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Divisors and square-free divisors involving the floor function

B. Feng and J. Wu[2]

[1] School of Mathematics and Big Data, Chongqing University of Education
Chongqing 400065, P. R. China
e-mail: binfengcq@163.com

[2] CNRS
Université Paris-Est Créteil
Université Gustave Eiffel
LAMA 8050
F-94010 Créteil, France
e-mail: jie.wu@u-pec.fr

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Abstract: Let $\tau(n)$ (respectively, $\tau_\sharp(n)$) be the number of divisors (respectively, square free divisors) of natural number n, and let $[t]$ be the integral part of real number t. In this short note, we prove that for any $\varepsilon > 0$ the asymptotic formula

$$\sum_{n \le x^{1/c}} g\Big(\Big[\frac{x}{n^c}\Big]\Big) = \Xi_{g,c} x^{1/c} + O_\varepsilon(x^{\theta_c^{\mathrm{FW}}+\varepsilon})$$

holds for $x \to \infty$, where $g = \tau$ or $\tau_\sharp$ and

$$\Xi_{g,c} := \sum_{d=1}^{\infty} g(d)\Big(\frac{1}{d^{1/c}} - \frac{1}{(d+1)^{1/c}}\Big), \qquad \theta_c^{\mathrm{FW}} := \begin{cases} \frac{2}{3c+2}, & \text{if } 0 < c \le \frac{2}{5}, \\ \frac{5}{5c+6}, & \text{if } c \ge \frac{2}{5}. \end{cases}$$

This improves and generalises the corresponding results of Feng–Wu for $\tau(n)$ and of Zhang for $\tau_\sharp(n)$ with $c = 1$, respectively.

Keywords: Exponential sums, Square free integers, Asymptotic results on arithmetic functions.
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1 Introduction

As usual, denote by $\tau(n)$ the classic divisor function and by $[t]$ the integral part of $t \in \mathbf{R}$. The well-known divisor problem of Dirichlet states as follows

$$\sum_{n\le x}\tau(n)=\sum_{n\le x}\Big[\frac{x}{n}\Big]=x\log x+(2\gamma-1)x+O(x^{1/3})\quad(x\to\infty),$$

where γ is the Euler constant. In [2], Bordellès, Dai, Heyman, Pan and Shparlinski considered a more general function of summation:

$$S_f(x):=\sum_{n\le x}f\Big(\Big[\frac{x}{n}\Big]\Big),$$

by establishing its asymptotic formula under some simple assumptions of f. Subsequently, Wu [11] and Zhai [13] improved their results independently. In particular, if $f(n)\ll_\varepsilon n^\varepsilon$ for any $\varepsilon>0$ and all $n\ge 1$, then [11, Theorem 1.2(i)] or [13, Theorem 1] yields the following asymptotic formula

$$S_f(x)=\bigg(\sum_{d=1}^{\infty}\frac{f(d)}{d(d+1)}\bigg)x+O_\varepsilon(x^{1/2+\varepsilon}) \tag{1}$$

as $x\to\infty$. Many authors are interested in breaking the $\frac{1}{2}$-barrier for the error term of (1) for some classical arithmetic functions [1,6,8–10,14]. In particular, Ma and Sun [8] showed that

$$S_\tau(x)=\bigg(\sum_{d=1}^{\infty}\frac{\tau(d)}{d(d+1)}\bigg)x+O_\varepsilon(x^{11/23+\varepsilon})\qquad(x\to\infty). \tag{2}$$

Subsequently, the exponent $\frac{11}{23}$ has been improved to $\frac{19}{40}$ by Bordellès [1] and to $\frac{9}{19}$ by Liu, Wu and Yang [6]. Finally Stucky [10] observed that the bound for exponential sum with the coefficients $\tau(n)$ of Jutila [5] (see also Lemma 2.2 below) allows to obtain a better exponent $\frac{5}{11}$.

