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On q-generalized hyperharmonic numbers with two parameters

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Abstract: In this paper, we introduce q-generalized harmonic numbers with two parameters ω and ξ, $H_{\mu,\lambda,q}(\omega,\xi)$ for integers μ,λ such that $\lambda \geq \mu$. With the help of these numbers, we define a new family of numbers which is called q-generalized hyperharmonic numbers with two parameters ω and ξ of order r, $H^r_{\mu,\lambda,q}(\omega,\xi)$ for integer r. Then, we consider special matrices whose entries are given by these numbers and give some matrix multiplications. Additionally, we derive some combinatorial identities for $H_{\mu,\lambda,q}(\omega,\xi)$ and $H^r_{\mu,\lambda,q}(\omega,\xi)$ by matrix methods.
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1 Introduction

Harmonic numbers, denoted by H_μ, are a sequence of numbers that arise in various mathematical contexts. The harmonic numbers are described as the sum of the reciprocals of the first μ positive integers, that is

$$H_\mu = \sum_{k=1}^{\mu} \frac{1}{k}$$

and $H_0 = 0$. These numbers have been studied extensively in number theory, combinatorics, and calculus. They play a crucial role in analyzing the behavior of series and in estimating the growth of certain functions. These numbers are closely related to the Reimann zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}},$$

where the product is over all primes p. It is known that an expression for difference between harmonic numbers and the logarithm function as following:

$$\lim_{\mu\to\infty} (H_\mu - \log\mu) = 1 - \frac{1}{2}(\zeta(2)-1) - \frac{1}{3}(\zeta(3)-1) - \cdots = \xi,$$

where ξ is called the Euler's constant and $\xi = 0.57721\ldots$.

Harmonic numbers and their generalizations can be utilized in various applied sciences, and some special numbers related to harmonic numbers can also find applications in the same fields. Studies concerning other special number families associated with harmonic numbers and their generalizations are frequently encountered in the literature. The relationships between harmonic numbers and numbers used in applied sciences, such as Stirling numbers of the first kind, binomial coefficients, Fibonacci numbers, and Bernoulli numbers, etc., are important (see [4–8, 11, 13, 15, 23]). For instance,

$$\begin{aligned}
|S_1(\ell+1,2)| &= \ell! H_\ell^{(1)},\\
|S_1(\ell+1,3)| &= \frac{\ell!}{2}((H_\ell^{(1)})^2 - H_\ell^{(2)}),\\
|S_1(\ell+1,4)| &= \frac{\ell!}{6}((H_\ell^{(1)})^3 - 3H_\ell^{(1)}H_\ell^{(2)} + 2H_\ell^{(3)}),
\end{aligned}$$

and so on. In here, $S_1(\ell,k)$ are called Stirling numbers of the first kind and $H_\mu^{(\lambda)} = \sum_{k=1}^{\mu} \frac{1}{k^\lambda}$ are λ-order harmonic numbers [5, 11]. Notice that $H_\mu^{(1)} = H_\mu$. From the above identities, it can be observed that a generalization of harmonic numbers is related to the first kind Stirling numbers. These connections highlight the significance of harmonic numbers and their generalizations, offering direction for future research.

Recently, several authors have proposed generalizations of harmonic numbers. Studies have explored harmonic numbers and their generalizations [5, 10, 12, 17, 18, 21, 22].

Let $(\omega,\mu) \in \mathbb{R}^+ \times \mathbb{N}$. Genčev [10] defined the $H_\mu(\omega)$ numbers as

$$H_\mu(\omega) = \sum_{k=1}^{\mu} \frac{1}{k\omega^k}$$

and $H_0(\omega) = 0$. A well-known integral representation is given in the form

$$H_\mu(\omega) = \int_{\lambda(\omega)}^{1} \frac{1-(1-z)^\mu}{z} dz$$

with $\lambda(\omega) := \frac{\omega-1}{\omega}$.

Guo and Chu [12] presented the $H_{\mu}(\varsigma)$ numbers for $\varsigma \in \mathbb{R}$,

$$H_{\mu}(\varsigma) = \sum_{k=1}^{\mu} \frac{\varsigma^k}{k} \text{ for } \mu \geq 1.$$

and $H_0(\varsigma) = 0$. For positive integers μ and r, Benjamin et al. [2] introduced the hyperharmonic numbers of order r H_{μ}^{r}, as

$$H_{\mu}^{r} = \sum_{k=0}^{\mu} H_{k}^{r-1},$$

where for $\mu \geq 1$, $H_{\mu}^{0} = \frac{1}{\mu}$ and for $r < 0$ or $\mu \leq 0$, $H_{\mu}^{r} = 0$.

Ömür and Bilgin [21] gave the $H_{\mu}^{r}(\omega)$ numbers as for $\mu, r \geq 1$,

$$H_{\mu}^{r}(\omega) = \sum_{k=1}^{\mu} H_{k}^{r-1}(\omega),$$

where $H_{\mu}^{0}(\omega) = \frac{1}{\mu\omega^{\mu}}$ and for $r < 0$ or $\mu \leq 0$, $H_{\mu}^{r}(\omega) = 0$.

Boyadzhiev [3] presented the following expansion near zero,

$$-\frac{\log(1-ct)}{1-dt} = \sum_{k=1}^{\infty}\sum_{i=1}^{k} \frac{1}{i} c^{i} d^{k-i} t^{k},$$

where $c, d \in \mathbb{R}$.

