Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2026, Volume 32, Number 1, 5–14
DOI: 10.7546/nntdm.2026.32.1.5-14

Note on the irrationality of certain infinite series

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Received: 1 October 2025 **Revised:** 15 December 2025
Accepted: 6 February 2026 **Online First:** 12 February 2026

Abstract: The aim of this paper is to introduce new criteria for real infinite series that satisfy a specific property and yield an irrational sum. These criteria are based on an extension of previous ideas proposed by Erdős. The paper includes several illustrative examples.
Keywords: Irrationality, Transcendence, Liouville number, Infinite series.
2020 Mathematics Subject Classification: 11J72, 11J82.

1 Introduction

Infinite series are a powerful tool in the analysis of irrational numbers. In this paper, we focus on those numbers that can be expressed as the sum of infinite series consisting of positive rationals and including a special subseries which converges very rapidly. The basis for the results could be considered the following theorem which was proven by Erdős in 1950.

Theorem 1.1 (Erdős, [2]). *Let (a_n) be a strictly increasing sequence of positive integers such that*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_1 a_2 \cdots a_n} = \infty. \tag{1}$$

Then the sum of the series $\sum_{n=1}^{\infty} 1/a_n$ is an irrational number.

There have been many generalisations and modifications of this theorem. In 1984, Sándor in [16] investigated the existence of a certain subsequence (a_n) which diverges very fast to $+\infty$. Moreover, he stipulated an additional condition for a sequence (a_n) to increase at a rate comparable to a geometric sequence.

Theorem 1.2 (Sándor, [16]). *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of positive integers such that*

$$\limsup_{n\to\infty} \frac{a_{n+1}}{a_1 a_2 \cdots a_n} \cdot \frac{1}{b_{n+1}} = \infty$$

and

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \cdot \frac{b_n}{b_{n+1}} > 1.$$

Then the sum of the series

$$\sum_{n=1}^{\infty} \frac{b_n}{a_n}$$

is an irrational number.

This result is closely related in nature to our Theorem 3.1, which involves similar $\limsup$ and $\liminf$ assumptions and deals with a broader class of infinite series. These results were followed up by Badea in 1987, who derived other interesting results concerning the irrationality of infinite series, see [1]. Several generalizations of Sándor's theorem have been proposed subsequently, including those by Hančl and Rucki [10], and Hančl, Rucki, and Šustek [11], which extend Sándor's criterion. Furthermore, Sándor's approach also inspired results on irrationality criteria for infinite products, see the work of Laohakosol and Kuhapatanakul [12].

Hančl and Rucki [8, 9] also proposed sufficient conditions under which the sum of an infinite series is a transcendental or Liouville number. These conditions are predicated on certain results from the theory of Diophantine approximations, particularly Roth's theorem, which will be utilised later in this paper.

Theorem 1.3 (Roth, [15]). *Let ξ be an algebraic number of degree $n \geq 2$. Then the inequality*

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^{2+\delta}}$$

has only finitely many solutions in coprime integers p, q with $q \neq 0$ for each $\delta > 0$.

For additional results concerning the transcendence of rapidly converging series, see the papers of Nyblom [13, 14]. There have been some intriguing results in recent times. In 2017, Hančl and Nair demonstrated in [6] that if (a_n) is a non-decreasing sequence of positive integers such that $\lim_{n\to\infty} \log^2 a_n / 2^{n^2} = \infty$, then the sum of the series $\sum_{n=1}^{\infty} 1/\sqrt{a_n}$ is an irrational number. In 2019, Hančl and Luca proved in [5] that if (a_n) is a non-decreasing sequence of positive integers such that $\lim_{n\to\infty} a_n^{1/3^n} = \infty$, then the sum of the series $\sum_{n=1}^{\infty} 1/\left(\sqrt{2} + a_n\right)$ is an irrational number. In 2023, Sghiouer, Belhroukia, and Kacha established new results in [17] concerning a transcendental measure of the series $\sum_{n=1}^{\infty} 1/\sqrt{a_n}$. Furthermore, the concept of proving the irrationality of rapidly converging series can also be extended to infinite products, see [3, 4].

The results of this paper are not as stringent as Theorem 1.1. We will consider infinite series that can converge significantly slower, such as p-series and nested logarithm series.

