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Transformations of Pythagorean triples generated by generalized Fibonacci numbers

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Abstract: We present matrices that transform Pythagorean triples arising from generalized Fibonacci sequences into other such triples. We also show that entries in the powers of such matrices can be expressed in terms of generalized Fibonacci sequences.
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1 Introduction and preliminaries

A *Pythagorean triple* (PT) is an ordered triple of positive integers (a, b, c) such that $a^2 + b^2 = c^2$. A PT is called *primitive* provided $\gcd(a, b, c) = 1$. Any primitive PT (PPT) can be written in the form $(m^2 - n^2, 2mn, m^2 + n^2)$ for positive integers m and n with $m > n$, $m + n$ odd, and $\gcd(m, n) = 1$ (see, e.g., [3], p. 248). A *Pythagorean triple preserving matrix (PTPM)* is a 3×3 matrix that transforms any given PT into another when the PTs are expressed as column vectors (see, e.g., [6]). We define a *generalized Fibonacci sequence*, $f_k(a, b)$, by $f_0(a, b) = 0$, $f_1(a, b) = 1$ and $f_k(a, b) = af_{k-1}(a, b) + bf_{k-2}(a, b)$ for $k \geq 2$, where a, b are positive integers. For readability, we will typically write f_k instead of $f_k(a, b)$.

Palmer *et al.* [6] found the general form of a PTPM, as detailed in the following theorem.

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Theorem 1.1. *Let r, s, t, u, m, and n be any real or complex numbers. Then*

$$\begin{bmatrix} \frac{(r^2-t^2)-(s^2-u^2)}{2} & rt-su & \frac{(r^2+t^2)-(s^2+u^2)}{2} \\ rs-tu & ru+st & rs+tu \\ \frac{(r^2-t^2)+(s^2-u^2)}{2} & rt+su & \frac{(r^2+t^2)+(s^2+u^2)}{2} \end{bmatrix} \begin{bmatrix} m^2-n^2 \\ 2mn \\ m^2+n^2 \end{bmatrix} = \begin{bmatrix} v^2-w^2 \\ 2vw \\ v^2+w^2 \end{bmatrix}$$

where $v = mr + nt$ and $w = ms + nu$.

While it has not been fully considered in the literature, it is worth noting that for certain choices of r, s, t, and u, $(v^2 - w^2, 2vw, v^2 + w^2)$ is not guaranteed to be a PT. For example, when $r = 3$, $t = -3$, $s = 1$, and $u = 0$, the PT $(5, 12, 13)$ is transformed into the triple $(0, 18, 18)$, which is not a PT. We will avoid this slight complication in the discussion that follows by choosing our values for r, s, t, and u carefully.

Austin and Schneider [2] showed that the general matrix form in Theorem 1.1 gives rise to the following matrix that transforms the PT generated by $m = f_k$ and $n = f_{k-1}$ to the PT generated by $v = f_{k+1}$ and $w = f_k$ by choosing $r = a, t = b, s = 1$ and $u = 0$:

$$M_{a,b} = \begin{bmatrix} \dfrac{a^2-b^2-1}{2} & ab & \dfrac{a^2+b^2-1}{2} \\ a & b & a \\ \dfrac{a^2-b^2+1}{2} & ab & \dfrac{a^2+b^2+1}{2} \end{bmatrix}.$$

In other words, $M_{a,b}$ transforms a PT generated by consecutive generalized Fibonacci numbers from $f_k(a, b)$ into the next such PT. We will only consider positive values of a and b to guarantee $M_{a,b}$ is a PTPM. Austin and Schneider provided a form for the jth power of this matrix for the case when $b = 1$ [2]. The case when $a = 1$ arises when examining families of PPTs presented by Leyendekkers and Shannon [5], but no form for the jth power of these matrices has been presented [1].

In this paper, we give a form for the jth power of $M_{a,b}$ for any positive integers a and b. We also construct a more general collection of PTPMs that transform a PT generated by f_k and f_{k-g} to one generated by f_{k+g} and f_k (where $1 \le g \le k$).

2 The form of $M_{a,b}^j$

We will need the following lemma regarding $f_k(a, b)$ which can be found in [4].

Lemma 2.1. *For positive integers c and d such that $c \ge d$, $f_c = f_d f_{c-d+1} + b f_{d-1} f_{c-d}$.*

Note that since $M_{a,b}$ transforms a PT generated by consecutive generalized Fibonacci numbers to the next such PT, $M_{a,b}^j$ transforms the PT generated by $m = f_k$ and $n = f_{k-1}$ to the PT generated by $v = f_{k+j}$ and $w = f_{k+j-1}$. Applying Lemma 2.1 with $c = k + j$ and $d = j + 1$, we have that in the resulting PT $(v^2 - w^2, 2vw, v^2 + w^2)$, $v = f_{j+1} f_k + b f_j f_{k-1}$. Similarly, using $c = k + j - 1$ and $d = j$, we obtain that $w = f_j f_k + b f_{j-1} f_{k-1}$.