Binomial coefficients play a crucial role in mathematics and applied sciences. The binomial coefficients are defined by $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ for non-negative integers $n \geq k$ and $\binom{n}{k} = 0$ for $k > n$. These coefficients are widely used in calculating combinations as well as in permutations and probability calculations. Binomial coefficients hold significant importance across a broad range of mathematical applications, spanning from statistics to graph theory and computation. For instance, in probability theory, the likelihood of achieving exactly j successes in n independent Bernoulli trials is expressed by the probability mass function as follows [9, 20]:

$$f(j, n, \psi) = \binom{n}{j} \psi^{j} (1-\psi)^{n-j} \text{ for } \psi \in [0, 1].$$

The q-analogue of μ is defined as $[\mu]_q = 1 + q + q^2 + \cdots + q^{\mu-1}$, where $\lim_{q\to 1}[\mu]_q = \mu$. The q-factorial of these numbers is defined as

$$[\mu]_q! = \begin{cases} 1, & \text{if } \mu = 0, \\ [1]_q[2]_q \cdots [\mu-1]_q[\mu]_q, & \text{if } \mu \geq 1. \end{cases}$$

The q-binomial coefficients play an important role in mathematical combinatorics and number theory. These coefficients are generalizations of the classical binomial coefficients, which are their q-analogues. The q-binomial coefficients are used in various mathematical problems and are of great importance especially in the analysis of q-series and q-measures. In addition, q-binomial coefficients are also widely used in many mathematical fields such as number theory, probability theory, combinatorial design theory, and graph theory. Therefore, it is important to emphasize that q-binomial coefficients are of great importance in mathematical research and applications. For any $\mu, \ell \in \mathbb{N}$, the q-binomial coefficients are defined by

$$\begin{bmatrix} \mu \\ \ell \end{bmatrix}_q = \frac{[\mu]_q!}{[\mu-\ell]_q! \, [\ell]_q!},$$

if $\mu \geq \ell$, and if $\mu < \ell$, then $\begin{bmatrix} \mu \\ \ell \end{bmatrix}_q = 0$. It is clear that $\lim_{q \to 1} \begin{bmatrix} \mu \\ \ell \end{bmatrix}_q = \binom{\mu}{\ell}$, where $\binom{\mu}{\ell}$ is the usual binomial coefficient.

The recurrence relation is satisfied by the q-binomial coefficients

$$\begin{bmatrix} \mu \\ \ell \end{bmatrix}_q = \begin{bmatrix} \mu-1 \\ \ell-1 \end{bmatrix}_q + q^{\ell} \begin{bmatrix} \mu-1 \\ \ell \end{bmatrix}_q,$$

$$\sum_{\ell=1}^{\mu} q^{\mu-\ell} \begin{bmatrix} \mu+j-\ell \\ j \end{bmatrix}_q = \begin{bmatrix} \mu+j \\ j+1 \end{bmatrix}_q, \tag{1}$$

and

$$\sum_{\ell=p}^{\mu-i} q^{(\ell-p)(i+1)} \begin{bmatrix} \ell \\ p \end{bmatrix}_q \begin{bmatrix} \mu-\ell \\ i \end{bmatrix}_q = \begin{bmatrix} \mu+1 \\ p+i+1 \end{bmatrix}_q. \tag{2}$$

As an application field of harmonic numbers and generalizations of them, matrices induced by these numbers and their results are studied by many researchers. For example, Bahşi and Solak [1] defined the $n \times k$ matrix $G_{n,k}^{(r)}$ with entries consist of H_n^r. They gave relation between Pascal matrix and $G_{n,k}^{(r)}$, and the determinant of $G_{n,n}^{(r)}$.

Ömür and Koparal [22] gave two $n \times n$ matrices A_n with $a_{i,j} = H_{i,j}^r$ and B_n with $b_{i,j} = H_{i,\lambda}^j$. The authors gave matrix multiplications of these matrices and some identities, i.e., for $\mu, \lambda, r > 0$,

$$H_{\mu,\lambda}^r = \sum_{k=1}^{\mu} \binom{\mu-k+r-1}{r-1} \frac{1}{k^{\lambda}}.$$

Kızılateş and Tuğlu [14] introduced a q-analogue of several established results and derived certain identities by employing the q-difference operator. For example for $\mu, m \geq 0$

$$\sum_{k=m}^{\mu-1} q^{k-m} \begin{bmatrix} k \\ m \end{bmatrix}_q \widetilde{H}_{k,q} = \begin{bmatrix} \mu \\ m+1 \end{bmatrix}_q \left(\widetilde{H}_{\mu,q} - \frac{q^{m+1}}{[m+1]_q} \right),$$

where $\widetilde{H}_{\mu,q} = \sum\limits_{k=1}^{\mu} \frac{q^k}{[k]_q}$.

Mansour and Shattuck [19] defined the q-analogue of H_{μ}^r, denoted by $H_{\mu,q}^r$, and derived some sums for these numbers. For instance, for $r, \mu \in \mathbb{N}$ and $0 \leq m \leq r$,

$$H_{\mu,q}^r = \sum_{k=1}^{m} q^{k(\mu-m+k)} \begin{bmatrix} m \\ k \end{bmatrix}_q H_{\mu-m+k,q}^{r-k}.$$

Koparal et al. [16] defined the numbers $H_{\mu,q}(\omega)$ as follows:

$$H_{\mu,q}(\omega) = \sum_{k=1}^{\mu} \frac{1}{[k]_q \, \omega^{[k]_q}}.$$

With the help of $H_{\mu,q}(\omega)$, the generalized hyperharmonic numbers of order r, $H^r_{\mu,q}(\omega)$ are defined by for $\mu, r \geq 1$,

$$H^r_{\mu,q}(\omega) = \sum_{k=1}^{\mu} q^k H^{r-1}_{k,q}(\omega),$$

and for $\mu \leq 0$ or $r < 0$, $H^r_{\mu,q}(\omega) = 0$, where $H^0_{\mu,q}(\omega) = \frac{q^{-\mu}}{[\mu]_q \omega^{[\mu]_q}}$.The authors obtained some combinatorial identities involving these numbers. For example, for $0 < m < r < \mu$,

$$\sum_{k=1}^{m} q^{k(k-m+\mu)} \begin{bmatrix} m \\ k \end{bmatrix}_q H^{r-k}_{\mu-m+k,q}(\omega) = H^r_{\mu,q}(\omega) - H^r_{\mu-m,q}(\omega).$$

In this paper, some generalizations of harmonic numbers have been defined using q-analogue in addition to previous studies in the literature, and their identities and matrix equalities have been analyzed using algebraic and combinatorial methods. It is believed that the results obtained in this work will be utilized and provide guidance in future mathematical and applied areas. Furthermore, nature of these numbers is expected to reveal their connections with generalizations of special numbers commonly used in applied sciences, which will facilitate further research in these areas.

2 q-Generalized hyperharmonic numbers of order r with two parameters, $H^r_{\mu,\lambda,q}(\omega, \xi)$

In this section, firstly we will give definitions of various generalizations of harmonic numbers with the help of q-analogue. Secondly, we will introduce the q-generalized hyperharmonic numbers of order r with two parameters and obtain some identities and matrix equalities involving these numbers.

