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A note on periodic linear recurrence relations

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Abstract: We provide an elementary proof of the fact that a sequence defined by a linear recurrence relation with integer coefficients is periodic if and only if all characteristic roots are distinct roots of unity. Additionally, we discuss the case in which the coefficients of the recurrence relation are restricted to the set $\{-1, 0, 1\}$.
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1 Introduction

We study homogeneous linear recurrence relations of the form

$$s_{n+k} = c_{k-1}s_{n+k-1} + c_{k-2}s_{n+k-2} + \cdots + c_0 s_n \tag{1}$$

with integer coefficients $c_0, \ldots, c_{k-1}$, where $c_0 \neq 0$. A sequence (s_n) is uniquely determined by its initial values $s_0, \ldots, s_{k-1}$.

In a recent paper, Atanassov and Shannon [1] proved that sequences (s_n) defined by the coefficients $c_{k-1} = 1, c_{k-2} = -1, c_{k-3} = 1, c_{k-4} = -1, \ \ldots$ are periodic. They also gave explicit formulas for s_n, distinguishing between the cases where the order k is odd or even. A similar result was obtained by Gryszka [5] for sequences defined by coefficients of the types $0, -1, 0, -1, \ldots$ and $0, 1, 0, -1, 0, 1, 0, -1, \ldots$. It is worth mentioning that the characteristic polynomials for the sequences mentioned above are

$$P_1(x) = x^k - x^{k-1} + x^{k-2} - \cdots + (-1)^k \,,$$
$$P_2(x) = x^{2k} + x^{2k-2} + x^{2k-4} + \cdots + 1 \,,$$

and

$$P_3(x) = x^{2k} - x^{2k-2} + x^{2k-4} \cdots + (-1)^k \,,$$

respectively. All of them are divisors of a polynomial $x^m - 1$ for a suitable integer m.

This note aims to characterize linear recurrence relations with integer coefficients that generate periodic sequences. Our main result follows, for example, from more general results discussed in Chapter 3 of the book by Everest *et al.* [3].

2 Results

Theorem 1. *Let the sequence* (s_n) *be defined by the recurrence relation* (1) *with integer coefficients* c_i *for* $i = 0, 1, \ldots, k-1$, *where* $c_0 \neq 0$. *Then* (s_n) *is periodic for any initial values* $s_0, \ldots, s_{k-1}$ *if and only if all the roots of the characteristic polynomial*

$$P(x) = x^k - c_{k-1}x^{k-1} - c_{k-2}x^{k-2} - \cdots - c_1 x - c_0$$

are distinct and and lie on the unit circle.

Proof. First, suppose the characteristic polynomial $P(x)$ has only simple roots $\lambda_1, \lambda_2, \ldots, \lambda_k$ lying on the unit circle.

It is known that the closed-form solution for s_n is

$$s_n = a_1\lambda_1^n + a_2\lambda_2^n + \cdots + a_k\lambda_k^n \,.$$

Furthermore, for a given set of k initial conditions, the constants $a_1, a_2, \ldots, a_k$ are uniquely determined. By Kronecker's theorem, every algebraic integer λ on the unit circle is a root of unity (see, e.g., [5]). Therefore, for each root λ_j ($j = 1, 2, \ldots, k$), there exists a positive integer m_j (its multiplicative order) such that $\lambda_j^{m_j} = 1$. Let m be the least common multiple of $m_1, m_2, \ldots, m_k$. Then for any nonnegative integer n and for all j, we have $\lambda_j^{m+n} = \lambda_j^n$, and consequently, $s_{n+m} = s_n$, which proves the sequence (s_n) is periodic.

On the other hand, suppose, to the contrary, that the characteristic polynomial $P(x)$ has a multiple root or at least one root that does not lie on the unit circle and (s_n) is periodic for any initial values. The constant term c_0 is a nonzero integer. It follows from Vieta's formulas that if there is a root inside the unit circle, then there must also be a root outside the unit circle.

So, without loss of generality, we may assume that a root λ_1 of the polynomial $P(x)$ lies outside the unit circle or has multiplicity at least two.

Let $P(x)$ have distinct roots $\lambda_1, \lambda_2, \ldots, \lambda_r$ with multiplicities $t_1, t_2, \ldots, t_r$, respectively. In this case, the general closed-form solution for (s_n) is

$$s_n = a_1(n)\lambda_1^n + \cdots + a_r(n)\lambda_r^n \,, \tag{2}$$

where each $a_j(n)$ is a polynomial whose degree is less than t_j.

