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On the sum of partition norms and its connection to norms of partitions with parts greater than one

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Abstract: We study the *partition norm*—the product of the parts of a partition—with emphasis on partitions whose parts all exceed 1. We obtain two bivariate recurrences for the sum of norms over partitions into distinct parts, including a refinement that separates the contributions of the part 1 and of the prime 2. To count partitions of n with parts greater than 1 having norm i, we introduce the *restricted norm-counting function* $r_{>1}(i, n)$, give its two-parameter generating function, and derive recurrences and other relations. Finally, we formulate analogues of Goldbach's and twin-prime conjectures in the language of partition norms.

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1 Introduction

The partition $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_{\ell(\boldsymbol{\lambda})})$ of size $|\boldsymbol{\lambda}| := \sum_{k=1}^{\ell(\boldsymbol{\lambda})} \lambda_k = n$ is denoted by $\boldsymbol{\lambda} \vdash n$, where $\lambda_k > 0\ \forall\, k$ and $\ell(\boldsymbol{\lambda})$ is the length of $\boldsymbol{\lambda}$. For instance, the partitions of 4 are $\{4, 3+1, 2+2, 2+1+1, 1+1+1+1\}$. Replacing the '$+$' with '$\times$' produces the corresponding norms: $\{4, 3\times 1, 2\times 2, 2\times 1\times 1, 1\times 1\times 1\times 1\} = \{4, 3, 4, 2, 1\}$. Sequences related to products of parts appear in OEIS as A006906 and A118851. The partition norm is formally defined below.

Definition 1.1 (Partition norm [5, 10])**.** *For a partition $\boldsymbol{\lambda} \vdash n$, its norm is $\|\boldsymbol{\lambda}\| := \prod_{k=1}^{\ell(\boldsymbol{\lambda})} \lambda_k$.*

The product of parts has appeared in several contexts [3, 4, 6]. The partition norm features in Sills' work [12] on MacMahon's partial fractions. Developments through 2019 are summarized by Schneider and Sills [10], which also mentions a combinatorial interpretation of the norm of a partition with dotted Young diagrams. In 2016–17, Schneider [8, 9] advanced the multiplicative theory of partitions via norms, recovering classical number-theoretic results as special cases of general combinatorial laws. In 2021, Schneider and Sills [11] continued work on the partition zeta function, while Dawsey *et al.* [2] and Lagarias [7] studied a new statistic called the *partition supernorm* which is defined below.

Definition 1.2 (Partition supernorm [2, 7])**.** *For a partition $\boldsymbol{\lambda} \vdash n$, its supernorm $\|\boldsymbol{\lambda}\|_S := \prod_{k=1}^{\ell(\boldsymbol{\lambda})} p_{\lambda_k}$, where p_j denotes the j-th prime.*

Another parallel line of analysis is the norm counting function approach which has a fascinating form of generating function resembling a combination of Dirichlet and Euler-type generating functions. The approach was developed and investigated for ordinary partitions in [5]. For ordinary partitions, we have

$$\mathcal{R}(s, q) = 1 + \sum_{n=1}^{\infty} q^n \sum_{\boldsymbol{\lambda} \vdash n} \|\boldsymbol{\lambda}\|^{-s} = \prod_{k=1}^{\infty} \left(1 - \frac{q^k}{k^s}\right)^{-1}, \tag{1}$$

for $|q| < 1$ and $s \in \mathbb{R}$. At $s = 0$, the ordinary partition function $p(n)$ can be recovered which has been substantially investigated in literature [1]. At $s = -1$, we recover the sum of norms, denoted by $\dot{p}(n) := \sum_{\boldsymbol{\lambda} \vdash n} \|\boldsymbol{\lambda}\|$, which has the following generating function:

$$\sum_{n=0}^{\infty} \dot{p}(n) q^n = \prod_{k=1}^{\infty} \frac{1}{1 - kq^k}, \quad \dot{p}(0) = 1. \tag{2}$$

Let us also define $\dot{D}(n)$ as the sum of norms for integer partitions of n with distinct parts, which has the generating function:

$$\sum_{n=0}^{\infty} \dot{D}(n) q^n = \prod_{k=1}^{\infty} (1 + kq^k).$$

Section 2 establishes recurrences for norm sums. We then remove the part 1 and, in Section 3, analyze partition norms with all parts greater than one, including relations between the restricted norm-counting function and multiplicative partitions.