Definition 2.1. *For $\mu, \lambda \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$ such that $\lambda \geq \mu$, let $H_{\mu,\lambda,q}(\omega, \xi)$ and $\widetilde{H}_{\mu,\lambda,q}(\omega, \xi)$ denote two families of numbers defined as follows:*

$$H_{\mu,\lambda,q}(\omega, \xi) = \sum_{k=1}^{\mu} \frac{q^k}{[k]_q} \omega^{[k]_q} \xi^{[\lambda-k]_q}$$

and

$$\widetilde{H}_{\mu,\lambda,q}(\omega, \xi) = \sum_{k=1}^{\mu} \frac{1}{[k]_q} \omega^{[k]_{1/q}} \xi^{[\lambda-k]_{1/q}},$$

and $H_{0,\lambda,q}(\omega, \xi) = \widetilde{H}_{0,\lambda,q}(\omega, \xi) = 0$, where ω, ξ are parameters.

Proposition 2.1. *Let μ, λ be non-negative integers such that $\lambda \geq \mu$. We have*

$$H_{\mu,\lambda,q}(\omega, \xi) = q\widetilde{H}_{\mu,\lambda,1/q}(\omega, \xi).$$

Proof. The demonstration follows a similar approach to that of Theorem 2.2 in [16]. □

Now, with the help of $H_{\mu,\lambda,q}(\omega,\xi)$, we will introduce q-generalized hyperharmonic numbers of order r with two parameters, $H^r_{\mu,\lambda,q}(\omega,\xi)$ as follows:

Definition 2.2. *For $(\mu,\lambda)\in\mathbb{Z}\times\mathbb{N}$, q-generalized hyperharmonic numbers of order r with two parameters, denoted as $H^r_{\mu,\lambda,q}(\omega,\xi)$, are defined by*

$$H^r_{\mu,\lambda,q}(\omega,\xi)=\begin{cases}\sum_{k=1}^{\mu} q^k H^{r-1}_{k,\lambda,q}(\omega,\xi), & \text{for } 1\le\mu\le\lambda \text{ and } r>0,\\ \dfrac{\omega^{[\mu]_q}\xi^{[\lambda-\mu]_q}}{[\mu]_q}, & \text{for } 1\le\mu\le\lambda \text{ and } r=0,\\ 0, & \text{for } \mu\le 0 \text{ or } r<0 \text{ or } \lambda<\mu.\end{cases}$$

The recurrence relation for $H^r_{\mu,\lambda,q}(\omega,\xi)$ is given by

$$H^r_{\mu,\lambda,q}(\omega,\xi)=H^r_{\mu-1,\lambda,q}(\omega,\xi)+q^\mu H^{r-1}_{\mu,\lambda,q}(\omega,\xi). \tag{3}$$

Theorem 2.1. *Let $\mu,\lambda,\in\mathbb{N}$ such that $\lambda\ge\mu$. We have*

$$\sum_{k=1}^{r} q^{\mu(r-k)}H^k_{\mu-1,\lambda,q}(\omega,\xi)=H^r_{\mu,\lambda,q}(\omega,\xi)-\frac{q^{\mu r}}{[\mu]_q}\omega^{[\mu]_q}\xi^{[\lambda-\mu]_q}.$$

Proof. We establish this equality using double induction on r and μ. For $r=\mu=1$, it follows directly from Definition 2.2. When $r=1$, suppose that the statement is valid for $\mu\ge 1$. In that

$$H_{\mu,\lambda,q}(\omega,\xi)=H_{\mu-1,\lambda,q}(\omega,\xi)+\frac{q^\mu}{[\mu]_q}\omega^{[\mu]_q}\xi^{[\lambda-\mu]_q}.$$

For $\mu+1$, utilizing the induction hypothesis and Definition 2.1, we have

$$\begin{aligned}H_{\mu,\lambda,q}(\omega,\xi)+\frac{q^{\mu+1}}{[\mu+1]_q}\omega^{[\mu+1]_q}\xi^{[\lambda-\mu-1]_q}&=H_{\mu-1,\lambda,q}(\omega,\xi)+\frac{q^\mu}{[\mu]_q}\omega^{[\mu]_q}\xi^{[\lambda-\mu]_q}+\frac{q^{\mu+1}}{[\mu+1]_q}\omega^{[\mu+1]_q}\xi^{[\lambda-\mu-1]_q}\\&=\cdots\\&=H_{0,\lambda,q}(\omega,\xi)+q\omega\xi^{[\lambda-1]_q}+\cdots+\frac{q^{\mu+1}}{[\mu+1]_q}\omega^{[\mu+1]_q}\xi^{[\lambda-\mu-1]_q}\\&=\sum_{k=1}^{\mu+1}\frac{q^k}{[k]_q}\omega^{[\lambda]_q}\xi^{[\lambda-k]_q}\\&=H_{\mu+1,\lambda,q}(\omega,\xi).\end{aligned}$$

Lastly, let r be a positive integer and assume that the statement holds for $r-1$. Considering that for all $\mu\ge 1$,

$$\sum_{k=1}^{r-1} q^{\mu(r-1-k)}H^k_{\mu-1,\lambda,q}(\omega,\xi)=H^{r-1}_{\mu,\lambda,q}(\omega,\xi)-\frac{q^{\mu(r-1)}}{[\mu]_q}\omega^{[\mu]_q}\xi^{[\lambda-\mu]_q}. \tag{4}$$

Therefore, we demonstrate that the statement is valid for r. By (3) and (4), we have

$$\sum_{k=1}^{r} q^{\mu(r-k)}H^k_{\mu-1,\lambda,q}(\omega,\xi)=H^r_{\mu-1,\lambda,q}(\omega,\xi)+\sum_{k=1}^{r-1} q^{\mu(r-k)}H^k_{\mu-1,\lambda,q}(\omega,\xi)$$

$$
\begin{aligned}
&= H^r_{\mu-1,\lambda,q}(\omega,\xi) + q^{\mu}\sum_{k=1}^{r-1} q^{\mu(r-1-k)} H^k_{\mu-1,\lambda,q}(\omega,\xi)\\
&= H^r_{\mu-1,\lambda,q}(\omega,\xi) + q^{\mu}\left(H^{r-1}_{\mu,\lambda,q}(\omega,\xi) - \frac{q^{\mu(r-1)}}{[\mu]_q}\omega^{[\mu]_q}\xi^{[\lambda-\mu]_q}\right)\\
&= H^r_{\mu-1,\lambda,q}(\omega,\xi) + q^{\mu}H^{r-1}_{\mu,\lambda,q}(\omega,\xi) - \frac{q^{\mu r}}{[\mu]_q}\omega^{[\mu]_q}\xi^{[\lambda-\mu]_q}\\
&= H^r_{\mu,\lambda,q}(\omega,\xi) - \frac{q^{\mu r}}{[\mu]_q}\omega^{[\mu]_q}\xi^{[\lambda-\mu]_q}.
\end{aligned}
$$