Recently Feng [3] investigated a more general problem than (2):

$$S_{\tau,c}(x):=\sum_{n\le x^{1/c}}\tau\Big(\Big[\frac{x}{n^c}\Big]\Big)=\Xi_{\tau,c}x^{1/c}+O(x^{\theta_c^{\mathrm{F}}+\varepsilon}),$$

where $c>0$ is a fixed real number,

$$\Xi_{\tau,c}:=\sum_{d=1}^{\infty}\tau(d)\bigg(\frac{1}{d^{1/c}}-\frac{1}{(d+1)^{1/c}}\bigg)$$

and

$$\theta_c^{\mathrm{F}}:=\begin{cases}\frac{2}{3c+2}, & \text{if } 0<c<\frac{2}{11},\\ \frac{11}{11c+12}, & \text{if } c\ge\frac{2}{11}.\end{cases} \tag{3}$$

Following Stucky's idea [10], Wu [12] applied Jutila's bound for exponential sum mentioned above to get a better error term

$$\theta_c^{\mathrm{FW}} := \begin{cases} \frac{2c+2}{(2c+1)(c+2)}, & \text{if } 0 < c \leq \frac{2}{3}, \\ \frac{5}{5c+6}, & \text{if } c \geq \frac{2}{3}. \end{cases} \tag{4}$$

This gives an improvement of (3) for $c > \frac{2}{9}$. Noticing that $\theta_1^{\mathrm{FW}} = \frac{5}{11}$, Wu's (4) generalises the work of Stucky [10].

The first aim of this short note is to propose better exponent than (3) and (4).

Theorem 1.1. *For any $\varepsilon > 0$ and $x \to \infty$, we have*

$$S_{\tau,c}(x) = \Xi_{\tau,c} x^{1/c} + O(x^{\theta_c^{\mathrm{FW}}+\varepsilon}), \tag{5}$$

where

$$\theta_c^{\mathrm{FW}} := \begin{cases} \frac{2}{3c+2}, & \textit{if } 0 < c \leq \frac{2}{5}, \\ \frac{5}{5c+6}, & \textit{if } c \geq \frac{2}{5}. \end{cases} \tag{6}$$

Note that

$$c > 0 \ \Rightarrow \ \frac{2}{3c+2} < \frac{2c+2}{(2c+1)(c+2)} \qquad \text{and} \qquad 0 < c < \frac{2}{3} \ \Rightarrow \ \frac{5}{5c+6} < \frac{2c+2}{(2c+1)(c+2)}$$

Theorem 1.1 improves Feng's (3) and Wu's (4) for $\frac{2}{11} < c < \frac{2}{3}$.

Denote by $\tau_\sharp(n)$ the number of square free divisors of n. In [1], Bordèlles proved the following asymptotic formula

$$S_{\tau_\sharp}(x) := \sum_{n \leq x} \tau_\sharp\Big(\Big[\frac{x}{n}\Big]\Big) = \bigg(\sum_{d=1}^{\infty} \frac{\tau_\sharp(d)}{d(d+1)}\bigg) x + O_\varepsilon(x^{97/203+\varepsilon}).$$

Subsequently the exponent $\frac{97}{203} \approx 0.47783$ has been reduced to $\frac{9}{19} \approx 0.47368$ by Liu–Wu–Yang [7] and $\frac{107}{229} \approx 0.46724$ by Zhang [14]. The second aim of this short note is to consider a more general sum

$$S_{\tau_\sharp,c}(x) := \sum_{n \leq x^{1/c}} \tau_\sharp\Big(\Big[\frac{x}{n^c}\Big]\Big).$$

Our result is as follows.

Theorem 1.2. *For any $\varepsilon > 0$ and $x \to \infty$, we have*

$$S_{\tau_\sharp,c}(x) = \Xi_{\tau_\sharp,c} x^{1/c} + O_{c,\varepsilon}(x^{\theta_c^{\mathrm{FW}}+\varepsilon}), \tag{7}$$

where θ_c^{FW} is given by (6) *and*

$$\Xi_{\tau_\sharp,c} := \sum_{d=1}^{\infty} \tau_\sharp(d) \bigg(\frac{1}{d^{1/c}} - \frac{1}{(d+1)^{1/c}}\bigg). \tag{8}$$

Noticing that $\theta_1^{\mathrm{FW}} = \frac{5}{11} \approx 0.45454$, a special case of Theorem 1.2 with $c = 1$ improves Zhang's exponent $\frac{107}{229} \approx 0.46724$. It is worth noticing that

$$\lim_{c\to 0^+} \frac{\theta_c^{\mathrm{FW}}}{(1/c)} = \lim_{c\to 0^+} \frac{2c}{3c+2} = 0.$$

This means that the error term is very small with respect to the main term when c is small. Such a situation is rather rare in the analytic number theory. Maybe it should be an interesting aspect of $S_{\tau,c}(x)$ and $S_{\tau_\sharp,c}(x)$.

