unify the Dirichlet and unitary convolutions, among others. A key feature of these generalized products is that their fundamental algebraic properties, such as associativity, are not guaranteed and depend on specific conditions on the kernel K or the divisor sets A_n.

The central question of this paper is: *How are L-additive functions characterized within these more general and structurally nuanced convolution rings?* We answer this question by providing rigorous proofs for several characterization theorems that take the form of Leibniz-type rules. In doing so, we correct significant logical errors present in earlier drafts of this work, particularly in the proofs of the converse directions of our main theorems. By first establishing the necessary algebraic context for these convolutions and then providing rigorous proofs and insightful examples, we aim to place the study of L-additive functions on a firm footing within the theory of generalized arithmetic convolutions.

2 Preliminaries

Let $\mathcal{A}$ denote the set of all arithmetic functions $f : \mathbb{N} \to \mathbb{C}$. This set forms a commutative ring under pointwise addition, $(f + g)(n) = f(n) + g(n)$, and the Dirichlet convolution, $(f * g)(n) = \sum_{d|n} f(d)g(n/d)$.

Definition 2.0.1 (Additive and Multiplicative Functions). *An arithmetic function f is said to be:*

- ***Additive*** *if* $f(mn) = f(m) + f(n)$ *whenever* $\gcd(m, n) = 1$.
- ***Completely Additive*** *if* $f(mn) = f(m) + f(n)$ *for all* $m, n \in \mathbb{N}$. *Such functions have been studied extensively; see, for example, [8].*
- ***Multiplicative*** *if* $f(1) = 1$ *and* $f(mn) = f(m)f(n)$ *whenever* $\gcd(m, n) = 1$.
- ***Completely Multiplicative*** *if* $f(1) = 1$ *and* $f(mn) = f(m)f(n)$ *for all* $m, n \in \mathbb{N}$.

Definition 2.0.2 (L-additive Function [3]). *An arithmetic function f is **L-additive** if there exists a completely multiplicative function h_f such that for all $m, n \in \mathbb{N}$,*

$$f(mn) = f(m)h_f(n) + f(n)h_f(m).$$

A key result connecting L-additive and completely additive functions is the following theorem.

Theorem 2.0.1 (Haukkanen [3]). *Let f be an arithmetic function. If f is L-additive and its associated function h_f is non-zero valued, then the function f/h_f is completely additive. Conversely, if there is a completely multiplicative non-zero valued function h such that f/h is completely additive, then f is L-additive and $h_f = h$.*

2.1 Generalized convolutions

We now define the two generalized convolutions that are the focus of this study.

Definition 2.1.1 (K-convolution). *Let $\mathcal{A}$ denote the set of all arithmetic functions $f : \mathbb{N} \to \mathbb{C}$. Let $K \in \mathcal{A}$ be a nonzero arithmetic function, called the* kernel. *The **K-convolution** (or K-product) of two functions $u, v \in \mathcal{A}$ is defined by*

$$(u *_K v)(n) = \sum_{d|n} u(d)\, v\Big(\frac{n}{d}\Big)\, K\Big(\gcd\Big(d, \frac{n}{d}\Big)\Big).$$

The K-convolution is always commutative. It is associative if and only if the kernel K satisfies the functional equation $K(\gcd(a, b))K(\gcd(ab, c)) = K(\gcd(a, bc))K(\gcd(b, c))$ for all $a, b, c \in \mathbb{N}$ [2].

Definition 2.1.2 (Narkiewicz's A-convolution [7]). *For each $n \in \mathbb{N}$, let A_n be a non-empty set of positive divisors of n. The **A-convolution** of two functions $u, v \in \mathcal{A}$ is defined as*

$$(u *_A v)(n) = \sum_{d \in A_n} u(d)v\left(\frac{n}{d}\right).$$

The algebraic properties of the A-convolution depend on the structure of the sets A_n. As established by Narkiewicz [7], the A-convolution is:

- **Commutative** if and only if for every $n \in \mathbb{N}$ and every $d \in A_n$, we also have $n/d \in A_n$.
- **Associative** if and only if for all $d, m, n \in \mathbb{N}$, the conditions '$d \in A_m$ and $m \in A_n$' are equivalent to the conditions '$d \in A_n$ and $m/d \in A_{n/d}$'.