2 Recurrences on sum of partition norms

Theorem 2.1. *Let $\dot{D}(m,n)$ be the sum of norms over partitions of n into m distinct parts, and let $\dot{D}_{>1}(m,n)$ be the analogous sum restricted to parts greater than 1. Then*

$$\dot{D}(m,n) = \dot{D}_{>1}(m,n) + \dot{D}_{>1}(m-1,n-1).$$

Proof. Beginning with the generating function of $\dot{D}(m,n)$,

$$\sum_{n,m\geq 0} \dot{D}(m,n)\, z^m q^n = \prod_{k=1}^{\infty}(1+kzq^k) = (1+zq)\prod_{k=2}^{\infty}(1+kzq^k).$$

Expanding the right-hand side gives

$$\sum_{n,m\geq 0} \dot{D}_{>1}(m,n) z^m q^n + \sum_{n,m\geq 0} \dot{D}_{>1}(m,n) z^{m+1} q^{n+1},$$

and coefficient extraction yields the claim. □

Remark 1. *The above identity can be proved combinatorially, too. Split the set of distinct-parts partitions of n into those that do and do not contain a part equal to 1. Including/removing the part 1 does not change the product (the norm), only the size and the number of parts. Removing the 1 transforms a partition of n with m parts into a partition of $n-1$ with $m-1$ parts.*

Theorem 2.2. *Let $\dot{D}_p(m,n)$ be the sum of norms over partitions of n into m distinct primes, and $\dot{D}_{po}(m,n)$ be the analogous sum over distinct odd primes. Then*

$$\dot{D}_p(m,n) \;=\; \dot{D}_{po}(m,n) \;+\; 2\,\dot{D}_{po}(m-1,n-2).$$

Proof. With p_j the j-th prime,

$$\begin{aligned}\sum_{n,m\geq 0} \dot{D}_p(m,n)\, z^m q^n &:= \prod_{j\geq 1}\big(1+p_j z q^{p_j}\big)\\ &= (1+2zq^2)\prod_{j\geq 2}\big(1+p_j z q^{p_j}\big)\\ &= (1+2zq^2)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \dot{D}_{po}(m,n) z^m q^n.\end{aligned}$$

The first factor splits into the odd-prime part plus the contribution from the prime 2, yielding the stated decomposition after coefficient comparison. □

Remark 2. *One can also give a combinatorial proof of the above recurrence. Split the set of distinct-parts prime partitions into two disjoint classes according to whether the part 2 is present. Those with no '2' contribute exactly $\dot{D}_{po}(m,n)$, since all parts are odd primes. For those that contain 2, delete the part 2; this decreases the size from n to $n-2$ and the number of parts from m to $m-1$. The norm of each original partition is exactly 2 times the norm of the reduced partition, so summing over this class contributes $2\dot{D}_{po}(m-1,n-2)$.*

Next, consider the case in which parts equal to μ are disallowed, then the generating function gives

$$\sum_{n\geq 0}\dot{p}_\mu(n)q^n=\prod_{\substack{k\geq 1\\ k\neq\mu}}\frac{1}{1-kq^k}=(1-\mu q^\mu)\prod_{k\geq 1}\frac{1}{1-kq^k}=\sum_{n\geq 0}\dot{p}(n)q^n-\mu q^\mu\sum_{n\geq 0}\dot{p}(n)q^n.$$

Comparing the coefficients on both sides gives $\dot{p}_\mu(n)=\dot{p}(n)-\mu\dot{p}(n-\mu)$. We are interested in the case $\mu=1$ (removing ones), then $\dot{p}_1(n)=\dot{p}(n)-\dot{p}(n-1)$. In the next section, we investigate the norms of restricted partitions with parts greater than one.

3 Partition norms without 1 as a part

Table 1 lists the restricted partitions (with all parts greater than 1) together with their norms. Notice different restricted partitions of the same n may share a norm; for example, when $n=6$, both $4+2$ and $2+2+2$ yield the norm 8. Therefore, we define the *restricted norm–counting function* $r_{>1}(i,n)$ as the number of restricted partitions of n (with no part equal to 1) whose norm equals i. This is the analogue, for restricted partitions, of the norm-counting function $r(i,n)$ introduced in [5] for the ordinary partitions.