2 Bilinear forms

The aim of this section is to establish good bound for the bilinear forms

$$\mathfrak{S}_{c,\delta}^{g}(x, D) := \sum_{D<d\le 2D} g(d)\psi\Big(\frac{x^{1/c}}{(d+\delta)^{1/c}}\Big), \tag{9}$$

where $g = \tau$ or $\tau_\sharp$ and $\psi(t) := t - [t] - \frac{1}{2}$. For this, we need two preliminary lemmas.

The first one is [4, Theorem A.6].

Lemma 2.1. *For $x \ge 1$ and $H \ge 1$, we have*

$$\psi(x) = -\sum_{1\le |h|\le H} \Phi\Big(\frac{h}{H+1}\Big)\frac{\mathrm{e}(hx)}{2\pi\mathrm{i}h} + R_H(x), \tag{10}$$

where $\mathrm{e}(t) := \mathrm{e}^{2\pi\mathrm{i}t}$, $\Phi(t) := \pi t(1-|t|)\cot(\pi t) + |t|$ and the second term $R_H(x)$ satisfies

$$|R_H(x)| \le \frac{1}{2H+2}\sum_{0\le |h|\le H}\Big(1 - \frac{|h|}{H+1}\Big)\mathrm{e}(hx). \tag{11}$$

The second one is due to Jutila [5, Theorem 4.6].

Lemma 2.2. *Let $D \ge 2$ and $D < D' \le 2D$. Let f be a holomorphic function in the domain*

$$\mathcal{D} := \{z : |z - x| < \eta D \text{ for some } x \in [D, D']\},$$

where $\eta > 0$ is a positive constant. Suppose that $f(x)$ is real for $D \le x \le D'$ and that

$$f(z) = Bz^{\alpha}\{1 + O(F^{-1/3})\} \qquad (z \in \mathcal{D}),$$

where $\alpha \ne 0, 1$ is a fixed real number and $F = |B|D^{\alpha}$. Suppose that

$$D^{3/4} \ll F \ll D^{3/2}. \tag{12}$$

Then for any $\varepsilon > 0$, we have

$$\sum_{D<d\le D'} \tau(d)\mathrm{e}(f(d)) \ll F^{1/3+\varepsilon}D^{1/2}. \tag{13}$$

The following proposition gives the desired bound for $\mathfrak{S}^g_{c,\delta}(x, D)$. It will play a key role in the proof of Theorems 1.1 and 1.2.

Theorem 2.1. *Let $g = \tau$ or $\tau_\sharp$. For any $\varepsilon > 0$, we have*

$$\mathfrak{S}^g_{c,\delta}(x, D) \ll_\varepsilon (x^2 D^{5c-2})^{1/8c} x^\varepsilon \tag{14}$$

uniformly for $x^{2/(3c+2)} \le D \le x^{4/(3c+4)}$ and $0 \le \delta \le \varepsilon^{-1}$.

Proof. First consider the case of $g = \tau$. Applying (10) and (11), for $H \ge 1$ we have

$$\mathfrak{S}^\tau_{c,\delta}(x, D) \ll \left\{ \frac{D}{H} + \sum_{h \le H} \frac{1}{h} \left| \sum_{D < d \le 2D} \tau(d) \mathrm{e}\left(\frac{x^{1/c} h}{(d+\delta)^{1/c}} \right) \right| \right\} x^\varepsilon. \tag{15}$$

We would like to use Lemma 2.2 to bound the last sum over d. Clearly we can write

$$f(z) = \frac{x^{1/c} h}{(z+\delta)^{1/c}} = x^{1/c} h z^{-1/c} \{1 + O(D^{-1})\}.$$

Thus we have $\alpha = -1/c$, $B = x^{1/c} h$ and $F = x^{1/c} h D^{-1/c}$. Under the condition

$$(x^{-4} D^{3c+4})^{1/4c} \ll h \ll (x^{-2} D^{3c+2})^{1/2c}, \tag{16}$$

the assumption (12) of Lemma 2.2 is satisfied and $D^{-1} \le F^{-1/3}$. Thus using this lemma, we have

$$\sum_{d \sim D} \tau(d) \mathrm{e}\left(\frac{x^{1/c} h}{(d+\delta)^{1/c}} \right) \ll (x^2 D^{3c-2})^{1/6c} h^{1/3} x^\varepsilon. \tag{17}$$