Therefore, if we choose $r = f_{j+1}$, $s = f_j$, $t = bf_j$, and $u = bf_{j-1}$, we have that $v = mr + nt$ and $w = ms + nu$. Since this is true for every positive integer k, we may easily verify that the linear transformation in question is uniquely determined. Hence, since $M_{a,b}$ is a PTPM, the matrix generated by Theorem 1.1 with these values of r, s, t and u yields the same linear transformation as $M_{a,b}^j$, for a given positive integer j. This directly implies the following theorem.

Theorem 2.1. *For any positive integer j,*

$$M_{a,b}^j = \begin{bmatrix} \frac{f_{j+1}^2-(b^2+1)f_j^2+b^2f_{j-1}^2}{2} & bf_j(f_{j+1}-f_{j-1}) & \frac{f_{j+1}^2+(b^2-1)f_j^2-b^2f_{j-1}^2}{2} \\ f_j(f_{j+1}-b^2f_{j-1}) & b(f_{j+1}f_{j-1}+f_j^2) & f_j(f_{j+1}+b^2f_{j-1}) \\ \frac{f_{j+1}^2-(b^2-1)f_j^2-b^2f_{j-1}^2}{2} & bf_j(f_{j+1}+f_{j-1}) & \frac{f_{j+1}^2+(b^2+1)f_j^2+b^2f_{j-1}^2}{2} \end{bmatrix}.$$

We can represent the j-th power of $M_{a,b}$ differently by using the definition of f_j and applying the following identities [4]:

$$f_{j-1}f_{j+1} - f_j^2 = (-1)^j b^{j-1} \tag{1}$$

and

$$f_{2j} = f_j f_{j+1} + bf_{j-1}f_j. \tag{2}$$

Identity (2) can be rewritten as

$$f_{2j} = f_j f_{j+1} + bf_{j-1}f_j = f_j(f_{j+1} + bf_{j-1}) = af_j^2 + 2bf_jf_{j-1}, \tag{3}$$

and the rest of the algebra, though a bit tedious, is straightforward. We obtain that

$$M_{a,b}^j = \begin{bmatrix} \left(\frac{a^2-(b-1)^2}{2}\right)f_j^2 + (-b)^j & \frac{a(b+1)f_j^2+(b-1)f_{2j}}{2} & \frac{af_{2j}+(b^2-1)f_j^2}{2} \\ \frac{a(b+1)f_j^2+(1-b)f_{2j}}{2} & 2bf_j^2 + (-b)^j & \frac{a(1-b)f_j^2+(b+1)f_{2j}}{2} \\ \frac{af_{2j}+(1-b^2)f_j^2}{2} & \frac{a(b-1)f_j^2+(b+1)f_{2j}}{2} & \left(\frac{a^2+(b+1)^2}{2}\right)f_j^2 + (-b)^j \end{bmatrix}.$$

Notice that this form of $M_{a,b}^j$ can be more easily compared to and agrees with Austin and Schneider's result when $b = 1$ [2]:

Theorem 2.2. *Let $\{t_j\}$ be the sequence defined by $t_0 = 0, t_1 = 1$ and $t_j = at_{j-1} + t_{j-2}$ for $j \geq 2$. Then, for $j \geq 1$,*

$$M_{a,1}^j = \begin{bmatrix} \frac{a^2}{2}t_j^2 + (-1)^j & at_j^2 & \frac{a}{2}t_{2j} \\ at_j^2 & 2t_j^2 + (-1)^j & t_{2j} \\ \frac{a}{2}t_{2j} & t_{2j} & \left(\frac{a^2}{2}+2\right)t_j^2 + (-1)^j \end{bmatrix}.$$

We can also express any PT generated by consecutive terms of $f_k(a, b)$ as $M_{a,b}^j \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$, where j is a positive integer. The triple $(1, 0, 1)$ is not a PT but is generated by the first two terms of $f_k(a, b)$.

3 Generalizing $M_{a,b}$

$M^j_{a,b}$ transforms a PT generated by consecutive terms of f_k to another such PT with an arbitrarily large gap (i.e., j) between the pairs of consecutive terms. Now let us turn to constructing a PTPM that allows us to transform PTs generated by elements of f_k that are not necessarily consecutive. In other words, given a positive integer g, $1 \leq g \leq k$, we can also construct a PTPM that transforms the PT generated by $m = f_k$ and $n = f_{k-g}$ into the PT generated by $v = f_{k+g}$ and $w = f_k$. Powers of such PTPMs have terms from generalized Lucas sequences [4], which we define as follows: $l_0(a,b) = 2$, $l_1(a,b) = a$, and $l_k(a,b) = al_{k-1} + bl_{k-2}$ for $k \geq 2$, where a and b are positive integers. We will also need the following lemma.