In other words, Theorem 3.1 and 3.2 introduce new criteria for determining the irrationality and transcendence of the sum of an infinite series. These criteria extend the results previously discussed in [7–9]. Additionally, the theorems and corollaries include specific examples of infinite series.

2 Notation and preliminaries

We will manipulate with the nested logarithmic function intensively. Therefore, we define it as follows

$$\log_k x := \begin{cases} x & \text{for } k = 0, \\ \log\log_{k-1} x & \text{for } k > 1, \end{cases}$$

provided that x is a sufficiently large real number. Since we will consider only the logarithm (to base e), there can be no ambiguity with k. It will always determine the number of nested logarithms.

To make further math expressions more transparent, we define an operator B_k as

$$\mathrm{B}_k\, x_n := (x_n - 1)\log_k n, \qquad k \in \mathbb{N}_0,\ n \geq n_0, \tag{2}$$

where (x_n) is a real sequence.

At the end of this section, let us remind the concept of the irrationality measure $\mu(\xi)$ of a real number ξ. It is the supremum of $m \in \mathbb{R}^+$ such that the relation

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^m}$$

is fulfilled for infintely many rationals p/q in lowest terms.

3 Main results

In the next theorem, we introduce two sufficient conditions under which the sum of the series is an irrational number.

Theorem 3.1. *Let (a_n), (b_n) be two sequences of positive integers and $k \in \mathbb{N}_0$ such that*

$$\limsup_{n\to\infty}\left(\frac{a_{n+1}}{b_{n+1}\prod_{i=0}^{k}\log_i n} \cdot \frac{1}{a_1 a_2 \cdots a_n}\right) = \infty \tag{3}$$

and

$$\liminf_{n\to\infty} \mathrm{B}_k\,\mathrm{B}_{k-1}\cdots\mathrm{B}_1\,\mathrm{B}_0\left(2 - \frac{b_{n+1}}{a_{n+1}} \cdot \frac{a_n}{b_n}\right) > 1. \tag{4}$$

Then the sum ξ of the series

$$\sum_{n=1}^{\infty} \frac{b_n}{a_n}$$

is an irrational number.

Example 3.1. *Let* $a_1 = 1$. *Suppose for every positive integer* n *that*

$$a_{n+1} = \begin{cases} n^2 \cdot a_1 a_2 \cdots a_n, & \textit{if } n = 10^{10^{m!}},\ m \in \mathbb{N}, \\ a_n + \lfloor H_n^2 + 2H_n \rfloor, & \textit{otherwise,} \end{cases}$$

where H_n *is the* n*-th harmonic number. Then* $\xi = \sum_{n=1}^{\infty} 1/a_n$ *is an irrational number.*

Let us put $k = 1$ in Theorem 3.1. The recurrence relation $\Delta a_n = H_n^2 + 2H_n$ is satisfied by a sequence (a_n) that can be expressed in the form

$$a_n = a_{10^{10^{m!}}} + n \log^2 n + \mathcal{O}(n \log n), \qquad 10^{10^{m!}} + 1 < n \leq 10^{10^{(m+1)!}}.$$

It implies immediately that the lower limit (4) equals 2, so all the conditions of Theorem 3.1 are fulfiled and ξ is an irrational number.

In the following part, we will present some resultshlconcerning the irrationality measure of real numbers in a special form.

The next theorem gives a lower bound on the irrationality measure of the sum of an infinite series consisting of positive rationals. It contains two necessary conditions with a similar meaning as in Theorem 3.1.

Theorem 3.2. *Let* (a_n), (b_n) *be two sequences of positive integers,* $\delta \geq 2$ *and* $k \in \mathbb{N}_0$. *Suppose that*

$$\limsup_{n\to\infty} \left[\log\left(\frac{a_{n+1}}{b_{n+1}}\right) \cdot \frac{1}{\log(a_1 a_2 \ldots a_n)} \right] = \delta \tag{5}$$

and

$$\liminf_{n\to\infty} \mathrm{B}_k\, \mathrm{B}_{k-1} \ldots \mathrm{B}_1\, \mathrm{B}_0 \left(2 - \frac{b_{n+1}}{a_{n+1}} \cdot \frac{a_n}{b_n}\right) > 1. \tag{6}$$

Then the sum ξ *of the series*

$$\xi = \sum_{n=1}^{\infty} \frac{b_n}{a_n}$$

has the irrationality measure $\mu(\xi) \geq \delta$.