Since $1 \le h \le H$, the condition (16) is satisfied provided

$$x^{2/(3c+2)} \le D \le x^{4/(3c+4)} \qquad \text{and} \qquad 1 \le H \le (x^{-2} D^{3c+2})^{1/2c}. \tag{18}$$

Assuming (18), we can insert (17) into (15) to get

$$\mathfrak{S}^\tau_{c,\delta}(x, D) \ll \big(D H^{-1} + (x^2 D^{3c-2})^{1/6c} H^{1/3} \big) x^\varepsilon. \tag{19}$$

Taking $H = (x^{-2} D^{3c+2})^{1/8c} \in [1, (x^{-2} D^{3c+2})^{1/2c}]$, we obtain the inequality (14) for $g = \tau$.

Next consider the case of $g = \tau_\sharp$. Let $\mu(n)$ be the Möbius function and

$$\tilde{\mu}(d) := \begin{cases} \mu(m), & \text{if } d = m^2, \\ 0, & \text{otherwise.} \end{cases}$$

In view of the relation $\tau_\sharp = \tau * \tilde{\mu}$, we can write $\mathfrak{S}^{\tau_\sharp}_{c,\delta}(x, D)$ as follows

$$\begin{aligned} \mathfrak{S}^{\tau_\sharp}_{c,\delta}(x, D) &= \sum_{D < m^2 n \le 2D} \mu(m) \tau(n) \psi\left(\frac{x^{1/c}}{(m^2 n + \delta)^{1/c}} \right) \\ &= \sum_{m \le (2D)^{1/2}} \mu(m) \sum_{D/m^2 < n \le 2D/m^2} \tau(n) \psi\left(\frac{(x/m^2)^{1/c}}{(n + \delta/m^2)^{1/c}} \right) \\ &= \sum_{m \le (2D)^{1/2}} \mu(m) \mathfrak{S}^\tau_{c,\delta/m^2}(x/m^2, D/m^2). \end{aligned} \tag{20}$$

In order to prove (14) for $g = \tau_\sharp$, we divide the last sum of (20) into two parts:

$$m \le (x^{-2}D^{3c+2})^{1/6c} \qquad \text{or} \qquad (x^{-2}D^{3c+2})^{1/6c} \le m \le 2D^{1/2}.$$

In the first case, we have $(x/m^2)^{2/(3c+2)} \le D/m^2 \le (x/m^2)^{4/(3c+4)}$. Thus we can use (14) with $g = \tau$ to derive that the contribution of $m \le (x^{-2}D^{3c+2})^{1/6c}$ is

$$\begin{aligned}
&\ll \sum_{m \le (x^{-2}D^{3c+2})^{1/6c}} ((x/m^2)^2(D/m^2)^{5c-2})^{1/8c}x^{\varepsilon} \\
&\ll \sum_{m \le (x^{-2}D^{3c+2})^{1/6c}} (x^2D^{5c-2}m^{-10c})^{1/8c}x^{\varepsilon} \\
&\ll (x^2D^{5c-2})^{1/8c}x^{\varepsilon}.
\end{aligned}$$

Trivially, the contribution of $(x^{-2}D^{3c+2})^{1/6c} \le m \le (2D)^{1/2}$ is

$$\ll \sum_{m \ge (x^{-2}D^{3c+2})^{1/6c}} Dm^{-2}x^{\varepsilon} \ll (x^2D^{3c-2})^{1/6c}x^{\varepsilon} \ll (x^2D^{5c-2})^{1/8c}x^{\varepsilon},$$

since $(x^2D^{3c-2})^{1/6c} \le (x^2D^{5c-2})^{1/8c}$ for $D \ge x^{2/(3c+2)}$. Inserting these into (20), we obtain the inequality (14) for $g = \tau_\sharp$. $\square$