Table 1. List of norms for restricted partitions for $n\in\{6,7,8\}$.

	Relevant partitions	**Norm i**	$\boldsymbol{r_{>1}(i,n)}$
$n=6$	6	6	1
	$4+2,\ 2+2+2$	8	2
	$3+3$	9	1
$n=7$	7	7	1
	$5+2$	10	1
	$4+3$	12	1
	$3+2+2$	12	1
$n=8$	8	8	1
	$6+2$	12	1
	$5+3$	15	1
	$4+4,\ 4+2+2,\ 2+2+2+2$	16	3
	$3+3+2$	18	1

Moreover, let $(t_n)_{n\geq 1}=\{1,2,3,4,6,9,12,\ldots\}$ denote the maximum value of norm for a particular n. The sequence is regarded as the maximum product sequence and is entry A000792 in OEIS. Let $p(n)$ denote the ordinary partition function that counts the number of ways of expressing n as a sum of positive integers, and $p^*(i)$ be the multiplicative partition function that counts the number of ways of factorization of i. For a fixed $n\geq 2$, $\Theta_n:=\{i\in\mathbb{N}:r_{>1}(i,n)>0\}$. It follows immediately that $\Theta_n\not\subseteq\Theta_{n+1}$ since any partition of n with the part 1 added to it is an invalid partition of $n+1$ since 1 is forbidden.

Similar to the analysis done in [5], we have the following generating function

$$1+\sum_{n=1}^{\infty} q^n \sum_{i=1}^{t(n)} \frac{r_{>1}(i,n)}{i^s} = \prod_{k=2}^{\infty}\left(1-\frac{q^k}{k^s}\right)^{-1}, \tag{3}$$

where $\sum_i \frac{r_1(i,1)}{i^s} = 0$. As $r_1(1,1)=0$ and $r_1(i,n)=0$ for $i>t(n)$, we can equivalently write:

$$1+\sum_{n=2}^{\infty}\sum_{i=1}^{\infty} \frac{r_{>1}(i,n)}{i^s} q^n = \prod_{k=2}^{\infty}\left(1-\frac{q^k}{k^s}\right)^{-1} =: \mathcal{K}(s,q), \tag{4}$$

over the domain $|q|<1$ and $s\in\mathbb{R}$. The series does not converge when $q=1$ and $s\le 1$ simultaneously. With the necessary notation introduced, we can now proceed with the analysis.

Proposition 3.1. *We have* $\sum_{i=1}^{t(n)} r_{>1}(i,n) = p(n)-p(n-1)$.

Proof. Setting $s=0$ in (3) gives

$$1+\sum_{n=1}^{\infty} q^n \sum_{i=1}^{t(n)} r_{>1}(i,n) = \prod_{k=2}^{\infty}(1-q^k)^{-1} = (1-q)\sum_{n=0}^{\infty} p(n)q^n.$$

Comparing the coefficients gives the final result. □

Proposition 3.2. *We have* $\sum_{i=1}^{t(n)} i\, r_{>1}(i,n) = \dot{p}(n)-\dot{p}(n-1)$.

Proof. Plug $s=-1$ in (3) and compare the coefficients of q^n to get the result. □

Proposition 3.3. *We have* $\sum_{n=1}^{\infty} r_{>1}(i,n) = p^*(i)$.

Proof. Plugging $q=1$ (for $s>1$) in (4) gives

$$\sum_{i=1}^{\infty}\frac{1}{i^s}\sum_{n=1}^{\infty} r_{>1}(i,n) = \prod_{k=2}^{\infty}(1-k^{-s})^{-1} = \sum_{i=1}^{\infty}\frac{p^*(i)}{i^s}$$

where the infinite product is the Dirichlet generating function of $p^*(i)$. □

Remark 3. *An alternative proof of Proposition 3.3 is as follows. Any $i\in\mathbb{N}$ has exactly $p^*(i)$ (unordered) multiplicative partitions into integers greater than 1. Fix a canonical nondecreasing ordering of factors and denote the corresponding restricted partitions by $\{\boldsymbol{\lambda}_k\}_{k=1}^{p^*(i)}$. In the absence of parts equal to 1, these are precisely the restricted partitions whose norm is i, and their sizes are $\{|\boldsymbol{\lambda}_k|\}_{k=1}^{p^*(i)}$. Hence, $\sum_{n\ge1} r_{>1}(i,n) = \sum_{n\ge1}\#\{k : |\boldsymbol{\lambda}_k| = n\} = p^*(i)$.*