The following corollaries are derived from Theorem 3.2 by setting $k = 0$ and $\delta > 2$, respectively, $\delta = \infty$.

Corollary 3.1. *Let* (a_n) *and* (b_n) *be two sequences of positive integers. Suppose that*

$$\limsup_{n\to\infty} \left[\log\left(\frac{a_{n+1}}{b_{n+1}}\right) \cdot \frac{1}{\log(a_1 a_2 \ldots a_n)} \right] > 2$$

and

$$\liminf_{n\to\infty} n \left(1 - \frac{b_{n+1}}{a_{n+1}} \cdot \frac{a_n}{b_n}\right) > 1. \tag{7}$$

Then the sum ξ *of the series*

$$\sum_{n=1}^{\infty} \frac{b_n}{a_n}$$

is a transcendental number.

The fact that the sum is a transcendental number follows immediately from Roth's Theorem (Theorem 1.3).

The following corollary presents sufficient conditions for an infinite series whose sum is a Liouville number. Recall that Liouville numbers are real numbers with an infinite irrationality measure.

Corollary 3.2. *Let (a_n) and (b_n) be two sequences of positive integers. Suppose that*

$$\limsup_{n\to\infty}\left[\log\left(\frac{a_{n+1}}{b_{n+1}}\right)\cdot\frac{1}{\log(a_1a_2\cdots a_n)}\right]=\infty$$

and

$$\liminf_{n\to\infty} n\left(1-\frac{b_{n+1}}{a_{n+1}}\cdot\frac{a_n}{b_n}\right)>1.$$

Then the sum ξ of the series

$$\sum_{n=1}^{\infty}\frac{b_n}{a_n}$$

is a Liouville number.

Example 3.2. *Let $a_1=b_1=1$. Suppose for every positive integer n that*

$$a_{n+1}=\begin{cases}(a_1a_2\cdots a_n)^{\varepsilon_n}, & \text{if } n=(m!)^{m!},\ m\in\mathbb{N},\\ (2n+1)a_n, & \text{otherwise,}\end{cases}$$

$$b_{n+1}=(2n-2)b_n,$$

For a constant sequence $\varepsilon_n=4$, $n\in\mathbb{N}$, we get that

$$\xi=\sum_{n=1}^{\infty}\frac{b_n}{a_n}$$

is a transcendental number. On the other hand, if we take $\varepsilon_n=p_n$ as the n-th prime, then the sum ξ is a Liouville number.

It easy to verify that $b_n/a_n=\mathcal{O}(n^{-3/2})$ as $n\to\infty$ and hence the lower limit (7) in Corollary 3.1 is at least $3/2$.

4 Auxiliary statements

The purpose of the auxiliary Lemma 4.1 is to find an appropriate estimate for tails of infinite series which are under our observation. We achieve this using a generalization of Bertrand's convergence test. By combining this approach with specific results from the theory of Diophantine approximation, we establish Theorem 3.1 which provides sufficient conditions for an infinite series to have an irrational sum. Additionally, we introduce Theorem 3.2 and its corollary which deal with the irrationality measure of the sum of a particular type of infinite series and offer a lower bound for this measure.

Lemma 4.1. *Let (x_n) be a sequence of positive real numbers such that $x_n \to 0$ as $n \to \infty$ and*

$$\liminf_{n\to\infty} \mathrm{B}_k\,\mathrm{B}_{k-1}\cdots\mathrm{B}_1\,\mathrm{B}_0\left(2-\frac{x_{n+1}}{x_n}\right) > 1, \qquad k \in \mathbb{N}_0. \tag{8}$$

Then there exists a real number $A > 1$, a positive integer n_0 and a sequence (c_n) of real numbers approaching 1 with $c_n > 1$ such that

$$\sum_{m=n+1}^{\infty} x_m < \frac{c_n}{A-1}\cdot x_{n+1}\cdot\prod_{j=0}^{k}\log_j n, \qquad \forall n > n_0. \tag{9}$$