Lemma 3.1. *For positive integers k and j with $k \geq j$, $f_{k+j} = l_j f_k - (-b)^j f_{k-j}$.*

Proof. Let λ and μ denote the roots of $x^2 - ax - b = 0$. Since $b > 0$, $\lambda \neq \mu$ and we have from [4] that

$$f_k = \frac{\lambda^k - \mu^k}{\lambda - \mu}$$

and

$$l_j = \lambda^j + \mu^j.$$

Then, using the fact that $\lambda\mu = -b$, we have

$$\begin{aligned} l_j f_k - (-b)^j f_{k-j} &= (\lambda^j + \mu^j)\left(\frac{\lambda^k - \mu^k}{\lambda - \mu}\right) - (-b)^j\left(\frac{\lambda^{k-j} - \mu^{k-j}}{\lambda - \mu}\right) \\ &= \frac{\lambda^{k+j} - \mu^{k+j}}{\lambda - \mu} + \frac{\lambda^{k-j}(\lambda\mu)^j - \mu^{k-j}(\lambda\mu)^j - (-b)^j(\lambda^{k-j} - \mu^{k-j})}{\lambda - \mu} \\ &= \frac{\lambda^{k+j} - \mu^{k+j}}{\lambda - \mu} + \frac{(-b)^j(\lambda^{k-j} - \mu^{k-j}) - (-b)^j(\lambda^{k-j} - \mu^{k-j})}{\lambda - \mu} \\ &= f_{k+j}. \end{aligned}$$

□

Now, apply Theorem 1.1 to construct a matrix with $r = l_g$, $t = -(-b)^g$, $s = 1$, and $u = 0$. We obtain a matrix that transforms a PT generated by $m = f_k$ and $n = f_{k-g}$ into a PT generated by $v = l_g m - (-b)^g n$ and $w = m$. By Lemma 3.1, $v = f_{k+g}$ and the resulting matrix, given below, therefore performs the desired transformation:

$$M_{a,b,g} = \begin{bmatrix} \dfrac{l_g^2 - b^{2g} - 1}{2} & -l_g(-b)^g & \dfrac{l_g^2 + b^{2g} - 1}{2} \\ l_g & -(-b)^g & l_g \\ \dfrac{l_g^2 - b^{2g} + 1}{2} & -l_g(-b)^g & \dfrac{l_g^2 + b^{2g} + 1}{2} \end{bmatrix}.$$

Note that $M_{a,b,1} = M_{a,b}$ and, since $l_1 = a$, the form of $M_{a,b,1}$ given here agrees with our previous expression for $M_{a,b}$. It should also be noted that $M_{a,b,g}$ is only guaranteed to be a PTPM when $-(-b)^g > 0$, i.e., when g is odd.

4 Conclusion

In this paper, we have derived forms for the powers of matrices that transform PTs generated by terms of generalized Fibonacci sequences into other such PTs. We have also found matrices that transform PTs generated by non-consecutive terms of generalized Fibonacci sequences into other such PTs.

There are several possible avenues for further research. First, since the matrices presented here are PTPMs, the interested reader may wish to explore whether the $M_{a,b}$ or $M_{a,b,g}$ matrices exhibit interesting properties when acting on other collections of PTs. Second, we have not examined the $M_{a,b,g}$ matrices for values of g greater than 1. In particular, the reader may wish to find an expression for the j-th power of $M_{a,b,g}$. Finally, we have not detailed how the PTPMs discussed here can be used to specifically generate PTs that are primitive.

References

[1] Austin, J. (2023). A note on generating primitive Pythagorean triples using matrices. *Notes on Number Theory and Discrete Mathematics*, 29(2), 402–406.

[2] Austin, J., & Schneider, L. (2020). Generalized Fibonacci numbers in Pythagorean triple preserving matrices. *The Fibonacci Quarterly*, 58(4), 340–350.

[3] Burton, D. M. (2010). *Elementary Number Theory*. McGraw Hill, New York.

[4] Kalman, D., & Mena, R. (2003). The Fibonacci numbers–exposed. *Mathematics Magazine*, 76(3), 167–181.

[5] Leyendekkers, J. V., & Shannon, A. G. (2016). Primitive Pythagorean triples and generalized Fibonacci sequences. *Notes on Number Theory and Discrete Mathematics*, 23(1), 54–62.

[6] Palmer, L., Ahuja, M., & Tikoo, M. (1998). Finding Pythagorean triple preserving matrices. *Missouri Journal of Mathematical Sciences*, 10(2), 99–105.