Using the principle of double induction, the desired result holds for all $\mu \geq 1$ and $r \geq 1$. This concludes the proof. □

Lemma 2.1. [16] *Let* $r, \lambda \in \mathbb{N}$ *such that* $0 < \lambda \leq r-1$. *For* $\mu \geq 0$,

$$
\sum_{j=0}^{\mu} q^{j(\lambda-r)} \begin{bmatrix} r-\lambda+j-1 \\ j \end{bmatrix}_q = q^{\mu(\lambda-r)} \begin{bmatrix} \mu+r-\lambda \\ \mu \end{bmatrix}_q.
$$

Theorem 2.2. *Let* $\mu, \lambda \in \mathbb{N}$ *such that* $\lambda \geq \mu$. *For a positive integer* r,

$$
H^r_{\mu,\lambda,q}(\omega,\xi) = \sum_{k=1}^{\mu} \begin{bmatrix} \mu-k+r-1 \\ r-1 \end{bmatrix}_q \frac{q^{kr}}{[k]_q}\omega^{[k]_q}\xi^{[\lambda-k]_q}. \tag{5}
$$

Proof. By Definition 2.2 and Lemma 2.1, using the iteration method, we have

$$
\begin{aligned}
H^r_{\mu,\lambda,q}(\omega,\xi) &= \sum_{k=1}^{\mu} q^k H^{r-1}_{k,\lambda,q}(\omega,\xi)\\
&= q^1 H^{r-1}_{1,\lambda,q}(\omega,\xi) + q^2 H^{r-1}_{2,\lambda,q}(\omega,\xi) + \cdots + q^{\mu} H^{r-1}_{\mu,\lambda,q}(\omega,\xi)\\
&= q^2\left(1+q+\cdots+q^{\mu-1}\right)H^{r-2}_{1,\lambda,q}(\omega,\xi) + q^4\left(1+q+\cdots+q^{\mu-2}\right)H^{r-2}_{2,\lambda,q}(\omega,\xi)\\
&\quad + \cdots + q^{2\mu}H^{r-2}_{\mu,\lambda,q}(\omega,\xi)\\
&= q^2\begin{bmatrix} \mu \\ 1 \end{bmatrix}_q H^{r-2}_{1,\lambda,q}(\omega,\xi) + q^4\begin{bmatrix} \mu-1 \\ 1 \end{bmatrix}_q H^{r-2}_{2,\lambda,q}(\omega,\xi) + \cdots + q^{2\mu}\begin{bmatrix} 1 \\ 1 \end{bmatrix}_q H^{r-2}_{\mu,\lambda,q}(\omega,\xi)\\
&= q^2\begin{bmatrix} \mu \\ 1 \end{bmatrix}_q qH^{r-3}_{1,\lambda,q}(\omega,\xi) + q^4\begin{bmatrix} \mu-1 \\ 1 \end{bmatrix}_q \left(qH^{r-3}_{1,\lambda,q}(\omega,\xi) + q^2H^{r-3}_{2,\lambda,q}(\omega,\xi)\right)\\
&\quad + \cdots + q^{2\mu}\begin{bmatrix} 1 \\ 1 \end{bmatrix}_q \left(q^1H^{r-3}_{1,\lambda,q}(\omega,\xi) + \cdots + q^{\mu}H^{r-3}_{\mu,\lambda,q}(\omega,\xi)\right)\\
&= q^3\begin{bmatrix} \mu+1 \\ 2 \end{bmatrix}_q H^{r-3}_{1,\lambda,q}(\omega,\xi) + q^6\begin{bmatrix} \mu \\ 2 \end{bmatrix}_q H^{r-3}_{2,\lambda,q}(\omega,\xi) + \cdots + q^{3\mu}\begin{bmatrix} 2 \\ 2 \end{bmatrix}_q H^{r-3}_{\mu,\lambda,q}(\omega,\xi)\\
&= \cdots\\
&= q^r\begin{bmatrix} \mu-1+r-1 \\ r-1 \end{bmatrix}_q H^0_{1,\lambda,q}(\omega,\xi) + q^{2r}\begin{bmatrix} \mu-2+r-1 \\ r-1 \end{bmatrix}_q H^0_{2,\lambda,q}(\omega,\xi)\\
&\quad + \cdots + q^{\mu r}\begin{bmatrix} \mu-\mu+r-1 \\ r-1 \end{bmatrix}_q H^0_{\mu,\lambda,q}(\omega,\xi).
\end{aligned}
$$

Thus, by Definition 2.2, the proof is complete. □

Recently, there have been some works including matrices with generalized harmonic numbers [1, 16, 19, 21, 22].

Let λ and n be non-negative integers such that $\lambda \geq n$. Now, we will construct an $n \times n$ matrix $M_n = [m_{i,j}]$ with $m_{i,j} = q^{-ij} H^j_{i,\lambda,q}(\omega, \xi)$, and an $n \times n$ matrix $E_n = [e_{i,j}]$ with

$$e_{i,j} = \begin{cases} \dfrac{\omega^{[i-j+1]_q}\xi^{[\lambda-i+j-1]_q}}{[i-j+1]_q} - \dfrac{\omega^{[i-j]_q}\xi^{[\lambda-i+j]_q}}{[i-j]_q}, & \text{if } i > j, \\ \omega\xi^{[\lambda-1]_q}, & \text{if } i = j, \\ 0, & \text{if } i < j. \end{cases}$$

The following theorem can be stated.