Theorem 3.1. *For $n\ge 2$, we have $r_{>1}(i,n) = r(i,n)-r(i,n-1)$.*

Proof. From (4), we have

$$\sum_{n=2}^{\infty}\sum_{i=1}^{\infty}\frac{r_{>1}(i,n)}{i^s} q^n = (1-q)\prod_{k=1}^{\infty}\left(1-\frac{q^k}{k^s}\right)^{-1} - 1.$$

Identifying the infinite product as the generating function of $r(i,n)$, we get

$$\begin{aligned}\sum_{n=2}^{\infty}\sum_{i=1}^{\infty}\frac{r_{>1}(i,n)}{i^s}q^n &= (1-q)\sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\frac{r(i,n)}{i^s}q^n - q\\ &= \sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\frac{r(i,n)}{i^s}q^n - q\sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\frac{r(i,n)}{i^s}q^n - q\\ &= \sum_{n=2}^{\infty}\sum_{i=1}^{\infty}\frac{r(i,n)}{i^s}q^n - \sum_{n=2}^{\infty}\sum_{i=1}^{\infty}\frac{r(i,n-1)}{i^s}q^n.\end{aligned}$$

This establishes the connection between the norm counts of ordinary and restricted partitions. □

Lemma 3.1. *For $|q|<1$ and $s\geq 0$, we have*

$$\ln\mathcal{K}(s,q) = \sum_{n=2}^{\infty}\frac{q^n}{n}\sum_{\substack{d|n,\\ d\neq n}}\Big(\frac{d}{n}\Big)^{ds-1}. \tag{5}$$

Proof. Taking logarithm on both sides of (4) and expanding as Taylor series gives

$$\ln\mathcal{K}(s,q) = -\sum_{k=2}^{\infty}\ln\Big(1-\frac{q^k}{k^s}\Big) = \sum_{k=2}^{\infty}\sum_{u=1}^{\infty}\frac{1}{u}\Big(\frac{q^k}{k^s}\Big)^u,$$

which converges absolutely for $|q|<1$ and $s\geq 0$. Setting $u=d$ and $kd=n$, we can write

$$\ln\mathcal{K}(s,q) = \sum_{\substack{d\geq 1,\\ n\geq 2d,\\ d|n}}\frac{q^n}{n}\left(\frac{d}{n}\right)^{ds-1} = \sum_{n=2}^{\infty}\frac{q^n}{n}\sum_{\substack{d|n,\\ d\leq\frac{n}{2}}}\left(\frac{d}{n}\right)^{ds-1} = \sum_{n=2}^{\infty}\frac{q^n}{n}\sum_{\substack{d|n,\\ d<n}}\left(\frac{d}{n}\right)^{ds-1},$$

which gives the final result. □

Theorem 3.2. *We have*

$$n\sum_{i=1}^{t(n)}\frac{r_{>1}(i,n)}{i^s} = \sum_{m=1}^{n}\left(\sum_{i=1}^{t(n-m)}\frac{r_{>1}(i,n-m)}{i^s}\cdot\sum_{\substack{d|m,\\ d\neq m}}\left(\frac{d}{m}\right)^{ds-1}\right). \tag{6}$$

Proof. Differentiating (5) with respect to q gives

$$\frac{\partial}{\partial q}\ln\mathcal{K}(s,q) = \frac{1}{\mathcal{K}(s,q)}\frac{\partial\mathcal{K}(s,q)}{\partial q} = \sum_{n=2}^{\infty}q^{n-1}\left(\sum_{d|n}\left(\frac{d}{n}\right)^{ds-1}-1\right).$$