Proof. Inequality (8) ensures the existence of a real constant $A > 1$ and a number $n_0 \in \mathbb{N}$ such that

$$\mathrm{B}_k\,\mathrm{B}_{k-1}\cdots\mathrm{B}_1\,\mathrm{B}_0\left(2-\frac{x_{n+1}}{x_n}\right) > A, \qquad \forall n > n_0.$$

Using this relation and (2) $(k+1)$-times, we can express the ratio x_{n+1}/x_n in the following way

$$\begin{aligned}
\mathrm{B}_{k-1}\,\mathrm{B}_{k-2}\cdots\mathrm{B}_1\,\mathrm{B}_0\left(2-\frac{x_{n+1}}{x_n}\right) &> \frac{A}{\log_k n}+1\\
\mathrm{B}_{k-2}\cdots\mathrm{B}_1\,\mathrm{B}_0\left(2-\frac{x_{n+1}}{x_n}\right) &> \frac{A}{\log_{k-1} n\cdot\log_k n}+\frac{1}{\log_{k-1} n}+1\\
&\vdots\\
\frac{x_{n+1}}{x_n} &< 1-\frac{A}{\prod_{i=0}^{k}\log_i n}-\sum_{j=0}^{k-1}\frac{1}{\prod_{i=0}^{j}\log_i n}
\end{aligned}$$

with $n > n_0$. In general, for every $s = 1, 2, \ldots$ we can estimate the term x_{n+s} using x_n and a certain function of n from above

$$\begin{aligned}
x_{n+s} &= x_n\prod_{m=0}^{s-1}\frac{x_{n+m+1}}{x_{n+m}}\\
&< x_n\prod_{m=0}^{s-1}\left(1-\frac{A}{\prod_{i=0}^{k}\log_i(n+m)}-\sum_{j=0}^{k-1}\frac{1}{\prod_{i=0}^{j}\log_i(n+m)}\right).
\end{aligned}$$

Now we simplify the product on the right-hand side by applying the relation $\log(1-t) \le -t$ with $t < 1$ and the elementary integral identity

$$\int\frac{\mathrm{d}t}{\prod_{i=0}^{k}\log_i t} = \log_{k+1} t + c, \quad k \ge 0,\ c \in \mathbb{R}.$$

Hence

$$\begin{aligned}
x_{n+1+s} &< x_{n+1} \exp\left(\sum_{m=0}^{s-1} \log\left(1 - \frac{A}{\prod_{i=0}^{k} \log_i(n+1+m)} - \sum_{j=0}^{k-1} \frac{1}{\prod_{i=0}^{j} \log_i(n+1+m)}\right)\right) \\
&\le x_{n+1} \exp\left(-\sum_{m=0}^{s-1} \left(\frac{A}{\prod_{i=0}^{k} \log_i(n+1+m)} + \sum_{j=0}^{k-1} \frac{1}{\prod_{i=0}^{j} \log_i(n+1+m)}\right)\right) \\
&< x_{n+1} \exp\left(-\int_{n+1}^{n+1+s} \left(\frac{A}{\prod_{i=0}^{k} \log_i t} + \sum_{j=0}^{k-1} \frac{1}{\prod_{i=0}^{j} \log_i t}\right) \mathrm{d}t\right) \\
&= x_{n+1} \exp\left(\log\left(\frac{\log_k(n+1)}{\log_k(n+1+s)}\right)^A + \sum_{j=0}^{k-1} \log\left(\frac{\log_j(n+1)}{\log_j(n+1+s)}\right)\right) \\
&= x_{n+1} \left(\frac{\log_k(n+1)}{\log_k(n+s+1)}\right)^A \cdot \prod_{j=0}^{k-1} \left(\frac{\log_j(n+1)}{\log_j(n+s+1)}\right).
\end{aligned}$$

Shifting the parameter $s+1$ to s, we get

$$x_{n+s} \le x_{n+1} \left(\frac{\log_k(n+1)}{\log_k(n+s)}\right)^A \cdot \prod_{j=0}^{k-1} \left(\frac{\log_j(n+1)}{\log_j(n+s)}\right), \qquad n > n_0,\ s \in \mathbb{N}. \tag{10}$$

We have estimated all the terms of the sequence (x_n) with large enough indices using x_n and x_{n+1}, respectively, with a fixed index n so far.