Theorem 2.3. *For $n \in \mathbb{N}$, we have*

$$M_n = E_n P_n, \tag{6}$$

where $P_n = [p_{i,j}]$ is an $n \times n$ q-analogue of Pascal matrix with

$$p_{i,j} = q^{j(1-i)} \begin{bmatrix} j+i-1 \\ j \end{bmatrix}_q .$$

Proof. It is easily seen that for $i = j = 1$,

$$m_{1,1} = \sum_{k=1}^{1} \frac{q^{k-1}}{[k]_q} \omega^{[k]_q} \xi^{[\lambda-k]_q} = \omega\xi^{[\lambda-1]_q} = q^{-1} H^1_{1,\lambda,q}(\omega, \xi).$$

For $i = 1$, $j > 1$, we write

$$m_{1,j} = e_{1,1}p_{1,j} = \omega\xi^{[\lambda-1]_q} \begin{bmatrix} 1+j-1 \\ j \end{bmatrix}_q = \omega\xi^{[\lambda-1]_q} = q^{-j} H^j_{1,\lambda,q}(\omega, \xi),$$

since $H^r_{1,\lambda,q}(\omega, \xi) = q^r \omega\xi^{[\lambda-1]_q}$. For $i > 1$, $j = 1$, we have

$$\begin{aligned} m_{i,1} &= \sum_{k=1}^{n} e_{i,k}p_{k,1} \\ &= \left(\frac{\omega^{[i]_q}\xi^{[\lambda-i]_q}}{[i]_q} - \frac{\omega^{[i-1]_q}\xi^{[\lambda-i+1]_q}}{[i-1]_q} \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix}_q + \left(\frac{\omega^{[i-1]_q}\xi^{[\lambda-i+1]_q}}{[i-1]_q} - \frac{\omega^{[i-2]_q}\xi^{[\lambda-i+2]_q}}{[i-2]_q} \right) q^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \\ &\quad + \cdots + \omega\xi^{[\lambda-1]_q} q^{1-i} \begin{bmatrix} i \\ 1 \end{bmatrix}_q \\ &= \frac{\omega^{[i]_q}\xi^{[\lambda-i]_q}}{[i]_q} + q^{-1} \frac{\omega^{[i-1]_q}\xi^{[\lambda-i+1]_q}}{[i-1]_q} + q^{-2} \frac{\omega^{[i-2]_q}\xi^{[\lambda-i+2]_q}}{[i-2]_q} + \cdots + q^{1-i}\omega\xi^{[\lambda-1]_q} \\ &= \sum_{k=1}^{i} \begin{bmatrix} i-k+1-1 \\ 1-1 \end{bmatrix}_q \frac{q^{k-i}}{[k]_q} \omega^{[k]_q}\xi^{[\lambda-k]_q} \\ &= q^{-i} H^1_{i,\lambda,q}(\omega, \xi). \end{aligned}$$

For $i>1$, $j>1$, we obtain

$$\begin{aligned} m_{i,j} &= \sum_{k=1}^{n} e_{i,k}p_{k,j} = \sum_{k=1}^{i-1} e_{i,k}p_{k,j} + e_{i,i}p_{i,j} \\ &= \sum_{k=1}^{i} q^{j-kj} \begin{bmatrix} k+j-1 \\ j \end{bmatrix}_q \frac{\omega^{[i-k+1]_q}\xi^{[\lambda-i+k-1]_q}}{[i-k+1]_q} - \sum_{k=1}^{i-1} q^{j-kj} \begin{bmatrix} k+j-1 \\ j \end{bmatrix}_q \frac{\omega^{[i-k]_q}\xi^{[\lambda-i+k]_q}}{[i-k]_q}. \end{aligned}$$

From (1), we have

$$\begin{aligned} m_{i,j} &= \sum_{k=1}^{i} q^{j-kj} \frac{\omega^{[i-k+1]_q}\xi^{[\lambda-i+k-1]_q}}{[i-k+1]_q} \sum_{t=1}^{k} q^{k-t} \begin{bmatrix} k+j-1-t \\ j-1 \end{bmatrix}_q \\ &\quad - \sum_{k=1}^{i-1} q^{j-kj} \frac{\omega^{[i-k]_q}\xi^{[\lambda-i+k]_q}}{[i-k]_q} \sum_{t=1}^{k} q^{k-t} \begin{bmatrix} k+j-1-t \\ j-1 \end{bmatrix}_q \\ &= q^{-ij} \left(\sum_{t=1}^{i-1} q^{i-t+1} \sum_{k=1}^{i-t+1} q^{k(j-1)} \frac{\omega^{[k]_q}\xi^{[i-k]_q}}{[k]_q} \begin{bmatrix} i-t-k+j \\ j-1 \end{bmatrix}_q \right. \\ &\quad \left. - \sum_{t=1}^{i-1} q^{i-t+j} \sum_{k=1}^{i-t} q^{k(j-1)} \frac{\omega^{[k]_q}\xi^{[i-k]_q}}{[k]_q} \begin{bmatrix} i-t-k+j-1 \\ j-1 \end{bmatrix}_q + q^j\omega\xi^{[\lambda-1]_q} \right). \end{aligned}$$

Taking $n=i-t-k+j$ and $k=j-1$ in (2), we have

$$\begin{aligned} m_{i,j} &= q^{-ij} \left(\sum_{t=1}^{i-1} q^{(i-t+1)(j-1)+i-t+1} \frac{\omega^{[i-t+1]_q}\xi^{[\lambda-i+t-1]_q}}{[i-t+1]_q} \frac{\omega^{[k]_q}\xi^{[\lambda-k]_q}}{[k]_q} \right. \\ &\quad \left. + \sum_{t=1}^{i-1} q^{i-t+1} \sum_{k=1}^{i-t} q^{k(j-1)} \begin{bmatrix} i-t-k+j-1 \\ j-2 \end{bmatrix}_q + q^j\omega\xi^{[\lambda-1]_q} \right) \\ &= q^{-ij} \left(\sum_{t=1}^{i-1} q^{i-t+1} \sum_{k=1}^{i-t+1} q^{k(j-1)} \frac{\omega^{[k]_q}\xi^{[\lambda-k]_q}}{[k]_q} \begin{bmatrix} i-t-k+j-1 \\ j-2 \end{bmatrix}_q + q^j\omega\xi^{[\lambda-1]_q} \right) \\ &= q^{-ij} \sum_{t=1}^{i} q^{i-t+1} \sum_{k=1}^{i-t+1} q^{k(j-1)} \frac{\omega^{[k]_q}\xi^{[\lambda-k]_q}}{[k]_q} \begin{bmatrix} i-t-k+j-1 \\ j-2 \end{bmatrix}_q \\ &= q^{-ij} \sum_{t=1}^{i} q^{i-t+1} H^{j-1}_{i-t+1,\lambda,q}(\omega,\xi) \\ &= q^{-ij} \sum_{t=1}^{i} q^{t} H^{j-1}_{t,\lambda,q}(\omega,\xi) = q^{-ij} H^{j}_{i,\lambda,q}(\omega,\xi), \end{aligned}$$