3 Main results

In this section, we present the characterization theorems for completely additive and L-additive functions with respect to the K- and A-convolutions. Throughout, we use the notation $(fu)(n) = f(n)u(n)$ for the pointwise product of two arithmetic functions.

3.1 Characterizations via K-convolution

Throughout this subsection, K denotes a given non-zero arithmetic function.

Theorem 3.1.1. *Let $K \in \mathcal{A}$ be an arithmetic function such that $K(n) \neq 0$ for all $n \in \mathbb{N}$. An arithmetic function f is completely additive if and only if for all arithmetic functions $u, v \in \mathcal{A}$, the identity*

$$f(u *_K v) = (fu) *_K v + u *_K (fv) \tag{2}$$

holds.

Proof. First, suppose f is a completely additive function. For any $n \in \mathbb{N}$, we have

$$\begin{aligned}
[f(u *_K v)](n) &= f(n)(u *_K v)(n) \\
&= f(n) \sum_{d|n} u(d) v\left(\frac{n}{d}\right) K\left(\gcd\left(d, \frac{n}{d}\right)\right) \\
&= \sum_{d|n} f\left(d \cdot \frac{n}{d}\right) u(d) v\left(\frac{n}{d}\right) K\left(\gcd\left(d, \frac{n}{d}\right)\right) \\
&= \sum_{d|n} \left(f(d) + f\left(\frac{n}{d}\right)\right) u(d) v\left(\frac{n}{d}\right) K\left(\gcd\left(d, \frac{n}{d}\right)\right) \\
&= \sum_{d|n} (fu)(d) v\left(\frac{n}{d}\right) K\left(\gcd\left(d, \frac{n}{d}\right)\right) + \sum_{d|n} u(d)(fv)\left(\frac{n}{d}\right) K\left(\gcd\left(d, \frac{n}{d}\right)\right) \\
&= [(fu) *_K v](n) + [u *_K (fv)](n).
\end{aligned}$$

This proves the "if" part of the theorem.

Conversely, suppose that equation (2) holds for all $u, v \in \mathcal{A}$. To show that f is completely additive, we must prove that $f(mn) = f(m) + f(n)$ for arbitrary $m, n \in \mathbb{N}$. We employ a standard technique by choosing specific functions for u and v. Let $\delta_k \in \mathcal{A}$ be the function defined by $\delta_k(j) = 1$ if $j = k$ and 0 otherwise.

Let $u = \delta_m$ and $v = \delta_n$. We evaluate both sides of (2) at the integer mn.

- **LHS:** For the left-hand side (LHS), we first compute the convolution $(u *_K v)(mn)$:

  $$(u *_K v)(mn) = (\delta_m *_K \delta_n)(mn) = \sum_{d|mn} \delta_m(d) \delta_n\left(\frac{mn}{d}\right) K\left(\gcd\left(d, \frac{mn}{d}\right)\right).$$

  The only non-zero term in this sum occurs when $d = m$, which implies $mn/d = n$. Thus, the sum reduces to a single term: $(\delta_m *_K \delta_n)(mn) = K(\gcd(m, n))$. The LHS is therefore $[f(u *_K v)](mn) = f(mn) \cdot K(\gcd(m, n))$.

- **RHS:** For the right-hand side (RHS), we compute the two convolution terms. The first term is $[(fu) *_K v](mn)$. Note that $fu = f \cdot \delta_m = f(m)\delta_m$.

$$[(f(m)\delta_m) *_K \delta_n](mn) = f(m) \sum_{d|mn} \delta_m(d)\delta_n\left(\frac{mn}{d}\right) K\left(\gcd\left(d, \frac{mn}{d}\right)\right)$$
$$= f(m)K(\gcd(m,n)).$$

Similarly, the second term is $[u *_K (fv)](mn)$. Since $fv = f \cdot \delta_n = f(n)\delta_n$, we have

$$[\delta_m *_K (f(n)\delta_n)](mn) = f(n) \sum_{d|mn} \delta_m(d)\delta_n\left(\frac{mn}{d}\right) K\left(\gcd\left(d, \frac{mn}{d}\right)\right)$$
$$= f(n)K(\gcd(m,n)).$$

The RHS is the sum of these two terms: $(f(m) + f(n))K(\gcd(m,n))$.