On rearranging and substituting for $\mathcal{K}(s,q)$, we have

$$\sum_{n=1}^{\infty}nq^{n-1}\sum_{i=1}^{t(n)}\frac{r_{>1}(i,n)}{i^s} = \sum_{n=0}^{\infty}q^n\sum_{i=1}^{t(n)}\frac{r_{>1}(i,n)}{i^s}\times\sum_{n=1}^{\infty}q^{n-1}\sum_{\substack{d|n,\\ d\neq n}}\left(\frac{d}{n}\right)^{ds-1}.$$

Multiplying the power series on the right-hand side and comparing the powers of q, we get

$$(n+1)\sum_{i=1}^{t(n+1)}\frac{r_{>1}(i,n+1)}{i^s}=\sum_{m=0}^{n}\left(\sum_{i=1}^{t(n-m)}\frac{r_{>1}(i,n-m)}{i^s}\times\sum_{\substack{d|m+1,\\ d\neq m+1}}\left(\frac{d}{m+1}\right)^{ds-1}\right).$$

Replace $n+1\to n$, then re-indexing from $m=1$ gives the final result. $\square$

Let ℓ be a prime, then it is only expressible as $\ell=\ell\times 1$. Thus, the partitions for which ℓ is a norm are ℓ, $\ell+1$, $\ell+1+1$, $\ell+1+1+1$, $\ldots$, and so on. Since we cannot have 1 as a part, the prime ℓ can never appear as a norm for any $n\neq\ell$. This analysis is shown to assert that norms formed from partitions can be remapped to partitions again. Regarding Goldbach Conjecture, we need to find the appropriate norm of $2n$ that gives the partition p_1+p_2 where p_1 and p_2 are primes. Let S_k be the set of semiprimes with k primes, and $\varnothing$ denote the empty set. We have:

- Goldbach's weak conjecture is true if any only if $(\forall\, n\geqslant 8)\,(\Theta_{2n+1}\cap\{i\in S_3 : 2\nmid i\}\neq\varnothing)$.
- Goldbach's conjecture is true if any only if $(\forall\, n\geqslant 6)\,(\Theta_{2n}\cap\{i\in S_2 : 2\nmid i\}\neq\varnothing)$.

Thus, we have an analogue of the Goldbach's conjecture in the theory of partition norms. Our concern will be semiprimes with two primes only, also referred to as biprimes (set S_2).

Theorem 3.3. *There exists an even number $2n$ with a biprime norm b for all b not divisible by 2.*

Proof. For any arbitrary biprime b (not divisible by 2), plugging $i=b$ in Proposition 3.3, we have

$$\sum_{k\geq 1} r_{>1}(b,k)=p^*(b)=2,$$

where $p^*(b)=2$ because b can only be factorized as itself or a product of two odd primes. The summation can be started from $k=4$ (since $b\geq 4$):

$$\sum_{\substack{k\geq 4\\ k\neq b}} r_{>1}(b,k)=1,$$

where we use $r_{>1}(b,b)=1$ because one of the factorization of b is b. Splitting the summation over the even and odd values of k, we have

$$\sum_{\substack{n\geq 2\\ 2n\neq b}} r_{>1}(b,2n)+\sum_{\substack{n\geq 2\\ 2n+1\neq b}} r_{>1}(b,2n+1)=1.$$

Since $r_{>1}(.,.)$ is a non-negative integer, the 1 on the right-hand side of the above equation comes from only one of the summations on the left-hand side. Notice that b can be generated as a norm for only two partitions, one is the number itself (which is excluded in the summations above) and the other is the sum of its two odd prime factors, which cannot be an odd number. Therefore, the second summation does not contribute; hence, b is a norm of a partition of some $2n$ ($\neq b$). $\square$

This shows that the assertion $\Theta_{2n}\cap\{b\in S_2 : 2\nmid b\}\neq\varnothing$ is not entirely false, but in order to prove the Goldbach's conjecture, a converse is required which demonstrates that the above holds for all n.

We can also write an equivalent form for the twin-prime conjecture. Recall that it states that there are infinite prime pairs (p, q) such that $p = q + 2$. So, there are infinite primes p with partition norm $2q$, where q is a prime. That is, there are infinite primes p with partition norm $b \in S_2$ divisible by 2. Equivalently, there are infinite primes p such that $\Theta_p \cap \{b \in S_2 : 2 \mid b\} \neq \varnothing$.

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