as claimed. □

Corollary 2.1. *Let $n, \lambda \in \mathbb{N}$ such that $\lambda \geq n$. We have*

$$\det M_n = q^{-2\binom{n+1}{3}} \omega^n \xi^{n[\lambda-1]_q}.$$

Proof. From Theorem 2.3 and $\det M_n = \det E_n \det P_n$, the proof is clear. □

Corollary 2.2. *Let $\mu, \lambda \in \mathbb{N}$ such that $\lambda \geq \mu$. For a positive integer r,*

$$\sum_{k=1}^{\mu-1} \left(\frac{\omega^{[\mu-k+1]_q} \xi^{[\lambda-\mu+k-1]_q}}{[\mu-k+1]_q} - \frac{\omega^{[\mu-k]_q} \xi^{[\lambda-\mu+k]_q}}{[\mu-k]_q} \right) q^{r(1-k)} \begin{bmatrix} k+r-1 \\ r \end{bmatrix}_q$$
$$= q^{-\mu r} H^r_{\mu,\lambda,q}(\omega, \xi) - \omega \xi^{[\lambda-1]_q} q^{r(1-\mu)} \begin{bmatrix} \mu+r-1 \\ r \end{bmatrix}_q .$$

Proof. Equating the (n, r)-entries of $M_n = E_n P_n$, the result is given. □

Theorem 2.4. *For $n \in \mathbb{N}$, we have*

$$P_n = L_n U_n, \tag{7}$$

where $L_n = [l_{i,j}]$ and $U_n = [u_{i,j}]$ are $n \times n$ matrices with

$$l_{i,j} = q^{j(j-i)} \begin{bmatrix} i \\ j \end{bmatrix}_q \text{ and } u_{i,j} = q^{j(1-i)} \begin{bmatrix} j-1 \\ i-1 \end{bmatrix}_q .$$

Corollary 2.3. *For $n \in \mathbb{N}$, we have*

$$q^{j(1-i)} \begin{bmatrix} i+j-1 \\ j \end{bmatrix}_q = \sum_{k=1}^{n} q^{k(k-i)+j(1-k)} \begin{bmatrix} i \\ k \end{bmatrix}_q \begin{bmatrix} j-1 \\ k-1 \end{bmatrix}_q .$$

Proof. With the help of (2.2) and matrix multiplication, we have the proof immediately. □

Corollary 2.4. *Let $P_n^{-1} = [p_{i,j}^{-1}]$. For positive integer $n \geq i, j$, we have*

$$p_{i,j}^{-1} = \sum_{k=1}^{n} (-1)^{i+j} q^{\binom{i-1}{2}+\binom{j}{2}+k-1} \begin{bmatrix} k-1 \\ i-1 \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q . \tag{8}$$

Proof. From (7), we write $P_n^{-1} = U_n^{-1} L_n^{-1}$, where $U_n^{-1} = [u_{i,j}^{-1}]$ with and $L_n^{-1} = [l_{i,j}^{-1}]$ with

$$u_{i,j}^{-1} = (-1)^{i+j} q^{\binom{j+1}{2}+\binom{i-1}{2}-1} \begin{bmatrix} j-1 \\ i-1 \end{bmatrix}_q$$

and

$$l_{i,j}^{-1} = (-1)^{i+j} q^{\binom{j}{2}-\binom{i}{2}} \begin{bmatrix} i \\ j \end{bmatrix}_q .$$

Using matrix multiplication, we have the proof. □

Corollary 2.5. *Let $m, n, t \in \mathbb{N}$ such that $n \geq m, t$. We have*

$$\sum_{i=1}^{n} \sum_{k=1}^{n} (-1)^i q^{\binom{i-1}{2}-mi+k} \begin{bmatrix} k-1 \\ i-1 \end{bmatrix}_q \begin{bmatrix} k \\ t \end{bmatrix}_q H^i_{m,\lambda,q}(\omega, \xi)$$
$$= (-1)^t q^{1-\binom{t}{2}} \begin{cases} \frac{\omega^{[m-t+1]_q} \xi^{[\lambda-m+t-1]_q}}{[m-t+1]_q} - \frac{\omega^{[m-t]_q} \xi^{[\lambda-m+t]_q}}{[m-t]_q}, & \text{if } m \neq t, \\ \omega \xi^{[\lambda-1]_q}, & \text{if } m = t. \end{cases}$$

Proof. By (6), (8), we write

$$M_n P_n^{-1} = E_n, \tag{9}$$

then, by equating (m, t)-entries of (9), this completes the proof. □

Let $A_n = [a_{i,j}]$ with $a_{i,j} = H^r_{i,\lambda,q}(j,\xi)$ and $B_n = [b_{i,j}]$ with $b_{i,j} = H^s_{i,\lambda,q}(j,\xi)$ be $n \times n$ matrices. A relationship between these matrices is presented below:

Theorem 2.5. *Let n, r, s be non-negative integers such that $r > s$. We have*

$$A_n = F_n B_n, \tag{10}$$

where the $n \times n$ matrix $F_n = [f_{i,j}]$ is given by

$$f_{i,j} = \begin{cases} q^{(r-s)j} \left[{i-j+r-s-1 \atop i-j} \right]_q, & \text{if } i \geq j, \\ 0, & \text{if } i < j. \end{cases}$$

Proof. The demonstration follows a similar approach to the proof of Theorem 2.3. □

Corollary 2.6. *Let μ, λ, r, s be non-negative integers such that $\lambda \geq \mu$ and $r > s$. We have*

$$H^r_{\mu,\lambda,q}(\omega,\xi) = \sum_{k=1}^{\mu} q^{(r-s)k} \begin{bmatrix} \mu - k + r - s - 1 \\ \mu - k \end{bmatrix}_q H^s_{k,\lambda,q}(\omega,\xi).$$