In the following part, we construct an estimate of the tail of the series that consists of these terms. From (10) and from the obvious formula

$$\int \frac{\mathrm{d}t}{\log_k^\alpha t \prod_{i=0}^{k-1} \log_i t} = -\frac{1}{\alpha-1} \cdot \frac{1}{\log_k^{\alpha-1} t} + c, \quad \alpha \ne 1,\ k \ge 0,\ c \in \mathbb{R},$$

we deduce that

$$\begin{aligned}
\sum_{s=1}^{\infty} x_{n+s} &\le x_{n+1} \sum_{s=1}^{\infty} \left(\frac{\log_k(n+1)}{\log_k(n+s)}\right)^A \cdot \prod_{j=0}^{k-1} \left(\frac{\log_j(n+1)}{\log_j(n+s)}\right) \\
&= x_{n+1} \cdot \log_k^A(n+1) \cdot \prod_{j=0}^{k-1} \log_j(n+1) \cdot \sum_{s=1}^{\infty} \frac{1}{\log_k^A(n+s) \cdot \prod_{j=0}^{k-1} \log_j(n+s)} \\
&\le x_{n+1} \cdot \log_k^A(n+1) \cdot \prod_{j=0}^{k-1} \log_j(n+1) \cdot \int_n^\infty \frac{\mathrm{d}t}{\log_k^A t \cdot \prod_{j=0}^{k-1} \log_j t} \\
&= x_{n+1} \cdot \frac{1}{A-1} \cdot \frac{\log_k^A(n+1)}{\log_k^{A-1} n} \cdot \prod_{j=0}^{k-1} \log_j(n+1), \quad n > n_0.
\end{aligned} \tag{11}$$

Now using the limit

$$\lim_{n\to\infty} \frac{\log_k^A(n+1)}{\log_k^A n} \cdot \prod_{j=0}^{k-1} \frac{\log_j(n+1)}{\log_j n} = 1,$$

we can infer that there exists a sequence (c_n) with $c_n > 1$ approaching 1 so that

$$\log_k^A(n+1) \cdot \prod_{j=0}^{k-1} \log_j(n+1) \le c_n \cdot \log_k^A n \cdot \prod_{j=0}^{k-1} \log_j n.$$

Finally, this fact and (11) yield the assertion of the lemma, thus

$$\sum_{m=n+1}^{\infty} x_m \le \frac{c_n}{A-1} \cdot x_{n+1} \cdot \prod_{j=0}^{k} \log_j n. \qquad \square$$

5 Proofs

Proof of Theorem 3.1. For contradiction suppose that $\xi = \sum_{n=1}^{\infty} b_n/a_n$ is a rational number in its simplest form p/q. Multiplying both sides of the following relation

$$\frac{p}{q} = \sum_{m=1}^{n} \frac{b_m}{a_m} + \sum_{m=n+1}^{\infty} \frac{b_m}{a_m}$$

by $qa_1a_2\cdots a_n$, we get the positive integer on the right-hand side and, consequently,

$$qa_1a_2\cdots a_n \sum_{m=n+1}^{\infty} \frac{b_m}{a_m} = pa_1a_2\cdots a_n - qa_1a_2\cdots a_n \sum_{m=1}^{n} \frac{b_m}{a_m}.$$

This implies immediately that the expression on the left-hand side containing the tail of the series must be a positive integer, as well. Since the sequence (b_n/a_n) fulfils all the conditions of Lemma 4.1, we use it to estimate the tail of the given series. In other words, there exist a real number $A > 1$, a positive integer n_0 and a sequence (c_n) with $c_n > 1$ approaching 1, such that

$$1 \le qa_1a_2\ldots a_n \sum_{m=n+1}^{\infty} \frac{b_m}{a_m} < \frac{qc_n}{A-1} \cdot a_1a_2\cdots a_n \cdot \frac{b_{n+1}}{a_{n+1}} \cdot \prod_{j=0}^{k} \log_j n, \qquad \forall n > n_0. \tag{12}$$