3 Proof of Theorems 1.1 and 1.2

Let $N \in [1, x^{1/(c+1)}]$ be a parameter to be chosen later and $g = \tau$ or $\tau_\sharp$. First we write

$$\sum_{n \le x^{1/c}} g\Big(\Big[\frac{x}{n^c}\Big]\Big) = \tilde{S}_{g,c}(x) + O(Nx^{\varepsilon}), \tag{21}$$

where

$$\tilde{S}_{g,c}(x) := \sum_{N < n \le x^{1/c}} g\Big(\Big[\frac{x}{n^c}\Big]\Big).$$

Noticing that $x/n^c - 1 < d = [x/n^c] \le x/n^c \Leftrightarrow (x/(d+1))^{1/c} < n \le (x/d)^{1/c}$, we can derive that

$$\begin{aligned}
\tilde{S}_{g,c}(x) &= \sum_{d \le x/N^c} g(d) \sum_{(x/(d+1))^{1/c} < n \le (x/d)^{1/c}} 1 \\
&= \sum_{d \le x/N^c} g(d)\bigg(\frac{x^{1/c}}{d^{1/c}} - \psi\Big(\frac{x^{1/c}}{d^{1/c}}\Big) - \frac{x^{1/c}}{(d+1)^{1/c}} + \psi\Big(\frac{x^{1/c}}{(d+1)^{1/c}}\Big)\bigg) \qquad (22) \\
&= \Xi_{g,c}x - \mathcal{R}^g_{c,0}(x,N) + \mathcal{R}^g_{c,1}(x,N) + O(Nx^{\varepsilon}),
\end{aligned}$$

where we have used the following bounds

$$\begin{aligned}
x^{1/c}\sum_{d > x/N^c} g(d)\Big(\frac{1}{d^{1/c}} - \frac{1}{(d+1)^{1/c}}\Big) &\ll Nx^{\varepsilon}, \\
\sum_{d \le N} g(d)\bigg(\psi\Big(\frac{x^{1/c}}{d^{1/c}}\Big) - \psi\Big(\frac{x^{1/c}}{(d+1)^{1/c}}\Big)\bigg) &\ll Nx^{\varepsilon},
\end{aligned}$$

and the notation

$$\mathcal{R}^{g,*}_{c,\delta}(x,N) := \sum_{N<d\le x/N^c} g(d)\psi\bigg(\frac{x^{1/c}}{(d+\delta)^{1/c}}\bigg).$$

When $c \ge \frac{2}{5}$, we take $N = x^{5/(5c+6)}$. In this case, it is easy to check that

$$\tfrac{5}{5c+6} \ge \tfrac{2}{3c+2} \qquad \text{and} \qquad 1 - \tfrac{5c}{5c+6} = \tfrac{6}{5c+6} \le \tfrac{4}{3c+4}. \tag{23}$$

Thus for $c \ge \frac{2}{5}$, we can apply Theorem 2.1 to derive that

$$\begin{aligned} \mathcal{R}^{g,*}_{c,\delta}(x, x^{5/(5c+6)}) &\ll x^{\varepsilon} \max_{N<D\le x/N^c} \big|\mathfrak{S}^{g}_{c,\delta}(x,D)\big| \\ &\ll (x^2(xN^{-c})^{5c-2})^{1/8c} x^{\varepsilon} \\ &\ll x^{5/(5c+6)+\varepsilon}. \end{aligned} \tag{24}$$

If $0 < c < \frac{2}{5}$, we take $N = x^{2/(3c+2)}$. It is easy to check that

$$1 - \tfrac{2c}{3c+2} = \tfrac{c+2}{3c+2} \le \tfrac{4}{3c+4} \;\Leftrightarrow\; c \le \tfrac{2}{3}. \tag{25}$$

Thus for $c < \frac{2}{5}$, we can apply Theorem 2.1 to derive that

$$\begin{aligned} \mathcal{R}^{g,*}_{c,\delta}(x, x^{2/(3c+2)}) &\ll x^{\varepsilon} \max_{N<D\le x/N^c} \big|\mathfrak{S}^{g}_{c,\delta}(x,D)\big| \\ &\ll \big((x^2N^{5c-2})^{1/8c} + (x^2N^{-c-2})^{1/2c}\big) x^{\varepsilon} \\ &\ll x^{2/(3c+2)+\varepsilon}. \end{aligned} \tag{26}$$

Now the required results (5) and (7) follow from (21), (22), (24) and (26). □

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