Proof. Equating (n, ω)-entries of (10) gives the claimed result. □

Let n, k and λ be non-negative integers such that $\lambda \geq n$. Define the $n \times k$ matrix $G^{(r)}_{n,k}$ and the $n \times n$ matrix A as follows:

$$G^{(r)}_{n,k} = \begin{bmatrix} H^r_{1,\lambda,q} & H^{r+1}_{1,\lambda,q} & \cdots & H^{r+k-1}_{1,\lambda,q} \\ H^r_{2,\lambda,q} & H^{r+1}_{2,\lambda,q} & \cdots & H^{r+k-1}_{2,\lambda,q} \\ \vdots & \vdots & \ddots & \vdots \\ H^r_{n,\lambda,q} & H^{r+1}_{n,\lambda,q} & \cdots & H^{r+k-1}_{n,\lambda,q} \end{bmatrix} \text{ and } A_n = \begin{bmatrix} q & 0 & \cdots & 0 \\ q & q^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ q & q^2 & \cdots & q^n \end{bmatrix}.$$

Lemma 2.2. *For $r \in \mathbb{N}$ we have $A^r_n = [a_{i,j}]$ with*

$$a_{i,j} = \begin{cases} q^{jr} \left[{i-j+r-1 \atop r-1} \right]_q, & i \geq j, \\ 0, & i < j. \end{cases}$$

Proof. By an inductive argument and using combinatorial identities, the desired result is obtained. □

Theorem 2.6. *Let r, n, k be non-negative integers. We have*

$$G^{(r+1)}_{n,k} = A^r_n G^{(1)}_{n,k}. \tag{11}$$

Proof. From matrix multiplication and Definition 2.2, we get

$$G^{(r+1)}_{n,k} = A_n G^{(r)}_{n,k}.$$

By using inductive argument, we have the proof. □

Corollary 2.7. *Let μ, m, r, λ be non-negative integers such that $\lambda \geq \mu$. Then,*

$$H^m_{\mu,\lambda,q}(\omega,\xi) = \sum_{k=1}^{\mu} (-1)^{\mu-k} q^{\binom{\mu-k}{2} - \mu r} \begin{bmatrix} r \\ \mu - k \end{bmatrix}_q H^{r+m}_{k,\lambda,q}(\omega,\xi).$$

Proof. Let the inverse matrix of $A_n^r = [a_{i,j}]$ be denoted by $(A_n^r)^{-1} = [a_{i,j}^{-1}]$. Then,

$$a_{i,j}^{-1} = \begin{cases} (-1)^{i-j} q^{\binom{i-j}{2} - ir} \begin{bmatrix} r \\ i-j \end{bmatrix}_q, & \text{if } i \geq j, \\ 0, & \text{if } i < j, \end{cases}$$

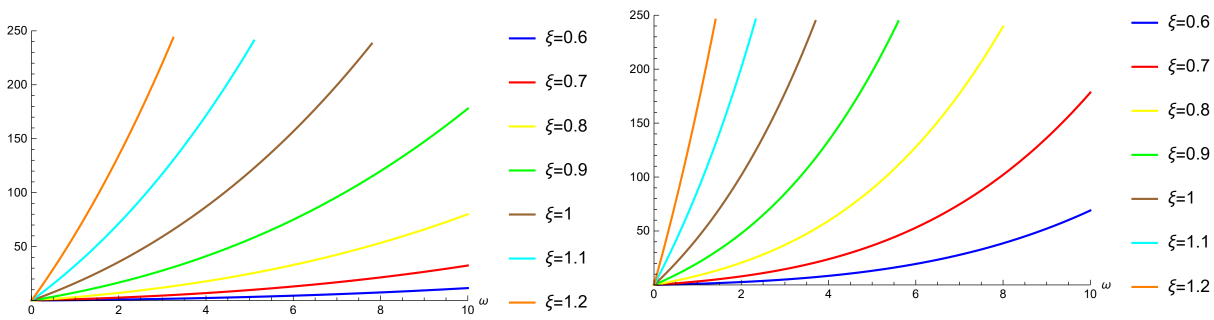
since

$$\sum_{k=1}^{\mu} a_{i,k} a_{k,j}^{-1} = \sum_{k=j}^{i} (-1)^{k-j} q^{\binom{k-j}{2}} \begin{bmatrix} i-k+r-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} r \\ k-j \end{bmatrix}_q = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

If we multiply both sides of (11) with $(A_n^r)^{-1}$, we get $(A_n^r)^{-1} G_{n,k}^{(r+1)} = G_{n,k}^{(1)}$. From here, by equating (n, m)-entries of this matrix equality, we obtain the claimed result. □

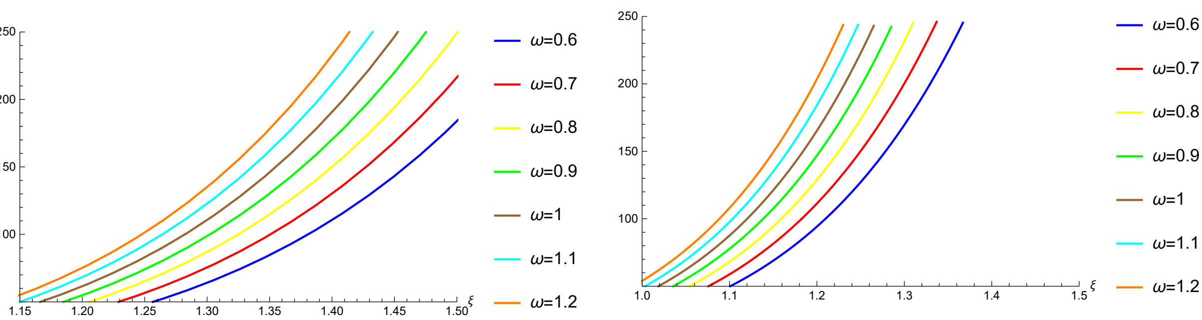
3 Visualizations of $H_{\mu,\lambda,q}^{r}(\omega, \xi)$

In this section, the numerical values of $H_{\mu,\lambda,q}^{r}(\omega, \xi)$ have been investigated using by using *Mathematica 11.2*. Firstly, the independent value ω is examined for $n = 3, 4$, $l = 10$, $r = 7$, $q = 0.95$, while keeping ξ constant. Then, the independent value ξ is examined while keeping ω constant. These formations of $H_{\mu,\lambda,q}^{r}(\omega, \xi)$ numbers are shown in Figure 1 and Figure 2.