Equating the LHS and RHS gives

$$f(mn)K(\gcd(m,n)) = (f(m) + f(n))K(\gcd(m,n)).$$

Since $K(\gcd(m,n)) \neq 0$ by hypothesis, dividing by $K(\gcd(m,n))$ yields $f(mn) = f(m)+f(n)$. Since m, n were arbitrary, f is completely additive. □

Example 3.1.1. To illustrate Theorem 3.1.1, let $f(n) = \Omega(n)$, the total number of prime factors of n (a completely additive function). Let the kernel be the constant function $K(n) \equiv 1$. Let u and v be the constant function $U(n) = 1$. We test the identity at $n = 12 = 2^2 \cdot 3$.

- **LHS:** $f(U *_1 U)(12) = \Omega(12) \cdot (U *_1 U)(12) = 3 \cdot \sum_{d|12} U(d)U(12/d)K(\gcd(d, 12/d))$. The divisors are $\{1, 2, 3, 4, 6, 12\}$. Since $U(\cdot) = 1$ and $K(\cdot) \equiv 1$, the sum equals $\sum_{d|12} 1 = 6$. Hence LHS $= 3 \cdot 6 = 18$.

- **RHS:** The first term is $\sum_{d|12} \Omega(d) \cdot 1 \cdot 1 = \sum_{d|12} \Omega(d)$. The second term is $\sum_{d|12} 1 \cdot \Omega(12/d) \cdot 1 = \sum_{d|12} \Omega(12/d) = \sum_{d|12} \Omega(d)$. Thus RHS $= 2\sum_{d|12} \Omega(d)$. Now $\Omega(1) = 0, \Omega(2) = 1, \Omega(3) = 1,\ \Omega(4) = 2,\ \Omega(6) = 2,\ \Omega(12) = 3$, so $\sum_{d|12} \Omega(d) = 0 + 1 + 1 + 2 + 2 + 3 = 9$. Hence RHS $= 2 \cdot 9 = 18 =$ LHS.

Therefore, the identity holds.

Theorem 3.1.2. *Let f be an arithmetic function. If f is L-additive and h_f is non-zero valued, then for all $u, v \in \mathcal{A}$,*

$$f(u *_K v) = (fu) *_K (h_f v) + (h_f u) *_K (fv). \tag{3}$$

Conversely, if there is a completely multiplicative non-zero valued function h such that for all $u, v \in \mathcal{A}$,

$$f(u *_K v) = (fu) *_K (hv) + (hu) *_K (fv),$$

then f is L-additive and $h_f = h$.

Proof. Suppose that f is L-additive with associated function h_f. By Theorem 2.0.1 (restated from Haukkanen [3]), the function $g = f/h_f$ is completely additive. By Theorem 3.1.1, g satisfies

$$g(u *_K v) = (gu) *_K v + u *_K (gv).$$

Multiplying pointwise by h_f (which is completely multiplicative) gives

$$h_f \cdot (g(u *_K v)) = h_f \cdot ((gu) *_K v) + h_f \cdot (u *_K (gv)).$$

Since $f = gh_f$, the LHS is $f(u *_K v)$. For the RHS, we expand the convolutions. For any $n \in \mathbb{N}$:

$$\begin{aligned}
[h_f \cdot ((gu) *_K v)](n) &= h_f(n) \sum_{d|n} g(d)u(d)v(n/d)K(\gcd(d, n/d)) \\
&= \sum_{d|n} h_f(d)h_f(n/d)g(d)u(d)v(n/d)K(\gcd(d, n/d)) \\
&= \sum_{d|n} (f(d)u(d))(h_f(n/d)v(n/d))K(\gcd(d, n/d)) \\
&= [(fu) *_K (h_f v)](n).
\end{aligned}$$

A similar calculation shows that $[h_f \cdot (u *_K (gv))](n) = [(h_f u) *_K (fv)](n)$. This proves the first part.