Now, suppose, for the sake of contradiction, that the sequence (s_n) defined by the original recurrence is periodic for any choice of initial values $s_0, \ldots, s_{k-1}$. We will show this leads to a contradiction. The coefficients $a_j(n)$ in (2) are not arbitrary; they are uniquely determined by the initial values. We are free to choose these initial values to construct a specific sequence that violates periodicity. Consider a choice of initial values such that in the closed form (2), we have $a_2(n) = a_3(n) = \cdots = 0$ for all n, and $a_1(n)$ is a non-zero constant polynomial. This is always possible by an appropriate choice of initial conditions.

Now, we analyze the two cases from our initial assumption:

- Case 1: λ_1 is a multiple root. In this case, we can choose initial values such that $a_1(n)$ is a non-constant polynomial (e.g., $a_1(n) = n$).
- Case 2: λ_1 is a simple root with $|\lambda_1| > 1$. In this case, $a_1(n)$ is a constant, which we can take to be 1.

In both cases, the resulting sequence simplifies to $s_n = a_1(n)\lambda_1^n$. We then have

$$\lim_{n\to\infty} |s_n| = +\infty \ .$$

A sequence that diverges to infinity cannot be periodic. This contradicts the assumption that (s_n) is periodic for any initial values. □

We recall the definition of cyclotomic polynomials. The primitive n-th roots of unity are the complex roots of $x^n - 1$ that are not roots of $x^d - 1$ for any positive divisor $d < n$; that is, their multiplicative order is exactly n. The n-th cyclotomic polynomial, denoted $\Phi_n(x)$, is the monic polynomial whose complex roots are precisely the primitive n-th roots of unity.

The first six cyclotomic polynomials are:

$$\Phi_1(x) = x - 1\ , \quad \Phi_2(x) = x + 1\ , \quad \Phi_3(x) = x^2 + x + 1\ , \quad \Phi_4(x) = x^2 + 1\ ,$$
$$\Phi_5(x) = x^4 + x^3 + x^2 + x + 1\ , \quad \Phi_6(x) = x^2 - x + 1\ .$$

A fundamental identity involving cyclotomic polynomials is:

$$x^n - 1 = \prod_{d|n} \Phi_d(x)\ .$$

where the product is taken over all positive divisors d of n.

As a consequence of Theorem 1, we immediately have that any sequence whose characteristic polynomial is a product of pairwise distinct cyclotomic polynomials is periodic.

We now turn our attention to the case when the coefficients of the linear recurrence relation (1) belong to the set $\{-1, 0, 1\}$. It is known that for any $n < 105$, the coefficients of the cyclotomic polynomial $\Phi_n(x)$ are -1, 0 or 1. This property also holds for $\Phi_{pq}(x)$, where p and q are distinct primes (see [2]).

Let $\mathcal{U}$ denote the set of all monic polynomials with coefficients in $\{-1, 0, 1\}$, all of whose roots are simple roots of unity. Then, for example, for any positive integer n we have

$$x^n - 1 \in \mathcal{U}\ , \quad x^n + 1 \in \mathcal{U}\ , \quad \Phi_n(x) \in \mathcal{U} \ \text{(for } n < 105)\ .$$

Giving a complete characterization of $\mathcal{U}$ seems to be a very hard problem. We give some simple sufficient conditions for a polynomial to belong to $\mathcal{U}$.

Proposition 1.

(i) Let $P(x) \in \mathcal{U}$ be a polynomial with coefficients in $\{0,1\}$. Then $(1-x)P(x) \in \mathcal{U}$.

(ii) Let $P(x) \in \mathcal{U}$ be a polynomial of degree h and let $Q(x) \in \mathcal{U}$ be of the form

$$Q(x) = x^{t_m} + a_{t_{m-1}} x^{t_{m-1}} + a_{t_{m-2}} x^{t_{m-2}} + \cdots + a_{t_1} x^{t_1} + a_0 ,$$

where $t_1 > h$ and $t_i > t_{i-1} + h$ for $i = 2, 3, \ldots, m$. If $P(x)$ and $Q(x)$ are coprime, then $P(x)Q(x) \in \mathcal{U}$.

The proofs are straightforward and are left to the reader.

To conclude, we give an example of a periodic sequence defined by the recurrence relation:

$$s_{n+2k+1} = -s_{n+2k} - s_{n+1} - s_n \quad (n = 0, 1, 2, \ldots, n) , \tag{3}$$

where k is a fixed non-negative integer. The corresponding characteristic polynomial is

$$x^{2k+1} + x^{2k} + x + 1 = (x+1)(x^{2k}+1) \in \mathcal{U} .$$

Let the initial conditions $s_0, s_1, \ldots, s_{2k}$ be given. It can be shown that the sequence defined by (3) is periodic with period $4k$. The sequence for one period is $(s_0, s_1 . s_2, \ldots, s_{4k-1})$, where

$$s_{2k+m} = (-1)^m (s_{2k} + s_0) - s_m \quad \text{for } m = 1, 2, \ldots, 2k-1 .$$

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