(a) $H_{3,10,0.95}^{7}(\omega, \xi)$ values for some fixed ξ. (b) $H_{4,10,0.95}^{7}(\omega, \xi)$ values for some fixed ξ.

Figure 1: Comparison of $H_{3,10,0.95}^{7}(\omega, \xi)$ and $H_{4,10,0.95}^{7}(\omega, \xi)$.

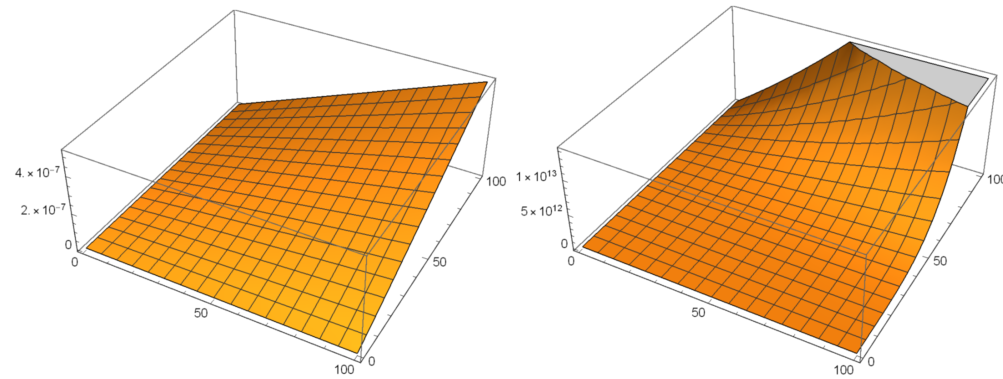


(a) $H_{3,10,0.95}^{7}(\omega, \xi)$ values for some fixed ω. (b) $H_{4,10,0.95}^{7}(\omega, \xi)$ values for some fixed ω.

Figure 2: Comparison of $H_{3,10,0.95}^{7}(\omega, \xi)$ and $H_{4,10,0.95}^{7}(\omega, \xi)$.

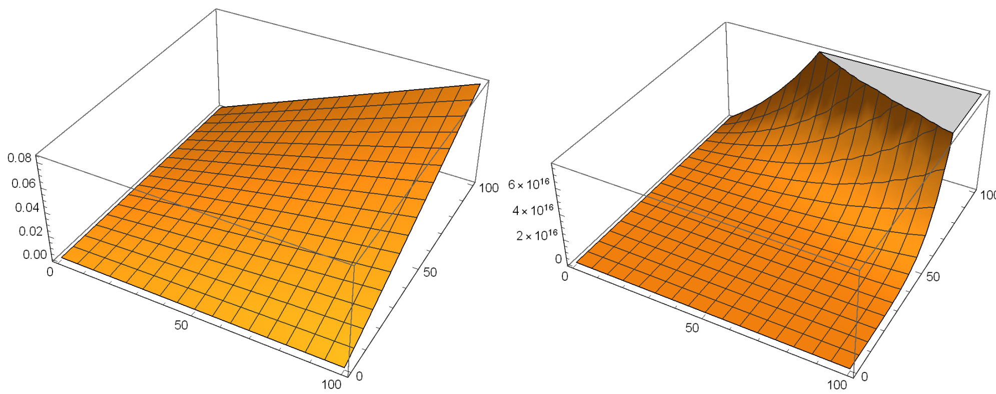
Secondly, we present some three-dimensional graphs illustrating the behavior of the $H^r_{\mu,\lambda,q}(\omega,\xi)$ function at specific values of the parameter q and non-negative integers μ, λ, r where $\lambda \geq \mu$. By visualizing the function for $q = 0.05$ and $q = 0.95$, we can observe the variations and trends of the function's surfaces and gain a comprehensive understanding of how the variables ω and ξ combine to shape the resulting surfaces.

The results clearly indicate that despite the varying values of μ, λ, and r, $H^r_{\mu,\lambda,q}(\omega,\xi)$ in both graphs in Figure 3(a) and 4(a) exhibit a flatter shape within the given range, while the graphs in Figure 3(b) and 4(b) demonstrate a steeper inclination. Notice that $0 \leq H^8_{6,6,0.05}(\omega,\xi) \leq 4\times10^{-7}$, $0 \leq H^8_{6,6,0.95}(\omega,\xi) \leq 4 \times 10^{13}$ when $0 \leq \omega,\xi \leq 100$ in Figure 3 and $0 \leq H^4_{7,9,0.05}(\omega,\xi) \leq 8 \times 10^{-2}$, $0 \leq H^4_{7,9,0.95}(\omega,\xi) \leq 6 \times 10^{16}$ in Figure 4.



(a) $\mu = \lambda = 6, r = 8$ and $q = 0.05$ (b) $\mu = \lambda = 6, r = 8$ and $q = 0.95$

Figure 3: Graphs of $H^8_{6,6,q}(\omega,\xi)$ for $q = 0.05$ and $q = 0.95$ values, where $0 \leq \omega,\xi \leq 100$.



(a) $\mu = 7, \lambda = 9, r = 4$ and $q = 0.05$ (b) $\mu = 7, \lambda = 9, r = 4$ and $q = 0.95$

Figure 4: Graphs of $H^4_{7,9,q}(\omega,\xi)$ for $q = 0.05$ and $q = 0.95$ values, where $0 \leq \omega,\xi \leq 100$.

4 Conclusions and future work

In this paper, we present a generalization of harmonic numbers using the q-analogue technique along with the parameters ω and ξ. We denote this generalization as $H_{\mu,\lambda,q}(\omega,\xi)$. In the context of this generalization, we introduce q-generalized hyperharmonic numbers of order r with two parameters, denoted by $H^{r}_{\mu,\lambda,q}(\omega,\xi)$, which further extend the concept. Moreover, we provide novel identities and matrix equations involving these numbers.

The derived results offer significant implications for studying generalizations of harmonic numbers and find applications in fields such as combinatorics, number theory, and mathematical analysis. Furthermore, by considering additional more parameters, we anticipate that future research can broaden the scope of this study and lead to further advancements.

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