For the converse, assume the identity holds for some completely multiplicative non-zero valued function h. Dividing pointwise by h, we get

$$\left(\frac{f}{h}\right)(u *_K v) = \left(\left(\frac{f}{h}\right)u\right) *_K v + u *_K \left(\left(\frac{f}{h}\right)v\right).$$

By Theorem 3.1.1, this implies that the function f/h is completely additive. By Theorem 2.0.1, it follows that f is L-additive with $h_f = h$. □

Corollary 3.1.3. *Let D denote the arithmetic derivative and let $N(n) = n$. If $u, v \in \mathcal{A}$, then*

$$D(u *_K v) = (Du) *_K (Nv) + (Nu) *_K (Dv).$$

Proof. The arithmetic derivative D is L-additive with $h_D(n) = n = N(n)$. The function N is completely multiplicative and non-zero valued.

The result follows directly from Theorem 3.1.2. □

3.2 Characterizations via A-convolution

We now present the analogous results for Narkiewicz's A-convolution.

Definition 3.2.1. *An arithmetic function f is* ***A-additive*** *if $f(mn) = f(m)+f(n)$ for all m, n with $m, n \in A_N$ and $mn = N$ [9]. A function f is* L_A***-additive*** *if there is a completely multiplicative function h_f such that $f(mn) = f(m)h_f(n) + f(n)h_f(m)$ for all m, n with $m, n \in A_N$ and $mn = N$.*

Theorem 3.2.1. *Let $*_A$ be a commutative A-convolution. An arithmetic function f is A-additive if and only if for all $u, v \in \mathcal{A}$,*

$$f(u *_A v) = (fu) *_A v + u *_A (fv).$$

Proof. The proof is analogous to that of Theorem 3.1.1. The direct part follows by expanding the definitions. For the converse, we again choose $u = \delta_m$ and $v = \delta_n$ and evaluate at an integer N such that $m, n \in A_N$ and $mn = N$. The convolution $(u *_A v)(N) = \sum_{d \in A_N} \delta_m(d)\delta_n(N/d)$ will be non-zero only if $m \in A_N$ and $n = N/m$. Commutativity ensures $n \in A_N$ as well. The argument then proceeds identically, yielding $f(N) = f(m) + f(n)$. □

Theorem 3.2.2. *Let $*_A$ be a commutative A-convolution. Let f be an arithmetic function. If f is L_A-additive and h_f is non-zero valued, then for all $u, v \in \mathcal{A}$,*

$$f(u *_A v) = (fu) *_A (h_f v) + (h_f u) *_A (fv).$$

Conversely, if there is a completely multiplicative non-zero valued function h such that the identity holds for all $u, v \in \mathcal{A}$, then f is L_A-additive and $h_f = h$.

Proof. The proof follows the same structure as that of Theorem 3.1.2, but relies on the A-convolution versions of the preceding theorems. □

Corollary 3.2.3. *Let $*_A$ be commutative. If f is an L_A-additive function and h_f is non-zero valued, then for all $u \in \mathcal{A}$,*

$$f(u *_A u) = 2((fu) *_A (h_f u)).$$

Proof. This follows by setting $v = u$ in Theorem 3.2.2 and using the commutativity of $*_A$. □

4 Conclusion

This paper has rigorously established characterization theorems for completely additive and L-additive functions within the frameworks of K-convolution and Narkiewicz's A-convolution. The core contribution is the demonstration that Leibniz-type rules serve as necessary and sufficient conditions for these classes of functions, extending classical results for the Dirichlet convolution to a much broader family of operations.

Future work could explore several avenues. One direction is to investigate higher-order L-additive functions or to seek analogous characterizations for other classes of arithmetic functions. Another promising direction is to study the analytic consequences of these identities, such as deriving asymptotic formulas for summatory functions involving these generalized convolutions, which would further illuminate the deep connections between the algebraic and analytic properties of arithmetic functions.

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