Assumption (3) implies that for every real $M > 0$ there exist infinitely many n's such that

$$\frac{a_{n+1}}{b_{n+1}\prod_{i=0}^{k}\log_i n} \cdot \frac{1}{a_1a_2\cdots a_n} > M. \tag{13}$$

Let us consider such indices $n \ge n_0$ for which relation (13) holds true. Taking a real constant $M > qc_n/(A-1)$ and using (12) and (13), we deduce that

$$1 \le qa_1a_2\cdots a_n \sum_{m=n+1}^{\infty} \frac{b_m}{a_m} < \frac{qc_n}{A-1} \cdot \frac{1}{M} < 1.$$

However, it is a contradiction. Hence ξ is an irrational number and the proof is complete. $\square$

Proof of Theorem 3.2. Similarly to the proof of Theorem 3.1, Lemma 4.1 ensures the existence of a real number $A > 1$, a positive integer n_0 and a sequence (c_n) with $c_n > 1$ approaching 1 such that

$$\sum_{m=n+1}^{\infty} \frac{b_m}{a_m} < \frac{c_n}{A-1} \cdot \frac{b_{n+1}}{a_{n+1}} \cdot \prod_{j=0}^{k} \log_j n, \qquad \forall n > n_0. \tag{14}$$

Considering assumption (5) in more detail, we deduce that for every real $\varepsilon > 0$ there exist infinitely many positive integers $n > n_0$ such that

$$\log\left(\frac{a_{n+1}}{b_{n+1}}\right) \cdot \frac{1}{\log(a_1 a_2 \cdots a_n)} > \delta - \varepsilon \qquad \Rightarrow \qquad \frac{b_{n+1}}{a_{n+1}} < \frac{1}{(a_1 a_2 \cdots a_n)^{\delta-\varepsilon}}. \tag{15}$$

As (a_n) is a sequence diverging to $+\infty$, we can find a positive integer n_1 such that $a_n \geq 2$ for every $n > n_1$ and

$$(a_1 a_2 \cdots a_N)^{\varepsilon} \geq 2^{\varepsilon(N-n_1)} \geq \frac{c_N}{A-1} \cdot \prod_{j=0}^{k} \log_j N \tag{16}$$

for every $N > n_1$ simultaneously. Let $n_2 = \max(n_0, n_1)$. From (14), (15) and (16) we infer for infinitely many $N > n_2$ that

$$\sum_{m=N+1}^{\infty} \frac{b_m}{a_m} \leq \frac{1}{(a_1 a_2 \cdots a_N)^{\delta-2\varepsilon}}. \tag{17}$$

Let us consider a sequence of rational approximations p_N/q_N generated from the N-th partial sums of $\xi = \sum_{n=1}^{\infty} b_n/a_n$ and expressed in reduced form. Using the obvious relation $q_N \leq \operatorname{lcm}(a_1 a_2 \cdots a_N) \leq a_1 a_2 \cdots a_N$ and inequality (17), we deduce that

$$\left|\xi - \frac{p_N}{q_N}\right| = \left|\xi - \sum_{m=1}^{N} \frac{b_m}{a_m}\right| < \frac{1}{(a_1 a_2 \cdots a_N)^{\delta-2\varepsilon}} \leq \frac{1}{q_N^{\delta-2\varepsilon}}$$

for infinitely many positive integers N. This and the fact that ε was chosen arbitrarily yield the estimate of the irrationality measure of ξ, thus $\mu(\xi) \geq \delta$. ☐

6 Conclusion

In this paper we have introduced new sufficient conditions for determining the irrationality and transcendence of sums of infinite series of rational terms. By extending the approach initiated by Erdős and subsequently developed by Sándor, Badea, Hančl, Rucki, and others, we established criteria that apply to a wider class of slowly converging series, including p-series and nested logarithmic series.

Theorem 3.1 provides explicit conditions under which the sum of a series must be irrational, while Theorem 3.2 gives lower bounds on the irrationality measure of such sums. From these results we derived corollaries identifying series with transcendental sums as well as series yielding Liouville numbers.

The relation between convergence rate and Diophantine properties of infinite series presents many open questions for future investigation of sums of infinite series and of analogous problems concerning infinite products.

Acknowledgements

The author would like to thank Marian Genčev for his valuable technical and language suggestions.

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